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ASYMPTOTIC ORDER OF THE QUANTIZATION ERRORS FOR A CLASS OF SELF-AFFINE MEASURES

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ABSTRACT. Let E be a Bedford-McMullen carpet determined by a set of affine mappings $(f_{ij})_{(i,j)\in G}$ and μ a self-affine measure on E associated with a probability vector $(p_{ij})_{(i,j)\in G}$. We prove that, for every $r \in (0,\infty)$, the upper and lower quantization coefficient are always positive and finite in its exact quantization dimension s_r. As a consequence, the nth quantization error for μ of order r is of the same order as $n^{-\frac{1}{s_r}}$. In sharp contrast to the Hausdorff measure for Bedford-McMullen carpets, our result is independent of the horizontal fibres of the carpets.

1. INTRODUCTION

Let m, n be two fixed positive integers with $2 \le m \le n$. Let G be a subset of

$$\{0,1,\ldots,n-1\} \times \{0,1,\ldots,m-1\}$$

with $N := \mathrm{card}\,(G) \ge 2$. We consider a family of affine mappings on $\mathbb{R}^2$:

$$f_{ij} : (x,y) \mapsto \big(n^{-1}x + n^{-1}i, m^{-1}y + m^{-1}j\big), \quad (i,j) \in G. \tag{1.1}$$

By [10], there exists a unique non-empty compact set E satisfying

$$E = \bigcup_{(i,j)\in G} f_{ij}(E).$$

The set E is the self-affine set determined by $(f_{ij})_{(i,j)\in G}$. We also call it a Bedford-McMullen carpet. Let $(p_{ij})_{(i,j)\in G}$ be a probability vector with $p_{ij} > 0$ for all $(i,j) \in G$; there exists a unique Borel probability measure μ satisfying

$$\mu = \sum_{(i,j)\in G} p_{ij}\mu \circ f_{ij}^{-1}. \tag{1.2}$$

The measure μ is referred to as the self-affine measure associated with $(p_{ij})_{(i,j)\in G}$ and $(f_{ij})_{(i,j)\in G}$. Self-affine sets and measures in the above-mentioned cases and some more general cases have been intensively studied in the past years; one may

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see [1–4, 14, 16, 18, 19, 21] for interesting results in this direction. Write

$$G_x := \{i : (i,j) \in G \text{ for some } j\}\,;\ \ G_y := \{j : (i,j) \in G \text{ for some } i\}\,,$$
$$G_{x,j} := \{i : (i,j) \in G\}\,,\ \ q_j := \sum_{i \in G_{x,j}} p_{ij}\ , j \in G_y;\ \ \theta := \frac{\log m}{\log n}.$$

We say that E has uniform horizontal fibres if $\mathrm{card}(G_{x,j})$ is constant for $j \in G_y$. By Peres [21], the Hausdorff measure of E is infinite in its Hausdorff dimension if E does not have uniform horizontal fibres; otherwise its Hausdorff measure is finite and positive.

In the present paper, we further study the quantization problem for self-affine measures as defined in (1.2). We refer to [12] for some previous work of the author and Kesseböhmer.

The quantization problem for probability measures originated in information theory and engineering technology (cf. [9, 24]). Mathematically, the problem consists in estimating the asymptotic error in the approximation of a given probability measure by discrete probability measures with finite support in terms of L_r-metrics. We refer to Graf and Luschgy [5] for rigorous mathematical foundations of quantization theory and [6–8, 15, 17, 20, 22, 23, 25] for more related results.

Let $\|\cdot\|$ be a norm on $\mathbb{R}^q$ and d the metric induced by this norm. For each $k \in \mathbb{N}$, we write $\mathcal{D}_k := \{\alpha \subset \mathbb{R}^q : 1 \le \mathrm{card}(\alpha) \le k\}$. Let ν be a Borel probability measure on $\mathbb{R}^q$. The kth quantization error for ν of order $r \in (0,\infty)$ is defined by

$$e_{k,r}(\nu) := \left(\inf_{\alpha \in \mathcal{D}_k} \int d(x,\alpha)^r d\nu(x)\right)^{\frac{1}{r}}. \tag{1.3}$$

The quantization error $e_{k,r}(\nu)$ coincides with the minimum error when approximating ν with discrete probability measures supported on at most k points. One may see [5] for several more equivalent interpretations for $e_{k,r}(\nu)$. If the infimum in (1.3) is attained at some $\alpha \in \mathcal{D}_n$, then we call α an n-optimal set for ν of order r. The collection of all n-optimal sets for ν of order r is denoted by $C_{n,r}(\nu)$. By [5], $C_{n,r}(\nu)$ is non-empty provided that the moment condition $\int |x|^r d\nu(x) < \infty$ is satisfied. This condition is clearly ensured if the support of the measure ν is compact. Also, under the moment condition, we have $e_{k,r}(\nu) \to 0$ as k tends to infinity (see Lemma 6.1 of [5]).

As natural characterizations of the asymptotics for the quantization error $e_{k,r}(\nu)$, we consider the s-dimensional upper and lower quantization coefficient for ν of order r, which are defined below:

$$\underline{Q}_r^s(\nu) := \liminf_{k\to\infty} k^{\frac{r}{s}} e_{k,r}^r(P),\ \ \overline{Q}_r^s(\nu) := \limsup_{k\to\infty} k^{\frac{r}{s}} e_{k,r}^r(\nu),\ \ s \in (0,\infty).$$

The upper and lower quantization dimension for ν of order r are defined by

$$\overline{D}_r(\nu) := \limsup_{k\to\infty} \frac{\log k}{-\log e_{k,r}(\nu)},\ \underline{D}_r(\nu) := \liminf_{k\to\infty} \frac{\log k}{-\log e_{k,r}(\nu)}. \tag{1.4}$$

These two quantities are exactly the critical points at which the upper and lower quantization coefficient jump from infinity to zero (cf. Proposition 11.3 of [5] and [22]). If $\overline{D}_r(\nu) = \underline{D}_r(\nu)$, the common value is called the quantization dimension for ν of order r and denoted by $D_r(\nu)$. Compared with the upper and lower quantization dimension, the upper and lower quantization coefficient provide us with more accurate information on the asymptotic properties of the quantization error. Accordingly, it is usually much more difficult to examine the finiteness and positivity of the upper and lower quantization coefficient.

Next, we recall our previous work on the quantization for self-affine measures in [12]. Let s_r be the unique solution of the following equation:

$$\left(\sum_{(i,j)\in G}(p_{ij}m^{-r})^{\frac{s_r}{s_r+r}}\right)^{\theta}\left(\sum_{j\in G_y}(q_jm^{-r})^{\frac{s_r}{s_r+r}}\right)^{1-\theta}=1. \tag{1.5}$$

In [12], Kesseböhmer and Zhu proved that, for every $r\in(0,\infty)$, the quantization dimension for μ of order r exists and equals s_r. Moreover, the s_r-dimensional upper and lower quantization coefficient are both positive and finite if one of the following conditions is fulfilled:

(a) $\sum_{i\in G_{x,j}}(p_{ij}q_j^{-1})^{\frac{s_r}{s_r+r}}$ is identical for all $j\in G_y$;
(b) q_j is identical for all $j\in G_y$.

While the quantization dimension is determined for μ in general, the finiteness and positivity of the upper and lower quantization coefficient are examined only for some particular values of r satisfying (a) and the extreme case (b); in these cases we could estimate the asymptotics of the quantization error by means of another self-affine measure. One may see [12] for more details.

As the upper and lower quantization coefficient indicate the convergence order of the quantization errors, they are of significant importance in quantization theory for probability measures. In view of our previous work in [12], a natural question is, what will happen if we drop the conditions in (a) and (b). With Peres' results [21] in mind, one might compare the quantization coefficient for μ with the Hausdorff measure of E and conjecture that the above assumption (a) or (b) is a necessary condition for the upper and lower quantization coefficient to be both positive and finite. However, as our main result of the present paper, we will prove

Theorem 1.1. *Let μ be the self-affine measure as defined in* (1.2)*. Then for every $r\in(0,\infty)$ we have $0<\underline{Q}_r^{s_r}(\mu)\leq\overline{Q}_r^{s_r}(\mu)<\infty$.*

By Theorem 1.1, one can see that the nth quantization error for μ of order r is of the same order as $n^{-\frac{1}{s_r}}$, independently of the horizontal fibres of E.

The main obstacle in the way of proving Theorem 1.1 lies in the fact that, without the assumptions (a) and (b), one can hardly transfer the sums over approximate squares (cf. Section 2) of different orders to those over approximate squares of the same order. Our main idea is to "embed" approximate squares into the product coding space $G^{\mathbb{N}}\times G_y^{\mathbb{N}}$ and estimate the asymptotic quantization errors for μ by means of a natural product measure on $G^{\mathbb{N}}\times G_y^{\mathbb{N}}$. As the coding space $G^{\mathbb{N}}\times G_y^{\mathbb{N}}$ is not completely compatible with the carpets, we will also need to take care of the overlapping cases which are induced by such embedding.

2. Preliminaries

In order to avoid degenerate cases, in the following, we always assume that

$$\operatorname{card}(G_x),\operatorname{card}(G_y)\geq 2. \tag{2.1}$$

Since norms on $\mathbb{R}^q$ are pairwise equivalent, we will always work with Euclidean metrics for convenience. For $x\in\mathbb{R}$, let $[x]$ denote the largest integer not exceeding

x. Set $\ell(k) := [k\theta]$ and

$$\Omega_k := \begin{cases} G_y^k, & \text{if } k < \theta^{-1} \\ G^{\ell(k)} \times G_y^{k-\ell(k)}, & \text{if } k \geq \theta^{-1} \end{cases}, \ \Omega^* := \bigcup_{k\geq 1} \Omega_k. \tag{2.2}$$

For $\sigma = \big((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)}), j_{\ell(k)+1}, \ldots, j_k\big) \in \Omega^*$, we define

$$|\sigma| := k,\ \sigma_a := \big((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)})\big),\ \sigma_b := \big(j_{\ell(k)+1}, \ldots, j_k\big).$$

We also write $\sigma = \sigma_a * \sigma_b$. To each word σ of the form

$$\sigma = \big((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)}), j_{\ell(k)+1}, \ldots, j_k\big) \in \Omega^*, \tag{2.3}$$

there correspond two numbers p, q:

$$p := \sum_{h=1}^{\ell(k)} i_h n^{\ell(k)-h},\ \ q := \sum_{h=1}^{k} j_h m^{k-h};$$

and a unique rectangle which is called an approximate square of order k:

$$F_\sigma := \left[\frac{p}{n^{\ell(k)}}, \frac{p+1}{n^{\ell(k)}}\right] \times \left[\frac{q}{m^k}, \frac{q+1}{m^k}\right]. \tag{2.4}$$

We call σ the location code for the approximate square F_σ. Set

$$\mu_\sigma := \mu(F_\sigma) = \prod_{h=1}^{\ell(k)} p_{i_h j_h} \prod_{h=\ell(k)+1}^{k} q_{j_h}. \tag{2.5}$$

For $\sigma, \tau \in \Omega^*$, we write $\sigma \prec \tau$ if $F_\tau \subset F_\sigma$; and write $\sigma = \tau^\flat$ if $\sigma \prec \tau$ and $|\tau| = |\sigma| + 1$. For a word of the form (2.3), $\sigma^\flat$ takes the following two possible forms:

$$\big((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)}), j_{\ell(k)+1}, \ldots, j_{k-1}\big), \qquad \text{if } \ell(k) = \ell(k-1), \tag{2.6}$$

$$\big((i_1, j_1), \ldots, (i_{\ell(k)-1}, j_{\ell(k)-1}), j_{\ell(k)}, \ldots, j_{k-1}\big), \quad \text{if } \ell(k) = \ell(k-1) + 1. \tag{2.7}$$

We say that $\sigma, \tau \in \Omega^*$ are incomparable if neither $\sigma \prec \tau$ nor $\tau \prec \sigma$. A finite set $\Gamma \subset \Omega^*$ is called a finite antichain if any two words $\sigma, \tau \in \Gamma$ are incomparable; a finite antichain Γ is called maximal if $E \subset \bigcup_{\sigma\in\Gamma} F_\sigma$.

Remark 2.1. We have the following simple facts about approximate squares:

(f1) Let $|A|$ denote the diameter of a set $A \subset \mathbb{R}^2$. One can easily see

$$m^{-|\sigma|} \leq |F_\sigma| \leq m^{-|\sigma|}\sqrt{n^2+1}.$$

(f2) For $\sigma, \tau \in \Omega^*$, by the definition, we have, either F_σ, F_τ are non-overlapping, or one is a subset of the other.

(f3) For $\sigma \in \Omega^*$, let μ_σ be as defined in (2.5). Then by (2.6), (2.7), we have

$$\frac{\mu_\sigma}{\mu_{\sigma^\flat}} \leq \max_{\hat{j}\in G_y} q_{\hat{j}}. \tag{2.8}$$

For $r > 0$ and each $k \geq 1$, we define

$$\underline{\eta}_r := \min\left\{p_{ij} q_k m^{-r} : (i,j) \in G, k \in G_y\right\};$$

$$\Upsilon_{k,r} := \left\{\sigma \in \Omega^* : \mu_{\sigma^\flat} m^{-|\sigma^\flat| r} \geq \underline{\eta}_r^k > \mu_\sigma m^{-|\sigma| r}\right\},\ \psi_{k,r} := \mathrm{card}(\Upsilon_{k,r}). \tag{2.9}$$

For two number sequences $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$, we write $a_k \asymp b_k$ if there exists a constant C independent of k such that $Cb_k \leq a_k \leq C^{-1}b_k$. As the proof of Lemma 4 in [12] shows, we have

$$e^r_{\psi_{k,r},r}(\mu) \asymp \sum_{\sigma\in\Upsilon_{k,r}} \mu_\sigma m^{-|\sigma|r}. \tag{2.10}$$

The set $\Upsilon_{k,r}$ possesses some kind of uniformity, which allows us to estimate the number of optimal points lying in disjoint neighborhoods of the approximate squares $F_\sigma, \sigma \in \Upsilon_{k,r}$. This uniformity allows us to think of (2.10) roughly as follows. For each $\sigma \in \Upsilon_{j,r}$, F_σ "owns" one point a_σ of a $\psi_{j,r}$-optimal set α and

$$\int_{F_\sigma} d(x,\alpha)^r d\mu(x) \asymp \mu_\sigma m^{-|\sigma|r}.$$

One may see [13] for some more intuitive interpretations on such estimates.

However, the structure of the set $\Upsilon_{k,r}$ is not clear enough for us to estimate the sum on the right side of (2.10). Let σ be given in (2.3). Assume that $\ell(k+1) = \ell(k)+1$. For $j \neq j_{\ell(k)+1}, \hat{j} \in G_y$ and $(i,j) \in G$, we write

$$\hat{\sigma} = \big((i_1,j_1),\ldots,(i_{\ell(k)},j_{\ell(k)}),(i,j),j_{\ell(k)+2},\ldots,j_k,\hat{j}\big).$$

One can see that $F_{\hat{\sigma}}$ is not a subset of F_σ. Roughly speaking, approximate squares do not enjoy enough "freedom" as far as sub-approximate squares are concerned. On the other hand, with (a) or (b) as stated in Section 1, the distribution of μ obeys a certain hereditary law between approximate squares and their sub-approximate squares (by (a)), or between approximate squares of the same order that are contained in a natural cylinder $f_{i_1j_1}\circ\cdots\circ f_{i_kj_k}(E)$ and those contained in its subcylinders (due to (b)). This allows us to estimate the quantization error for μ by means of another self-affine measure. However, without the assumptions (a) and (b), for distinct words of the form (2.3), the measure μ are distributed in different manners among sub-approximate squares of them and we do not have any certain laws as mentioned above. These facts seem to prevent us from constructing a suitable auxiliary measure via approximate squares.

In order to show the finiteness of the upper quantization coefficient for μ, we will "embed" the sets $\Upsilon_{j,r}$ into the product space $G^{\mathbb{N}}\times G_y^{\mathbb{N}}$, and then estimate the quantization errors by using a product measure W on $G^{\mathbb{N}}\times G_y^{\mathbb{N}}$ and counting all possible overlapping cases. To establish a lower bound for the lower quantization coefficient for μ, we will construct a new sequence of subsets $\mathcal{L}_{j,r}(2)$ such that, on one hand, they can play the same role as $\Upsilon_{j,r}$, and on the other hand, they enjoy enough "freedom" so that the corresponding integrals can be well estimated by means of the above-mentioned product measure W.

For convenience, in the remaining part of the paper, we write

$$\mathcal{E}_r(\sigma) := (\mu_\sigma m^{-|\sigma|r})^{\frac{s_r}{s_r+r}},\ \sigma\in\Omega^*.$$

Note that $\psi_{j,r} \asymp \psi_{j+1,r}$ by the proof of Lemma 1 in [12]. To study the finiteness and positivity of the upper and lower quantization coefficient for μ, it suffices to examine the asymptotics of the sequence $(e_{\psi_{j,r},r}(\mu))_{j=1}^{\infty}$. By Hölder's inequality with exponent less than one, the problem further reduces to the asymptotics of the following number sequence:

$$\sum_{\sigma\in\Upsilon_{j,r}} \mathcal{E}_r(\sigma),\ j\geq 1.$$

For the proof of the main theorem, we will need to go back and forth between words in $\Upsilon_{j,r}$ and subsets of $G^{\mathbb{N}} \times G_y^{\mathbb{N}}$.

3. The finiteness of the upper quantization coefficient for μ

Let $\sigma := ((i_1, j_1), \ldots, (i_k, j_k)) \in G^k$. We define

$$|\sigma| = k, \ \ \sigma|_h = ((i_1, j_1), \ldots, (i_h, j_h)), \ 1 \leq h \leq k; \ \ \sigma^- := \sigma|_{k-1}. \tag{3.1}$$

For $\sigma, \omega \in G^* := \bigcup_{k=0}^{\infty} G^k$ with $\sigma = \omega|_{|\sigma|}$, we write $\sigma \prec \omega$. We define $\sigma|_h$ similarly for $\sigma \in G^{\mathbb{N}}$ and $h \geq 1$. If $\omega \in G^*$ and $\sigma \in G^{\mathbb{N}}$ satisfy $\omega = \sigma|_{|\omega|}$, then we also write $\omega \prec \sigma$. For $\sigma = ((i_1, j_1), \ldots, (i_k, j_k))$ and $\omega = ((i_{k+1}, j_{k+1}), \ldots, (i_{k+h}, j_{k+h})) \in G$, we write

$$\sigma * \omega := ((i_1, j_1), \ldots, (i_k, j_k), (i_{k+1}, j_{k+1}), \ldots, (i_{k+h}, j_{k+h})).$$

For $\rho, \tau \in G_y^*$, we define $\rho^-, \rho * \tau$ and a partial order "$\prec$" in the same manner as we did for words in G^*. For $r \in (0, \infty)$, we write

$$P_r := \sum_{(i,j) \in G} (p_{ij} m^{-r})^{\frac{s_r}{s_r + r}}, \ Q_r := \sum_{j \in G_y} (q_j m^{-r})^{\frac{s_r}{s_r + r}}.$$

It is noted in the proof of Lemma 5 of [12] that $P_r \geq 1 \geq Q_r$.

Set $\overline{q} := \max_{j \in G_y} q_j$ and $\overline{\eta}_r := \overline{q} m^{-r}$. We define

$$H_{1,r} := \min\{h : \overline{\eta}_r^h < \underline{\eta}_r\}. \tag{3.2}$$

As we mentioned before, to obtain an upper bound for the upper quantization coefficient for μ, we will embed approximate squares into $G^{\mathbb{N}} \times G_y^{\mathbb{N}}$. For every $k \geq 1$ and $\sigma = ((i_1, j_1), \ldots, (i_k, j_k)) \in G^k$ and $\tau = (j_1, \ldots, j_k) \in G_y^k$, we write

$$[\sigma] = [(i_1, j_1), \ldots, (i_k, j_k)] := \{\omega \in G^{\mathbb{N}} : \omega|_k = \sigma\};$$
$$[\tau] = [j_1, \ldots, j_k] := \{\rho \in G_y^{\mathbb{N}} : \rho|_k = \tau\}.$$

Now for every $\sigma \in \Upsilon_{j,r}$, we associate F_σ in the following way:

$$\sigma = \sigma_a * \sigma_b \in \Upsilon_{j,r} \mapsto [\sigma_a] \times [\sigma_b] \subset G^{\mathbb{N}} \times G_y^{\mathbb{N}}.$$

Let G and G_y be endowed with discrete topology and $G^{\mathbb{N}}, G_y^{\mathbb{N}}$ be endowed with the corresponding product topology. We denote by $\mathcal{B}_1, \mathcal{B}_2$ the Borel sigma-algebra on $G^{\mathbb{N}}, G_y^{\mathbb{N}}$. For every $(i, j) \in G$ and $j \in G_y$, we define

$$\widetilde{p}_{ij} := P_r^{-1} (p_{ij} m^{-r})^{\frac{s_r}{s_r + r}}, \ \ \widetilde{q}_j := Q_r^{-1} (q_j m^{-r})^{\frac{s_r}{s_r + r}}.$$

By the Kolmogrov consistency theorem, there exist a unique Borel probability measure λ on $G^{\mathbb{N}}$ and a unique ν on $G_y^{\mathbb{N}}$ such that

$$\lambda([\sigma]) = \prod_{h=1}^k \widetilde{p}_{i_h j_h}, \text{ for every } \ \sigma = (i_1, j_1), \ldots, (i_k, j_k) \in G^k;$$
$$\nu([\tau]) = \prod_{h=1}^k \widetilde{q}_{j_h}, \text{ for every } \ \tau = (j_1, \ldots, j_k) \in G_y^k.$$

Thus, we obtain a unique product measure W on $G^{\mathbb{N}} \times G_y^{\mathbb{N}}$ such that

$$W(A \times B) = \lambda(A) \cdot \nu(B) \ \ A \in \mathcal{B}_1, B \in \mathcal{B}_2.$$

We know that words in $\Upsilon_{j,r}$ are pairwise incomparable and $F_\sigma, \sigma \in \Upsilon_{j,r}$, are non-overlapping. However, it can happen that $[\sigma_a^{(1)}] \times [\sigma_b^{(1)}]$ and $[\sigma_a^{(2)}] \times [\sigma_b^{(2)}]$ are overlapping. We use the following lemma to treat such overlapping cases.

Lemma 3.1. *For every $\sigma \in \Upsilon_{j,r}$, we write*

$$S_1(\sigma) := \{\tau \in \Upsilon_{j,r} : \sigma_a \prec \tau_a, \sigma_b \prec \tau_b\}.$$

Then for $H_{1,r}$ as defined in (3.2)*, we have*

$$\sum_{\tau \in S_1(\sigma)} W([\tau_a] \times [\tau_b]) \leq H_{1,r} W([\sigma_a] \times [\sigma_b]).$$

Proof. For every $\tau \in \Upsilon_{j,r}$, by (2.9), we have

$$\underline{\eta}_r^{\frac{s_r}{s_r+r}} \mathcal{E}_r(\sigma) \leq \mathcal{E}_r(\tau) \leq \underline{\eta}_r^{\frac{-s_r}{s_r+r}} \mathcal{E}_r(\sigma). \tag{3.3}$$

Suppose that $|\tau| \geq |\sigma| + H_{1,r}$ for some $\tau \in S_1(\sigma)$. We would have

$$\mathcal{E}_r(\tau) \leq \overline{\eta}_r^{\frac{H_{1,r} s_r}{s_r+r}} \mathcal{E}_r(\sigma) < \underline{\eta}_r^{\frac{s_r}{s_r+r}} \mathcal{E}_r(\sigma).$$

This contradicts (3.3). Thus, for every $\tau \in S_1(\sigma)$, we have $|\tau| \leq |\sigma| + H_{1,r}$. Hence,

$$\bigcup_{\tau \in S_1(\sigma)} [\tau_a] \times [\tau_b] \subset \bigcup_{h=1}^{H_{1,r}} \Gamma_h([\sigma_a] \times [\sigma_b]), \tag{3.4}$$

where, for every $h \geq 1$, the set $\Gamma_h([\sigma_a] \times [\sigma_b])$ is defined by

$$\{[\rho] \times [\omega] \subset [\sigma_a] \times [\sigma_b] : \rho \times \omega \in G^* \times G_y^*, |\rho| + |\omega| = |\sigma| + h, [(|\rho| + |\omega|)\theta] = |\rho|\}.$$

Note that the words in $\Gamma_1([\sigma_a] \times [\sigma_b])$ take exactly one of the following two forms:

$$[\sigma_a * (i,j)] \times [\sigma_b], \text{ or } [\sigma_a] \times [\sigma_b * \hat{j}], \ (i,j) \in G, \ \hat{j} \in G_y.$$

Using this fact and mathematical induction, for every $h \geq 1$, we obtain

$$\sum_{\rho \times \omega \in \Gamma_h([\sigma_a] \times [\sigma_b])} W(\Gamma_h([\sigma_a] \times [\sigma_b])) = W([\sigma_a] \times [\sigma_b]). \tag{3.5}$$

For distinct words $\sigma^{(1)}, \sigma^{(2)} \in \Upsilon_{j,r}$, we have either $\sigma_a^{(1)} \neq \sigma_a^{(2)}$, or $\sigma_b^{(1)} \neq \sigma_b^{(2)}$. So,

$$\sigma_a^{(1)} \times \sigma_b^{(1)} \neq \sigma_a^{(2)} \times \sigma_b^{(2)}. \tag{3.6}$$

Thus, the lemma follows by (3.4)-(3.6). □

Next, we show the finiteness of the upper quantization coefficient for μ, by using Lemma 3.1 and the auxiliary measure W.

Proposition 3.2. *Let μ be the measure as defined in* (1.2)*. Then $\overline{Q}_r^{s_r}(\mu) < \infty$.*

Proof. For a word $\sigma \in \Upsilon_{j,r}$, by the definition, it takes the form:

$$\sigma = \big((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)}), j_{\ell(k)+1}, \ldots, j_k\big) \in \Omega^*.$$

We associate σ with the following subset of $G^{\mathbb{N}} \times G_y^{\mathbb{N}}$:

$$[\sigma_a] \times [\sigma_b] = [(i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)})] \times [j_{\ell(k)+1}, \ldots, j_k].$$

Note that for all $k \geq \theta^{-1}$, we have $P_r^{-1} Q_r \leq P_r^{\ell(k)} Q_r^{(k-\ell(k))} \leq 1$. We deduce

$$\begin{aligned} W([\sigma_a] \times [\sigma_b]) \quad &= \prod_{h=1}^{\ell(k)} \widetilde{p}_{i_h j_h} \prod_{h=\ell(k)+1}^{k} \widetilde{q}_{j_h} \\ &= P_r^{-\ell(k)} Q_r^{-(k-\ell(k))} (\mu_\sigma m^{-|\sigma| r})^{\frac{s_r}{s_r+r}} \\ &\begin{cases} \leq P_r Q_r^{-1} \mathcal{E}_r(\sigma) \\ \geq \mathcal{E}_r(\sigma) \end{cases} . \end{aligned} \tag{3.7}$$

For distinct words $\sigma^{(1)}, \sigma^{(2)} \in \Upsilon_{j,r}$, we have either $\sigma_a^{(1)} \neq \sigma_a^{(2)}$ or $\sigma_b^{(1)} \neq \sigma_b^{(2)}$. Thus, they are associated to distinct subsets of $G^{\mathbb{N}} \times G_y^{\mathbb{N}}$. We write

$$\mathcal{W}_{j,r} := \{[\sigma_a] \times [\sigma_b] : \ \sigma = \sigma_a * \sigma_b \in \Upsilon_{j,r}\}.$$

We distinguish two cases:

Case (i). Either $\sigma_a^{(1)}, \sigma_a^{(2)}$ or, $\sigma_b^{(1)}, \sigma_b^{(2)}$ are incomparable. In this case, we have

$$([\sigma_a^{(1)}] \times [\sigma_b^{(1)}]) \cap ([\sigma_a^{(2)}] \times [\sigma_b^{(2)}]) = \emptyset.$$

Case (ii). Both $\sigma_a^{(1)}, \sigma_a^{(2)}$ and $\sigma_b^{(1)}, \sigma_b^{(2)}$ are comparable. Note that

$$[(|\sigma_a| + |\sigma_b|)\theta] = |\sigma_a|, \text{ for all } \ \sigma = \sigma_a * \sigma_b \in \Upsilon_{j,r}.$$

Thus, whenever $|\sigma_a^{(1)}| > |\sigma_a^{(2)}|$, we have $|\sigma_b^{(1)}| \geq |\sigma_b^{(2)}|$. Hence, we may assume that

$$\sigma_a^{(1)} \prec \sigma_a^{(2)} \ \text{ and } \ \sigma_b^{(1)} \prec \sigma_b^{(2)}.$$

In this case we have

$$([\sigma_a^{(1)}] \times [\sigma_b^{(1)}]) \supset ([\sigma_a^{(2)}] \times [\sigma_b^{(2)}]).$$

Let $H_{1,r}$ be as defined in (3.2). Then by the proof of Lemma 3.1, we have

$$|\sigma^{(2)}| \leq |\sigma^{(1)}| + H_{1,r}.$$

We may choose a subset $\mathcal{F}_{j,r}$ of $\Upsilon_{j,r}$ such that $\Upsilon_{j,r} = \bigcup_{\widetilde{\sigma} \in \mathcal{F}_{j,r}} S_1(\widetilde{\sigma})$ and for every pair of distinct words $\widetilde{\sigma}, \widetilde{\omega} \in \mathcal{F}_{j,r}$, we have

$$([\widetilde{\sigma}_a] \times [\widetilde{\sigma}_b]) \cap ([\widetilde{\omega}_a] \times [\widetilde{\omega}_b]) = \emptyset. \tag{3.8}$$

By Lemma 3.1, we have

$$\sum_{\omega \in S_1(\widetilde{\sigma})} W([\omega_a] \times [\omega_b]) \leq H_{1,r} W([\widetilde{\sigma}_a] \times [\widetilde{\sigma}_b]). \tag{3.9}$$

Combining this with (3.7)-(3.8), we deduce

$$\begin{aligned}
\sum_{\sigma \in \Upsilon_{j,r}} \mathcal{E}_r(\sigma) &\leq \sum_{\sigma \in \Upsilon_{j,r}} W([\sigma_a] \times [\sigma_b]) \\
&= \sum_{\widetilde{\sigma} \in \mathcal{F}_{j,r}} \sum_{\sigma \in \mathcal{S}_1(\widetilde{\sigma})} W([\sigma_a] \times [\sigma_b]) \\
&\leq H_{1,r} \sum_{\widetilde{\sigma} \in \mathcal{F}_{j,r}} W([\widetilde{\sigma}_a] \times [\widetilde{\sigma}_b]) \\
&\leq H_{1,r}. \qquad (3.10)
\end{aligned}$$

This, together with (2.9), implies

$$\psi_{j,r} \underline{\eta}_r^{\frac{(j+1)s_r}{s_r+r}} \leq H_{1,r}, \ \text{ implying } \ \underline{\eta}_r^{\frac{jr}{s_r+r}} \leq H_{1,r}^{\frac{r}{s_r}} \underline{\eta}_r^{\frac{-r}{s_r+r}} \psi_{j,r}^{-\frac{r}{s_r}}. \tag{3.11}$$

Using this, (3.10) and (2.10), we have

$$\begin{aligned}
e^r_{\psi_{j,r},r}(\mu) &\asymp \sum_{\sigma \in \Upsilon_{j,r}} \mu_\sigma m^{-|\sigma| r} = \sum_{\sigma \in \Upsilon_{j,r}} \mathcal{E}_r(\sigma) (\mu_\sigma m^{-|\sigma| r})^{\frac{r}{s_r+r}} \\
&\leq \sum_{\sigma \in \Upsilon_{j,r}} \mathcal{E}_r(\sigma) \underline{\eta}_r^{\frac{jr}{s_r+r}} \leq H_{1,r}^{1+\frac{r}{s_r}} \underline{\eta}_r^{\frac{-r}{s_r+r}} \psi_{j,r}^{-\frac{r}{s_r}}.
\end{aligned}$$

By Lemma 1 in [12], we have $\psi_{j,r} \leq \psi_{j+1,r} \leq (mn)^{H_{1,r}}\psi_{j,r}$. For each $k \geq \psi_{1,r}$, there exists some j such that $\psi_{j,r} \leq k < \psi_{j+1,r}$. It follows that

$$\begin{aligned}\overline{Q}_r^{s_r}(\mu) &= \limsup_{k\to\infty} k^{\frac{r}{s_r}} e^r_{k,r}(\mu) \leq \limsup_{j\to\infty} \psi_{j+1,r}^{\frac{r}{s_r}} e^r_{\psi_{j,r},r}(\mu) \\ &\leq (mn)^{\frac{rH_{1,r}}{s_r}} \limsup_{j\to\infty} \psi_{j,r}^{\frac{r}{s_r}} e^r_{\psi_{j,r},r}(\mu) \\ &\leq (mn)^{\frac{rH_{1,r}}{s_r}} H_{1,r}^{1+\frac{r}{s_r}} \underline{\eta}_r^{\frac{-r}{s_r+r}}. \qquad (3.12)\end{aligned}$$

This completes the proof of the proposition. □

4. The positivity of the lower quantization coefficient for μ

Let $\Upsilon_{j,r}$ be as defined in (2.9). We write

$$k_{1j} := \min_{\sigma\in\Upsilon_{j,r}} |\sigma|, \; k_{2j} := \max_{\sigma\in\Upsilon_{j,r}} |\sigma|; \; \Lambda_{j,r}(k) := \Upsilon_{j,r} \cap \Omega_k.$$

For $\sigma \in G^*$ and $\omega \in G_y^*$, we write $\sigma \times \omega$ for the corresponding word in $G^* \times G_y^*$. We consider words of $G^* \times G_y^*$ which take the following form:

$$\sigma \times \omega, \; |\sigma| + \ell(k_{1j}) = \ell(|\sigma| + |\omega| + k_{1j}), \; \sigma \in G^*, \omega \in G_y^*.$$

Let $\mathcal{H}_{j,r}$ denote the set of all such words. Note that

$$\begin{aligned}\ell(|\sigma| + |\omega| + k_{1j} - 1) &= [(|\sigma| + |\omega| + k_{1j} - 1)\theta] \geq (|\sigma| + |\omega| + k_{1j} - 1)\theta - 1 \\ &= (|\sigma| + |\omega| + k_{1j})\theta - (1+\theta) > |\sigma| + [k_{1j}\theta] - 2.\end{aligned}$$

Thus, $\ell(|\sigma|+|\omega|+k_{1j}-1)$ takes two possible values: $|\sigma|+\ell(k_{1j})$, or, $|\sigma|+\ell(k_{1j})-1$. This allows us to define $(\sigma \times \omega)^\flat \in \mathcal{H}_{j,r}$:

$$(\sigma \times \omega)^\flat := \begin{cases} \sigma \times \omega^- & \text{if } \ell(|\sigma| + |\omega| + k_{1j} - 1) = |\sigma| + \ell(k_{1j}), \\ \sigma^- \times \omega & \text{if } \ell(|\sigma| + |\omega| + k_{1j} - 1) = |\sigma| + \ell(k_{1j}) - 1. \end{cases} \qquad (4.1)$$

We write $P(\sigma \times \omega) := [\sigma] \times [\omega]$ and $P((\sigma \times \omega)^\flat) := [\sigma] \times [\omega^-]$ or $[\sigma^-] \times [\omega]$ in accordance with (4.1). One can easily see

$$P_r^{-1} \underline{\eta}_r^{\frac{s_r}{s_r+r}} W(P(\sigma \times \omega)^\flat) \leq W(P(\sigma \times \omega)) \leq W(P(\sigma \times \omega)^\flat). \qquad (4.2)$$

By the definition, for two words $\sigma^{(i)} \times \omega^{(i)} \in \mathcal{H}_{j,r}, i = 1, 2$, if $|\sigma^{(1)}| < |\sigma^{(2)}|$, we have $|\omega^{(1)}| \leq |\omega^{(2)}|$. Thus, whenever $\sigma^{(1)} \prec \sigma^{(2)}$ and $\sigma^{(1)} \neq \sigma^{(2)}$, we have $\omega^{(1)} \prec \omega^{(2)}$.

We write $\sigma^{(1)} \times \omega^{(1)} \prec \sigma^{(2)} \times \omega^{(2)}$, if $\sigma^{(1)} \prec \sigma^{(2)}$ and $\omega^{(1)} \prec \omega^{(2)}$; if neither $\sigma^{(1)} \times \omega^{(1)} \prec \sigma^{(2)} \times \omega^{(2)}$, nor $\sigma^{(1)} \times \omega^{(1)} \prec \sigma^{(2)} \times \omega^{(2)}$, then we say that $\sigma^{(i)} \times \omega^{(i)} \in \mathcal{H}_{j,r}, i = 1, 2$ are incomparable. A finite set $\Gamma \subset \mathcal{H}_{j,r}$ is called a finite maximal antichain, if the words in Γ are pairwise incomparable, and for every word $\sigma \times \omega$ in $G^{\mathbb{N}} \times G_y^{\mathbb{N}}$, there exists some word $\sigma' \times \omega' \in \Gamma$ such that $\sigma' \prec \sigma$ and $\omega' \prec \omega$. For a finite maximal antichain Γ in $\mathcal{H}_{j,r}$, we have

$$\bigcup_{\sigma\times\omega\in\Gamma} [\sigma] \times [\omega] = G^{\mathbb{N}} \times G_y^{\mathbb{N}}. \qquad (4.3)$$

For such a Γ and every pair of distinct words $\sigma^{(1)} \times \omega^{(1)}, \sigma^{(2)} \times \omega^{(2)} \in \Gamma$, we have

$$([\sigma^{(1)}] \times [\omega^{(1)}]) \cap ([\sigma^{(2)}] \times [\omega^{(2)}]) = \emptyset.$$

In order to establish a lower bound for the lower quantization coefficient for μ, we will construct a family of subsets of $G^{\mathbb{N}} \times G_y^{\mathbb{N}}$ and associate them with approximate squares. Recall that for two words $\sigma, \omega \in \Omega^*$, $\sigma \prec \omega$ means $F_\omega \subset F_\sigma$. The following lemma will be used to estimate the possible overlapping cases in this process.

Lemma 4.1. *Let $\sigma \in \Omega^*$ and $H_{2,r} := P_r^3 Q_r^{-2} \underline{\eta}_r^{-\frac{s_r}{s_r+r}}$. We write*

$$S_2(\sigma) := \{\omega \in \Omega^* : \ \sigma \prec \omega, \mathcal{E}_r(\omega) \geq H_{2,r}^{-1} \mathcal{E}_r(\sigma)\}. \tag{4.4}$$

Then there exists a constant $H_{3,r}$, which is independent of σ, such that

$$\sum_{\omega \in S_2(\sigma)} \mathcal{E}_r(\omega) \leq H_{3,r} \mathcal{E}_r(\sigma).$$

Proof. Let $\overline{\eta}_r$ be as defined in Section 3. Write

$$\Lambda(\sigma, h) := \{\omega \in \Omega^* : |\omega| = |\sigma| + h, \sigma \prec \omega\}, \ h \geq 1;$$
$$M_r := \min\{h \in \mathbb{N} : \overline{\eta}_r^{\frac{h s_r}{s_r+r}} < H_{2,r}^{-1}\}.$$

Then for every $\omega \in \Lambda(\sigma, M_r)$, by (2.8), we have

$$\mathcal{E}_r(\omega) \leq \overline{\eta}_r^{\frac{M_r s_r}{s_r+r}} \mathcal{E}_r(\sigma) < H_{2,r}^{-1} \mathcal{E}_r(\sigma).$$

By the hypothesis, we conclude that $|\omega| \leq |\sigma| + M_r$. Hence,

$$S_2(\sigma) \subset \bigcup_{h=0}^{M_r} \Lambda(\sigma, h).$$

Note that $0 < Q_r \leq 1$. By (2.6), we also have

$$\begin{aligned}
\sum_{\omega \in \Lambda(\tau,1)} \mathcal{E}_r(\omega) &\leq Q_r \sum_{i \in G_{x, j_{\ell k+1}}} \left(\frac{p_{i j_{\ell k+1}}}{q_{j_{\ell k+1}}}\right)^{\frac{s_r}{s_r+r}} \mathcal{E}_r(\tau) \\
&\leq \max_{j \in G_y} \sum_{i \in G_{x,j}} \left(\frac{p_{ij}}{q_j}\right)^{\frac{s_r}{s_r+r}} \mathcal{E}_r(\tau) =: \xi_r \mathcal{E}_r(\tau).
\end{aligned}$$

Using this fact and finite induction, we further deduce

$$\sum_{\omega \in S_2(\sigma)} \mathcal{E}_r(\omega) \leq \sum_{h=0}^{M_r} \sum_{\omega \in \Lambda(\tau,h)} \mathcal{E}_r(\omega) \leq \sum_{h=0}^{M_r} \xi_r^h \mathcal{E}_r(\sigma).$$

Setting $H_{3,r} := \sum_{h=0}^{M_r} \xi_r^h$, the lemma follows. $\square$

Using Lemma 4.1 and the product measure W, we are now able to prove the positivity of the lower quantization coefficient for μ. The proof consists of the following three steps.

First, we will construct a finite maximal antichain in $G^{\mathbb{N}} \times G_y^{\mathbb{N}}$ and consider the associated approximate squares. By using the preceding lemma, we will obtain a non-overlapping family of approximate squares.

Secondly, benefiting from Lemma 2 of [12], we choose a pairwise disjoint family of approximate squares of approximately equal "energy" $\mathcal{E}_r(\sigma)$; this is done according to the geometric structure of the carpet E and corresponding codings in terms of words in Ω^*.

Finally, the auxiliary measure W, together with our previous method, enables us to show the positivity of the lower quantization coefficient. One may see [13] for more details on the estimation for the upper and lower quantization coefficient.

Proposition 4.2. *Let μ be as defined in* (1.2). *Then* $\overline{Q}_r^{s_r}(\mu) > 0$.

Proof. For every $\tau \in \Omega_{k_{1j}} \setminus \Lambda_{j,r}(k_{1j})$, we set $\epsilon(\tau) := \underline{\eta}_r^{\frac{js_r}{s_r+r}} \mathcal{E}_r(\tau)^{-1}$ and define

$$\Gamma(\tau) := \{\sigma \times \omega \in \mathcal{H}_{j,r} : W(P((\sigma \times \omega)^\flat) \geq \epsilon(\tau) > W(P(\sigma \times \omega)\}.$$

Then $\Gamma(\tau)$ is a finite maximal antichain in $\mathcal{H}_{j,r}$. Using (4.3), we deduce

$$\begin{aligned}
\sum_{\sigma\times\omega\in\Gamma(\tau)} W([\tau_a * \sigma] \times [\tau_b * \omega]) &= \sum_{\sigma\times\omega\in\Gamma(\tau)} \lambda([\tau_a * \sigma])\nu([\tau_b * \omega]) \\
&= \sum_{\sigma\times\omega\in\Gamma(\tau)} \lambda([\tau_a])\lambda([\sigma])\nu([\tau_b])\nu([\omega]) \\
&= W([\tau_a * \tau_b]) \sum_{\sigma\times\omega\in\Gamma(\tau)} W([\sigma] \times [\omega]) \\
&= W([\tau_a * \tau_b]).
\end{aligned}$$

For distinct $\sigma^{(i)} \times \omega^{(i)} \in \Gamma(\tau), i = 1, 2$, we have distinct words which are location codes for approximate squares (cf. (2.4)), namely,

$$(\tau_a * \sigma^{(1)}) * (\tau_b * \omega^{(1)}) \neq (\tau_a * \sigma^{(2)}) * (\tau_b * \omega^{(2)}).$$

Furthermore, by the definition of $\Gamma(\tau)$, $(\tau_a*\sigma^{(1)})\times(\tau_b*\omega^{(1)})$ and $(\tau_a*\sigma^{(2)})\times(\tau_b*\omega^{(2)})$ are incomparable.

Also, for different $\tau^{(i)} \in \Omega_{k_{1j}} \setminus \Lambda_{j,r}(k_{1j})$, and $\sigma^{(i)} \times \omega^{(i)} \in \Gamma(\tau_i), i = 1, 2$, we have

$$|\tau_a^{(1)}| = |\tau_a^{(2)}|, \quad |\tau_b^{(1)}| = |\tau_b^{(2)}|.$$

Since $\tau^{(1)} \neq \tau^{(2)}$, we have either $\tau_a^{(1)}, \tau_a^{(2)}$ are incomparable, or $\tau_b^{(1)}, \tau_b^{(2)}$ are incomparable. It follows that $(\tau_a^{(1)} * \sigma^{(1)}) \times (\tau_b^{(1)} * \omega^{(1)})$ and $(\tau_a^{(2)} * \sigma^{(2)}) \times (\tau_b^{(2)} * \omega^{(2)})$ are incomparable and

$$(\tau_a^{(1)} * \sigma^{(1)}) * (\tau_b^{(1)} * \omega^{(1)}) \neq (\tau_a^{(2)} * \sigma^{(2)}) * (\tau_b^{(2)} * \omega^{(2)}).$$

However, it may happen that

$$F_{\tau_a^{(1)}*\sigma^{(1)})*(\tau_b^{(1)}*\omega^{(1)})} \subset F_{\tau_a^{(2)}*\sigma^{(2)})*(\tau_b^{(2)}*\omega^{(2)})}.$$

We denote by $\mathcal{L}_{j,r}(1)$ the set of all the words $(\tau_a*\sigma)*(\tau_b*\omega)$ and words in $\Lambda_{j,r}(k_{1j})$:

$$\mathcal{L}_{j,r}(1) := \Lambda_{j,r}(k_{1j}) \cup \Bigg(\bigcup_{\tau\in\Omega_{k_{1j}}\setminus\Lambda_{j,r}(k_{1j})} \big\{(\tau_a * \sigma) * (\tau_b * \omega) : \sigma \times \omega \in \Gamma(\tau)\big\} \Bigg).$$

For every $\tau \in \Omega_{k_{1j}} \setminus \Lambda_{j,r}(k_{1j})$ and $\sigma \times \omega \in \Gamma(\tau)$, using (3.7) and (4.2), we have

$$\begin{aligned}
\mathcal{E}_r((\tau_a * \sigma) * (\tau_b * \omega)) &= (\mu_{(\tau_a*\sigma)*(\tau_b*\omega)} m^{-|(\tau_a*\sigma)*(\tau_b*\omega)|r})^{\frac{s_r}{s_r+r}} \\
&\geq P_r^{-1} Q_r W([\tau_a * \sigma] \times [\tau_b * \omega]) \\
&= P_r^{-1} Q_r W([\tau_a] \times [\tau_b]) W([\sigma] \times [\omega]) \\
&= P_r^{-1} Q_r W([\tau_a] \times [\tau_b]) W(P(\sigma \times \omega)) \\
&\geq P_r^{-1} Q_r \mathcal{E}_r(\tau) W(P(\sigma \times \omega)) \\
&\geq P_r^{-1} Q_r \mathcal{E}_r(\tau) P_r^{-1} \underline{\eta}_r^{\frac{s_r}{s_r+r}} W(P((\sigma \times \omega)^\flat)) \\
&\geq P_r^{-1} Q_r \mathcal{E}_r(\tau) P_r^{-1} \underline{\eta}_r^{\frac{s_r}{s_r+r}} \underline{\eta}_r^{\frac{js_r}{s_r+r}} \mathcal{E}_r(\tau)^{-1} \\
&= P_r^{-2} Q_r \underline{\eta}_r^{\frac{(j+1)s_r}{s_r+r}}.
\end{aligned}$$

Analogously, one can see that $\mathcal{E}_r((\tau_a * \sigma) * (\tau_b * \omega)) \leq P_r Q_r^{-1} \underline{\eta}_r^{\frac{js_r}{s_r+r}}$. In addition, for every $\tau \in \Lambda_{j,r}(k_{1j})$, by (2.9), one can see that

$$\underline{\eta}_r^{\frac{(j+1)s_r}{s_r+r}} \leq \mathcal{E}_r(\tau) < \underline{\eta}_r^{\frac{js_r}{s_r+r}}.$$

Thus, for all words $\rho \in \mathcal{L}_{j,r}(1)$, we have

$$P_r^{-2} Q_r \underline{\eta}_r^{\frac{(j+1)s_r}{s_r+r}} \leq \mathcal{E}_r(\rho) < P_r Q_r^{-1} \underline{\eta}_r^{\frac{js_r}{s_r+r}}. \tag{4.5}$$

We may choose a subset $\mathcal{L}_{j,r}(2)$ of $\mathcal{L}_{j,r}(1)$ such that $\mathcal{L}_{j,r}(1) = \bigcup_{\rho \in \mathcal{L}_{j,r}(2)} S_2(\rho)$ and $F_\rho, \rho \in \mathcal{L}_{j,r}(2)$, are pairwise non-overlapping. By Lemma 4.1 and (3.7),

$$\begin{aligned}
\sum_{\rho \in \mathcal{L}_{j,r}(2)} \mathcal{E}_r(\rho) &\geq H_{3,r}^{-1} \sum_{\rho \in \mathcal{L}_{j,r}(2)} \sum_{\omega \in S_2(\rho)} \mathcal{E}_r(\omega) = H_{3,r}^{-1} \sum_{\rho \in \mathcal{L}_{j,r}(1)} \mathcal{E}_r(\rho) \\
&\geq H_{3,r}^{-1} P_r^{-1} Q_r \sum_{\tau \in \Lambda_{j,r}(k_{1j})} W([\tau_a] \times [\tau_b]) \\
&\quad + H_{3,r}^{-1} P_r^{-1} Q_r \sum_{\tau \in \Omega_{k_{1j}} \setminus \Lambda_{j,r}(k_{1j})} \sum_{\sigma \times \omega \in \Gamma(\tau)} W([\tau_a * \sigma] \times [\tau_b * \omega]) \\
&\geq H_{3,r}^{-1} P_r^{-1} Q_r \sum_{\tau \in \Lambda_{j,r}(k_{1j})} W([\tau_a] \times [\tau_b]) \\
&\quad + H_{3,r}^{-1} P_r^{-1} Q_r \sum_{\tau \in \Omega_{k_{1j}} \setminus \Lambda_{j,r}(k_{1j})} W([\tau_a] \times [\tau_b]) \\
&= H_{3,r}^{-1} P_r^{-1} Q_r \sum_{\tau \in \Omega_{k_{1j}}} W([\tau_a] \times [\tau_b]) = H_{3,r}^{-1} P_r^{-1} Q_r. \qquad (4.6)
\end{aligned}$$

Analogously, by (3.7), one may show that

$$\sum_{\rho \in \mathcal{L}_{j,r}(2)} \mathcal{E}_r(\rho) \leq 1. \tag{4.7}$$

We denote by $\phi_{j,r}$ the cardinality of $\mathcal{L}_{j,r}(2)$. By (4.5)-(4.7), we deduce

$$\phi_{j,r} P_r^{-2} Q_r \underline{\eta}_r^{\frac{s_r(j+1)}{s_r+r}} \leq 1; \; \phi_{j,r} P_r Q_r^{-1} \underline{\eta}_r^{\frac{js_r}{s_r+r}} \geq H_{3,r}^{-1} P_r^{-1} Q_r.$$

Set $H_{4,r} := P_r^2 Q_r^{-1}$ and $H_{5,r} := H_{3,r}^{-1} P_r^{-2} Q_r^2$. It follows that

$$H_{5,r} \underline{\eta}_r^{\frac{-js_r}{s_r+r}} \leq \phi_{j,r} \leq H_{4,r} \underline{\eta}_r^{\frac{-s_r(j+1)}{s_r+r}}. \tag{4.8}$$

Now let $H := 2([\theta^{-1}]+2)$ and $\delta := (n^2+1)^{-1/2}$. Using the method in the proof of Lemma 2 of [12], we may choose a $\widetilde{\rho}$ for every word $\rho \in \mathcal{L}_{j,r}(2)$ such that

$$\rho \prec \widetilde{\rho}, \quad |\widetilde{\rho}| \leq |\rho| + H$$

and for every pair of distinct words ρ, ω of $\mathcal{L}_{j,r}(2)$, we have

$$d(F_{\widetilde{\rho}}, F_{\widetilde{\omega}}) \geq \delta \max\{|F_{\widetilde{\rho}}|, |F_{\widetilde{\omega}}|\}.$$

Let $\alpha \in C_{\phi_{j,r},r}(\mu)$. Then by Lemma 3 of [12], we can find a constant $D > 0$, which is independent of j, such that

$$\begin{aligned} e^r_{\phi_{j,r},r}(\mu) &\geq \sum_{\rho \in \mathcal{L}_{j,r}(2)} \int_{F_\rho} d(x,\alpha)^r d\mu(x) \geq \sum_{\rho \in \mathcal{L}_{j,r}(2)} \int_{F_{\widetilde{\rho}}} d(x,\alpha)^r d\mu(x) \\ &\geq D \sum_{\rho \in \mathcal{L}_{j,r}(2)} \mu_{\widetilde{\rho}} m^{-|\widetilde{\rho}|r} \geq \widetilde{D} \sum_{\rho \in \mathcal{L}_{j,r}(2)} \mu_\rho m^{-|\rho|r}, \end{aligned} \tag{4.9}$$

where $\widetilde{D} := D\underline{\eta}_r^H$. Thus, by (4.6), (4.9) and Hölder's inequality with exponent less than one, we further deduce

$$\begin{aligned} e^r_{\phi_{j,r},r}(\mu) &\geq \widetilde{D}\Big(\sum_{\rho \in \mathcal{L}_{j,r}(2)} (\mu_\rho m^{-|\rho|r})^{\frac{s_r}{s_r+r}} \Big)^{\frac{s_r+r}{s_r}} \phi_{j,r}^{-\frac{r}{s_r}} \\ &= \widetilde{D}\Big(\sum_{\rho \in \mathcal{L}_{j,r}(2)} \mathcal{E}_r(\rho) \Big)^{\frac{s_r+r}{s_r}} \phi_{j,r}^{-\frac{r}{s_r}} \\ &\geq \widetilde{D}(H_{3,r}P_rQ_r^{-1})^{-\frac{s_r+r}{s_r}} \phi_{j,r}^{-\frac{r}{s_r}}. \end{aligned}$$

Unlike the cardinality of $\Upsilon_{j,r}$, the relationship between $\phi_{j,r}$ and $\phi_{j+1,r}$ is not straightforward. We need to construct a new subsequence of $(n)_{n=1}^\infty$ which enables us to estimate the lower quantization coefficient. By (4.8), we may choose a smallest integer $H_{6,r}$ such that for every j, we have

$$\phi_{j+H_{6,r},r} \geq H_{5,r}\underline{\eta}_r^{\frac{-(j+H_{6,r})s_r}{s_r+r}} > H_{4,r}\underline{\eta}_r^{\frac{-s_r(j+1)}{s_r+r}} \geq \phi_{j,r}.$$

For this integer $H_{6,r}$ and $j \geq 1$, we also have

$$\begin{aligned} \phi_{j+H_{6,r},r} &\leq H_{4,r}\underline{\eta}_r^{\frac{-(j+H_{6,r}+1)s_r}{s_r+r}} = H_{4,r}\underline{\eta}_r^{\frac{-(H_{6,r}+1)s_r}{s_r+r}} \underline{\eta}_r^{\frac{-js_r}{s_r+r}} \\ &\leq H_{5,r}^{-1} H_{4,r}\underline{\eta}_r^{\frac{-(H_{6,r}+1)s_r}{s_r+r}} \phi_{j,r}. \end{aligned}$$

We set $N_{j,r} := \phi_{[\theta^{-1}+jH_{6,r}],r}$ and $H_{7,r} := H_{5,r}^{-1} H_{4,r}\underline{\eta}_r^{\frac{-(H_{6,r}+1)s_r}{s_r+r}}$. Then we have

$$N_{j,r} < N_{j+1,r} \leq H_{7,r}N_{j,r}, \quad N_{j,r}^{\frac{r}{s_r}} e^r_{N_{j,r}}(\mu) \geq \widetilde{D}(H_{3,r}P_rQ_r^{-1})^{-\frac{s_r+r}{s_r}}.$$

For each $k \geq N_{1,r}$, we choose j such that $k \in (N_{j,r}, N_{j+1,r}]$. Then we have

$$\begin{aligned} \underline{Q}_r^{s_r}(\mu) &= \liminf_{k\to\infty} k^{\frac{r}{s_r}} e^r_{k,r}(\mu) \geq \liminf_{j\to\infty} N_{j,r}^{\frac{r}{s_r}} e^r_{N_{j+1,r},r}(\mu) \\ &\geq (H_{7,r})^{-\frac{r}{s_r}} \liminf_{j\to\infty} N_{j+1,r}^{\frac{r}{s_r}} e^r_{N_{j+1,r},r}(\mu) \\ &\geq (H_{7,r})^{-\frac{r}{s_r}} \widetilde{D}(H_{3,r}P_rQ_r^{-1})^{-\frac{s_r+r}{s_r}}. \end{aligned}$$

This completes the proof of the proposition. $\square$

Proof of Theorem 1.1. It is an immediate consequence of Propositions 3.2 and 4.2. □

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