A DIRECT IMAGE ERROR THEORY*

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1. In a previous paper[1] the author proposed a direct approach to the problem of geometrical optics. In this paper we shall give a new image error theory, to the fifth order, which seems to be more adapted to the practical problems than former theories. We are given a rotationally symmetric system. Let us choose two Cartesian systems, one in object space and the other in image space, such that the x, x' and y, y' axes have the same directions and the z, z' axes coincide with the optical axis of the system.

A ray is given in object (image) space by the coordinates x, y, (x', y') of its intersection point with the plane $z=0$, $(z'=0)$. The optical direction cosines (the direction cosines multiplied by the refractive indices n and n', respectively) may be designated by the Greek letters

$$\xi, \eta, \zeta = \sqrt{n^2 - (\xi^2 + \eta^2)}; \qquad \xi', \eta', \zeta' = \sqrt{n'^2 - (\xi'^2 + \eta'^2)}.$$

The fundamental problem of practical optics is to find x', y', ξ', η', when x, y, ξ, η are given. Because of the rotational symmetry, four functions, A, B, C, D, exist such that,

$$\begin{aligned} x' &= Ax + B\xi, & \xi' &= Cx + D\xi, \\ y' &= Ay + B\eta, & \eta' &= Cy + D\eta. \end{aligned} \tag{1}$$

where A, B, C, D depend only on the three symmetric functions u_1, u_2, u_3 of our coordinates:

$$u_1 = \tfrac{1}{2}(x^2 + y^2), \qquad u_2 = x\xi + y\eta, \qquad u_3 = \tfrac{1}{2}(\xi^2 + \eta^2). \tag{2}$$

We found in the previous paper[1] that, according to the laws of geometrical optics, A, B, C, D cannot be arbitrary functions, but must fulfill one finite and three differential equations, viz.,

$$AD - BC = 1$$

and

$$\begin{aligned} A\left(\frac{\partial C}{\partial u_2} - \frac{\partial D}{\partial u_1}\right) - C\left(\frac{\partial A}{\partial u_2} - \frac{\partial B}{\partial u_1}\right) + 2u_1\left(\frac{\partial A}{\partial u_1}\frac{\partial C}{\partial u_2} - \frac{\partial A}{\partial u_2}\frac{\partial C}{\partial u_1}\right) \\ + u_2\left(\frac{\partial A}{\partial u_1}\frac{\partial D}{\partial u_2} - \frac{\partial A}{\partial u_2}\frac{\partial D}{\partial u_1} + \frac{\partial B}{\partial u_1}\frac{\partial C}{\partial u_2} - \frac{\partial B}{\partial u_2}\frac{\partial C}{\partial u_1}\right) \\ + 2u_3\left(\frac{\partial B}{\partial u_1}\frac{\partial D}{\partial u_2} - \frac{\partial B}{\partial u_2}\frac{\partial D}{\partial u_1}\right) = 0, \end{aligned}$$

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[1] M. Herzberger, *Direct methods in geometrical optics*, Trans. Amer. Math. Soc., **53**, 218 (1943).

$$
\begin{aligned}
&A\left(\frac{\partial C}{\partial u_3}-\frac{\partial D}{\partial u_2}\right)+B\left(\frac{\partial C}{\partial u_2}-\frac{\partial D}{\partial u_1}\right)-C\left(\frac{\partial A}{\partial u_3}-\frac{\partial B}{\partial u_2}\right)\\
&\qquad -D\left(\frac{\partial A}{\partial u_2}-\frac{\partial B}{\partial u_1}\right)+2u_1\left(\frac{\partial A}{\partial u_1}\frac{\partial C}{\partial u_3}-\frac{\partial A}{\partial u_3}\frac{\partial C}{\partial u_1}\right)\\
&\qquad +u_2\left(\frac{\partial A}{\partial u_1}\frac{\partial D}{\partial u_3}-\frac{\partial A}{\partial u_3}\frac{\partial D}{\partial u_1}+\frac{\partial B}{\partial u_1}\frac{\partial C}{\partial u_3}-\frac{\partial B}{\partial u_3}\frac{\partial C}{\partial u_1}\right)\\
&\qquad +2u_3\left(\frac{\partial B}{\partial u_1}\frac{\partial D}{\partial u_3}-\frac{\partial B}{\partial u_3}\frac{\partial D}{\partial u_1}\right)=0,\\
&B\left(\frac{\partial C}{\partial u_3}-\frac{\partial D}{\partial u_2}\right)-D\left(\frac{\partial A}{\partial u_3}-\frac{\partial B}{\partial u_2}\right)+2u_1\left(\frac{\partial A}{\partial u_2}\frac{\partial C}{\partial u_3}-\frac{\partial A}{\partial u_3}\frac{\partial C}{\partial u_2}\right)\\
&\qquad +u_2\left(\frac{\partial A}{\partial u_2}\frac{\partial D}{\partial u_3}-\frac{\partial A}{\partial u_3}\frac{\partial D}{\partial u_2}+\frac{\partial B}{\partial u_2}\frac{\partial C}{\partial u_3}-\frac{\partial B}{\partial u_3}\frac{\partial C}{\partial u_2}\right)=0.^*
\end{aligned}
\tag{A}
$$

It is the purpose of this paper to develop from formulae (A) the theory of image errors. Developing A, B, C, D into a series with respect to u_1, u_2, u_3, we can write

$$
\begin{aligned}
A &= A_0 + A_1u_1 + A_2u_2 + A_3u_3\\
&\quad + \tfrac{1}{2}(A_{11}u_1^2 + 2A_{12}u_1u_2 + A_{22}u_2^2 + \cdots + A_{33}u_3^2),
\end{aligned}
\tag{3}
$$

and for B, C, D, correspondingly. Inserting (3) in (A) and comparing coefficients leads to the first-order equation:

$$A_0D_0 - B_0C_0 = 1; \tag{4}$$

the third-order equations

$$
\begin{aligned}
A_0D_1 + A_1D_0 &= B_0C_1 + B_1C_0,\\
A_0D_2 + A_2D_0 &= B_0C_2 + B_2C_0,\\
A_0D_3 + A_3D_0 &= B_0C_3 + B_3C_0;
\end{aligned}
\tag{5a}
$$

and

$$
\begin{aligned}
A_0(C_2 - D_1) - C_0(A_2 - B_1) &= 0,\\
A_0(C_3 - D_2) + B_0(C_2 - D_1) - C_0(A_3 - B_2) - D_0(A_2 - B_1) &= 0,\\
B_0(C_3 - D_2) - D_0(A_3 - B_2) &= 0;
\end{aligned}
\tag{5b}
$$

and finally the fifth-order equations:

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* If we differentiate the finite equation above with respect to u_1, u_2, u_3 and subtract from each of the equations (7) of the former paper,[1] equation (A) above results.

$$
\begin{aligned}
A_0D_{11} + D_0A_{11} - B_0C_{11} - C_0B_{11} &= 2(B_1C_1 - A_1D_1),\\
A_0D_{12} + D_0A_{12} - B_0C_{12} - C_0B_{12} &= B_1C_2 + B_2C_1 - A_1D_2 - A_2D_1,\\
A_0D_{13} + D_0A_{13} - B_0C_{13} - C_0B_{13} &= B_1C_3 + B_3C_1 - A_1D_3 - A_3D_1,\\
A_0D_{22} + D_0A_{22} - B_0C_{22} - C_0B_{22} &= 2(B_2C_2 - A_2D_2),\\
A_0D_{23} + D_0A_{23} - B_0C_{23} - C_0B_{23} &= B_2C_3 + B_3C_2 - A_2D_3 - A_3D_2,\\
A_0D_{33} + D_0A_{33} - B_0C_{33} - C_0B_{33} &= 2(B_3C_3 - A_3D_3),
\end{aligned}
\tag{6a}
$$

and

$$
\begin{aligned}
&A_0(C_{21}-D_{11})-C_0(A_{21}-B_{11})=C_1(A_2-B_1)-A_1(C_2-D_1)-2(A_1C_2-A_2C_1),\\
&A_0(C_{22}-D_{12})-C_0(A_{22}-B_{12})=C_2(A_2-B_1)-A_2(C_2-D_1)-(A_1D_2-A_2D_1)\\
&\qquad\qquad -(B_1C_2-B_2C_1),\\
&A_0(C_{23}-D_{13})-C_0(A_{23}-B_{13})=C_3(A_2-B_1)-A_3(C_2-D_1)-2(B_1D_2-B_2D_1),\\
&A_0(C_{31}-D_{21})+B_0(C_{21}-D_{11})-C_0(A_{31}-B_{21})-D_0(A_{21}-B_{11})\\
&\quad =-B_1(C_2-D_1)-A_1(C_3-D_2)+D_1(A_2-B_1)+C_1(A_3-B_2)-2(A_1C_3-A_3C_1),\\
&A_0(C_{32}-D_{22})+B_0(C_{22}-D_{12})-C_0(A_{32}-B_{22})-D_0(A_{22}-B_{12})\\
&\quad =-B_2(C_2-D_1)-A_2(C_3-D_2)+D_2(A_2-B_1)+C_2(A_3-B_2)\\
&\quad -(A_1D_3-A_3D_1)-(B_1C_3-B_3C_1),\\
&A_0(C_{33}-D_{23})+B_0(C_{23}-D_{13})-C_0(A_{33}-B_{23})-D_0(A_{23}-B_{13})\\
&\quad =-B_3(C_2-D_1)-A_3(C_3-D_2)+D_3(A_2-B_1)+C_3(A_3-B_2)-2(B_1D_3-B_3D_1),\\
&B_0(C_{31}-D_{21})-D_0(A_{31}-B_{21})=D_1(A_3-B_2)-B_1(C_3-D_2)-2(A_2C_3-A_3C_2),\\
&B_0(C_{32}-D_{22})-D_0(A_{32}-B_{22})=D_2(A_3-B_2)-B_2(C_3-D_2)\\
&\qquad\qquad -(A_2D_3-A_3D_2)-(B_2C_3-B_3C_2),\\
&B_0(C_{33}-D_{23})-D_0(A_{33}-B_{23})=D_3(A_3-B_2)-B_3(C_3-D_2)-2(B_2D_3-B_3D_2).
\end{aligned}
\tag{6b}
$$

Moreover, the (6+9) equations (6) are not independent; they are connected by the identity

$$
\begin{aligned}
&[A_0(C_{32}-D_{22})+B_0(C_{22}-D_{12})-C_0(A_{32}-B_{22})-D_0(A_{22}-B_{12})]\\
&\quad +[A_0D_{22}+D_0A_{22}-B_0C_{22}-C_0B_{22}]-[A_0D_{13}+D_0A_{13}-B_0C_{13}-C_0B_{13}]\\
&\quad -[B_0(C_{31}-D_{21})-D_0(A_{31}-B_{21})]-[A_0(C_{23}-D_{13})-C_0(A_{23}-B_{13})]\equiv 0.
\end{aligned}
\tag{7}
$$

2. Gaussian optics. Let us consider first the rays in the neighborhood of the axis. Let us assume that u_1, u_2, u_3 are so small that we can assume functions A, B, C, D to be equal to their constant members:

$$
\begin{aligned}
x' &= A_0x + B_0\xi, & \xi' &= C_0x + D_0\xi,\\
y' &= A_0y + B_0\eta, & \eta' &= C_0y + D_0\eta;
\end{aligned}
\tag{8}
$$

where $A_0D_0-B_0C_0=1$.

The evaluation of these equations and the investigation of the geometrical meaning of the coefficients form the content of Gaussian optics.

We shall not go into great detail here, but refer the reader to the discussion in the Journal of the Optical Society of America.[2]

Equations (8) can be inverted, and we obtain then

$$\begin{aligned} x &= D_0x' - B_0\xi', & \xi &= -C_0x' + A_0\xi', \\ y &= D_0y' - B_0\eta', & \eta &= -C_0y' + A_0\eta'. \end{aligned} \tag{9}$$

Let us investigate what happens if one of the coefficients vanishes.

$A_0=0$ means that for $\xi=\eta=0$, $x'=y'=0$, which means that the bundle of rays parallel to the axis converges to the image origin. We say that the image origin is at the focal point of the system.

$B_0=0$ means that for $x=y=0$, $x'=y'=0$. The rays through the object origin meet at the image origin. We say then that object and image origin are optically *conjugate.*

$C_0=0$ means that $\xi=\eta=0$ implies $\xi'=\eta'=0$, or, a bundle of rays entering the system parallel to the axis emerges parallel to the axis. The system is a *telescopic* system.

$D_0=0$ means that $x=y=0$ implies $\xi'=\eta'=0$. The rays through the object origin emerge parallel to the axis. The object origin is the object-side (front) *focal point.*

3. Image error functions. Let us for finite rays* project image point and direction back into the object space, according to Gaussian optics. That means we form equations (9) for our finite rays. The ensuing expressions may be called the equivalent object coordinates $\tilde{x}$, $\tilde{y}$, $\tilde{\xi}$, $\tilde{\eta}$. We have from (9) and (1)

$$\begin{aligned} \tilde{x} &= D_0x' - B_0\xi' = (D_0A - B_0C)x + (D_0B - B_0D)\xi = ax + b\xi, \\ \tilde{\xi} &= -C_0x' + A_0\xi' = (-C_0A + A_0C)x + (-C_0B + A_0D)\xi = cx + d\xi; \end{aligned} \tag{10}$$

and analogously,

$$\begin{aligned} \tilde{y} &= ay + b\eta, \\ \tilde{\eta} &= cy + d\eta. \end{aligned}$$

a, b, c, d are with A, B, C, D functions of u_1, u_2, u_3, and we have

$$\begin{aligned} a_0 &= d_0 = 1, \\ b_0 &= c_0 = 0, \end{aligned} \tag{11}$$

the values a_0, b_0, c_0, d_0, being the limits of a, b, c, d for $u_i=0$. If Gaussian optics were correct, we would have equation (11) for all values of u_i, that is for finite aperture and finite field. The deviation from its constant term as a

[2] M. Herzberger, ***On the fundamental optical invariant, the optical tetrality principle, and on the new development of Gaussian optics based on this law,*** J. Opt. Soc. Amer. **25**, 295–304 (1935).

* **The expression finite is used in distinction from paraxial rays, rays near the axis.**

function of aperture and field is therefore a measure of the image errors. We call a, b, c, d the error functions.

In the nomenclature of matrix algebra we can express these equations as follows:

Let

$$\begin{pmatrix} x' \\ \xi' \end{pmatrix} = M\begin{pmatrix} x \\ \xi \end{pmatrix}, \qquad \begin{pmatrix} x_0' \\ \xi_0' \end{pmatrix} = M_0\begin{pmatrix} x \\ \xi \end{pmatrix},$$
$$\begin{pmatrix} y' \\ \eta' \end{pmatrix} = M\begin{pmatrix} y \\ \eta \end{pmatrix}, \qquad \begin{pmatrix} y_0' \\ \eta_0' \end{pmatrix} = M_0\begin{pmatrix} y \\ \eta \end{pmatrix}. \tag{12}$$

x_0', y_0', ξ_0', η_0' would be the coordinates of the image ray if Gaussian optics were valid. Equations (12) combine equations (1) and (8), M being the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and M_0 being the matrix $\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$.

We have then

$$\begin{pmatrix} x \\ \tilde{\xi} \end{pmatrix} = M_0^{-1}M\begin{pmatrix} x \\ \xi \end{pmatrix} = m\begin{pmatrix} x \\ \xi \end{pmatrix},$$

where

$$m = M_0^{-1}M, \qquad M = M_0m. \tag{13}$$

From (13) it is obvious that the determinant of m is equal to unity. Therefore,

$$ad - bc = 1. \tag{14}$$

The reader can verify for himself that a, b, c, d fulfill equations (A) and therefore equations (5) and (6), which simplify considerably, owing to the fact that (11) is fulfilled.

Equations (13) can be written explicitly

$$\begin{aligned} a &= D_0A - B_0C, & A &= A_0a + B_0c, \\ b &= D_0B - B_0D, & B &= A_0b + B_0d, \\ c &= -C_0A + A_0C, & C &= C_0a + D_0c, \\ d &= -C_0B + A_0D, & D &= C_0b + D_0d. \end{aligned} \tag{15}$$

Differentiation and substitution in (A) prove our statement.

4. Third-order theory. The third-order image errors are usually called Seidel errors.

From our point of view, we obtain the image errors by inserting (11) into (5). Abbreviating $(\partial a/\partial u_k)_{u_i=0}$ by a_k

$$\begin{aligned} a_1 + d_1 &= 0 & c_2 &= d_1 \\ a_2 + d_2 &= 0 & c_3 &= b_1 \\ a_3 + d_3 &= 0 & a_3 &= b_2. \end{aligned} \tag{16}$$

Equations (16) lead to the conclusion that only six of these twelve coefficients are independent. Equations (16) are identically fulfilled by selecting six parameters k_{ik} with permutable indices such that

$$\begin{aligned} a_1 &= k_{21} & b_1 &= k_{31} & c_1 &= -k_{11} & d_1 &= -k_{21}\\ a_2 &= k_{22} & b_2 &= k_{32} & c_2 &= -k_{12} & d_2 &= -k_{22}\\ a_3 &= k_{23} & b_3 &= k_{33} & c_3 &= -k_{13} & d_3 &= -k_{23}. \end{aligned} \tag{17}$$

Geometrical investigation (which we omit) would show that (if object and image planes are optically conjugated), k_{33} may be interpreted as the coefficient of spherical aberration for the object origin; k_{23} as the coma coefficient; k_{22} and k_{13} as coefficients of the field errors; and k_{12} as the coefficient of the distortion for an object at the origin and an infinite entrance pupil.

On the other hand, k_{11} may be considered as the coefficient of spherical aberration for an infinite object; k_{12} as coma coefficient; k_{13} and k_{22} as field errors; and k_{23} as the coefficient of distortion for an infinite object and the entrance pupil at the object origin.

The connections between these errors are well-known laws of the Seidel theory.

The method developed here differs from the usual methods in that, first, we do not assume the coordinate origins to be in conjugated planes, and, second, we do not restrict ourselves to the consideration of the deviation of the object point, but investigate at the same time the deviation of the direction of the ray. Equations (10) give, within the limits of our Seidel region, the following equations:

$$\begin{aligned} \tilde{x} - x &= (k_{21}u_1 + k_{22}u_2 + k_{23}u_3)x + (k_{31}u_1 + k_{32}u_2 + k_{33}u_3)\xi,\\ \xi - \tilde{\xi} &= (k_{11}u_1 + k_{12}u_2 + k_{13}u_3)x + (k_{21}u_1 + k_{22}u_2 + k_{23}u_3)\xi; \end{aligned} \tag{18}$$

and y and η analogously.

We recommend a detailed study of these equations and their derivatives with respect to x and ξ for meridian rays ($y=\eta=0$), especially in the case where our origins are not conjugated.

5. Fifth-order aberrations. For the fifth-order aberrations we find from (6a) and (6b) the following fourteen independent equations between the twenty-four coefficients $a_{\iota\kappa}$, etc.

Making use of equations (11) and (17) we find that

$$\begin{aligned} a_{11}+d_{11} &= 2(k_{12}^2-k_{11}k_{13}), & a_{22}+d_{22} &= 2(k_{22}^2-k_{12}k_{23}),\\ a_{12}+d_{12} &= 2k_{12}k_{22}-k_{12}k_{13}-k_{11}k_{23}, & a_{23}+d_{23} &= 2k_{22}k_{23}-k_{13}k_{23}-k_{12}k_{33},\\ a_{13}+d_{13} &= 2k_{12}k_{23}-k_{13}^2-k_{11}k_{33}, & a_{33}+d_{33} &= 2(k_{23}^2-k_{13}k_{33}),\\ c_{21}-d_{11} &= 2k_{12}^2+k_{11}k_{13}-3k_{11}k_{22}, & b_{12}-a_{13} &= k_{13}^2+k_{13}k_{22}-2k_{12}k_{23},\\ c_{22}-d_{12} &= 2k_{12}k_{13}-k_{12}k_{22}-k_{11}k_{23}, & b_{22}-a_{23} &= 2k_{13}k_{23}-k_{12}k_{33}-k_{22}k_{23},\\ c_{23}-d_{13} &= k_{13}^2+k_{13}k_{22}-2k_{12}k_{23}, & b_{23}-a_{33} &= 2k_{23}^2+k_{13}k_{33}-3k_{22}k_{33},\\ c_{31}+b_{11} &= 3(k_{12}k_{13}-k_{11}k_{23}), & c_{33}+b_{13} &= 3(k_{13}k_{23}-k_{12}k_{33}). \end{aligned} \tag{19}$$

Here again, we can express the twenty-four quantities in terms of the third-order coefficients and nine parameters $k_{\iota\kappa\lambda}$.

6. The single sphere and the plane. In our previous paper we were able to calculate the functions A, B, C, D for the case of a plane and a sphere.

In the case of a plane, we put object and image origins at the point where the axis intersects the plane and found that

$$\begin{aligned} x' &= x, & y' &= y, \\ \xi' &= \xi, & \eta' &= \eta; \end{aligned} \tag{20}$$

or $A=D=1$, $B=C=0$. In this case we have $a=d=1$, $b=c=0$, and all the image errors vanish.

In the case of a spherical surface, we put the object and image origins at the center, and found that

$$\begin{aligned} x' &= Ax, & y' &= Ay, \\ \xi' &= Cx + D\xi, & \eta' &= Cy + D\eta; \end{aligned} \tag{21}$$

where

$$\begin{aligned} C &= \frac{1}{r}\left\{\sqrt{n'^2 - \frac{2n^2u_1 - u_2^2}{r^2}} - \sqrt{n^2 - \frac{2n^2u_1 - u_2^2}{r^2}}\right\}, \\ n^2D &= \frac{2n^2u_1 - u_2^2}{r^2} + \sqrt{\left(n'^2 - \frac{2n^2u_1 - u}{r^2}\right)\left(n^2 - \frac{2n^2u_1 - u}{r^2}\right)} - Cu_2, \\ A &= \frac{1}{D}\cdot \end{aligned} \tag{22}$$

If we develop A, B, C as functions of u_1, u_2, u_3, we obtain the Seidel and fifth-order coefficients. Taking care of (5) and observing that

$$\begin{aligned} A_0 &= \frac{n}{n'}, & B_0 &= 0, \\ C_0 &= \frac{n' - n}{r}, & D_0 &= \frac{n'}{n}, \end{aligned} \tag{23}$$

we finally find the image-error coefficients:

$$\begin{aligned} a_1 &= \frac{n^2}{r^2}\left(\frac{1}{n'} - \frac{1}{n}\right)^2, & a_2 &= \frac{1}{r}\left(\frac{1}{n'} - \frac{1}{n}\right), & a_3 &= 0, \\ b_1 &= 0, & b_2 &= 0, & b_3 &= 0, \\ c_1 &= -\frac{n^4}{r^3}\left(\frac{1}{n'^3} - \frac{4}{n'^2 n} - \frac{4}{n'n^2} + \frac{1}{n^3}\right), & c_2 &= -\frac{n^2}{r^2}\left(\frac{1}{n'} - \frac{1}{n}\right)^2, & c_3 &= 0, \\ d_1 &= -\frac{n^2}{r^2}\left(\frac{1}{n'} - \frac{1}{n}\right)^2, & d_2 &= -\frac{1}{r}\left(\frac{1}{n'} - \frac{1}{n}\right), & d_3 &= 0; \end{aligned} \tag{24a}$$

and the fifth-order coefficients:

$$a_{11}=\frac{n^4}{r^4}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{3}{n'^2}-\frac{2}{n'n}+\frac{3}{n^2}\right), \qquad a_{13}=0,$$

$$a_{12}=\frac{n^2}{r^3}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{2}{n'^2}-\frac{3}{n'n}+\frac{2}{n^2}\right), \qquad a_{23}=0,$$

$$a_{22}=\frac{1}{r^2}\left(\frac{1}{n'}-\frac{1}{n}\right)^2, \qquad a_{33}=0,$$

$$b_{11}=b_{12}=b_{22}=b_{13}=b_{23}=b_{33}=0,$$

$$c_{11}=-\frac{3n^6}{r^5}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{1}{n'^4}-\frac{3}{n'^3n}+\frac{3}{n'^2n^2}-\frac{3}{n'n^3}+\frac{1}{n^4}\right), \qquad c_{13}=0,$$

$$c_{12}=-\frac{n^4}{r^4}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{2}{n'^2}-\frac{3}{n'n}+\frac{2}{n^2}\right), \qquad c_{23}=0, \tag{24b}$$

$$c_{22}=-\frac{n^2}{r^3}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{1}{n'^2}-\frac{1}{n'n}+\frac{1}{n^2}\right), \qquad c_{33}=0,$$

$$d_{11}=-\frac{n^4}{r^4}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^2, \qquad d_{13}=0,$$

$$d_{12}=-\frac{n^2}{r^3}\left(\frac{1}{n'}-\frac{1}{n}\right)\frac{1}{n'n}, \qquad d_{23}=0,$$

$$d_{22}=\frac{1}{r^2}\left(\frac{1}{n'}-\frac{1}{n}\right)^2, \qquad d_{33}=0,$$

equations which fulfill all the conditions of equations (6).

The nonvanishing seventh-order coefficients for one surface would be:

$$a_{111}=\frac{3n^6}{r^6}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{5}{n'^4}-\frac{6}{n'^3n}+\frac{10}{n'^2n^2}-\frac{6}{n'n^3}+\frac{5}{n^4}\right),$$

$$a_{112}=\frac{n^4}{r^5}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{8}{n'^4}-\frac{19}{n'^3n}+\frac{25}{n'^2n^2}-\frac{19}{n'n^3}+\frac{8}{n^4}\right),$$

$$a_{122}=\frac{n^2}{r^4}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{3}{n'^2}-\frac{2}{n'n}+\frac{3}{n^2}\right),$$

$$a_{222}=-\frac{3}{r^2}\left(\frac{1}{n'}-\frac{1}{n}\right)\frac{1}{n'n},$$

$$c_{111}=-\frac{3n^8}{r^7}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{5}{n'^6}-\frac{17}{n'^5n}+\frac{26}{n'^4n^2}-\frac{33}{n'^3n^3}+\frac{26}{n'^2n^4}-\frac{17}{n'n^5}+\frac{5}{n^6}\right),$$

$$
\begin{aligned}
c_{112} &= -\frac{n^6}{r^6}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{8}{n'^4}-\frac{19}{n'^3n}+\frac{25}{n'^2n^2}-\frac{19}{n'n^3}+\frac{8}{n^4}\right),\\
c_{122} &= -\frac{n^4}{r^5}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{3}{n'^4}-\frac{7}{n'^3n}+\frac{11}{n'^2n^2}-\frac{7}{n'n^3}+\frac{3}{n^4}\right),\\
c_{222} &= -\frac{3n^2}{r^4}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\frac{1}{n'n},\\
d_{111} &= -\frac{3n^6}{r^6}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{1}{n'^4}+\frac{2}{n'^3n}+\frac{2}{n'^2n^2}+\frac{2}{n'n^3}+\frac{1}{n^4}\right),\\
d_{112} &= -\frac{n^4}{r^5}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{1}{n'^2}+\frac{1}{n'n}+\frac{1}{n^2}\right)\frac{1}{n'n},\\
d_{122} &= \frac{n^2}{r^4}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{1}{n'^2}+\frac{2}{n'n}+\frac{1}{n^2}\right),\\
d_{222} &= \frac{3}{r^3}\left(\frac{1}{n'}-\frac{1}{n}\right)\frac{1}{n'n}.
\end{aligned}
\tag{25}
$$