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WEIGHTED NORM INEQUALITIES FOR MULTILINEAR FOURIER MULTIPLIERS WITH CRITICAL BESOV REGULARITY

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ABSTRACT. In this paper, weighted norm inequalities for multilinear Fourier multipliers with Besov regularity are discussed. As a result, we obtain a limiting case of Hörmander type multiplier theorem for multilinear operators.

1. Introduction

For $m \in L^\infty(\mathbb{R}^{Nn})$, the N-linear Fourier multiplier operator T_m is defined by

$$T_m(f_1, \dots, f_N)(x) = \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi) \widehat{f_1}(\xi_1) \dots \widehat{f_N}(\xi_N)\, d\xi$$

for $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$, where $x \in \mathbb{R}^n$, $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$ and $d\xi = d\xi_1 \dots d\xi_N$. In the unweighted case, Coifman and Meyer [4] proved the boundedness of T_m under the condition

$$|\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} m(\xi_1, \dots, \xi_N)| \le C_{\alpha_1,\dots,\alpha_N} (|\xi_1| + \cdots + |\xi_N|)^{-(|\alpha_1|+\cdots+|\alpha_N|)}$$

for sufficiently many multi-indices $\alpha_1, \dots, \alpha_N \in \mathbb{N}_0^n = \{0, 1, 2, \dots\}^n$. Weighted norm inequalities for multilinear Fourier multipliers of sufficient smoothness were also discussed by, for example, Grafakos and Torres [12], and Lerner, Ombrosi, Pérez, Torres and Trujillo-González [14].

In this paper, we are interested in multilinear Fourier multipliers of limited smoothness. Let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\text{(1.1)} \qquad \operatorname{supp} \Psi \subset \left\{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\right\}, \quad \sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

We set

$$\text{(1.2)} \qquad m_j(\xi_1, \dots, \xi_N) = m(2^j\xi_1, \dots, 2^j\xi_N)\Psi(\xi_1, \dots, \xi_N), \quad j \in \mathbb{Z},$$

where Ψ is as in (1.1) with $d = Nn$, and denote by $\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)}$ the smallest constant C satisfying

$$\|T_m(f_1, \dots, f_N)\|_{L^p(w)} \le C \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}, \quad f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n).$$

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The definition of function spaces will be given in Section 2. In the unweighted case, Tomita [21] proved a Hörmander type multiplier theorem for multilinear operators, namely, if $s > Nn/2$ then

$$\|T_m\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_N}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^s(\mathbb{R}^{Nn})} \tag{1.3}$$

for $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \cdots + 1/p_N = 1/p$. Grafakos and Si [11] extended this result to the case $p \le 1$ by using the L^r-based Sobolev spaces, $1 < r \le 2$. For further results in this direction, see [8–10, 17]. The purpose of this paper is to consider the limiting case $s = Nn/2$ in (1.3).

Let $1 < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. In the weighted case, Fujita-Tomita [6] proved that if $n/2 < s_j \le n$, $p_j > n/s_j$ and $w_j \in A_{p_j s_j/n}$, $1 \le j \le N$, then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)}, \tag{1.4}$$

where $w = w_1^{p/p_1} \dots w_N^{p/p_N}$ (see Li, Xue and Yabuta [16] for the case where some p_j are equal to infinity). In particular, for $Nn/2 < s \le Nn$, $p_j > Nn/s$ and $w_j \in A_{p_j s/(Nn)}$, $1 \le j \le N$, taking $s_1 = \cdots = s_N = s/N$ in (1.4), we have

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s/N,\dots,s/N)}((\mathbb{R}^n)^N)}. \tag{1.5}$$

Strengthening the condition on multipliers in the right-hand side of this inequality, Bui and Duong [3] and Li and Sun [15] proved

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^s(\mathbb{R}^{Nn})}$$

under the weaker condition $(w_1, \dots, w_N) \in A_{(s/Nn)\vec{P}}$ with $\vec{P} = (p_1, \dots, p_N)$ introduced in [14]. We remark that (1.5) does not hold in general for $(w_1, \dots, w_N) \in A_{(s/Nn)\vec{P}}$ ([7]). This means that both conditions on multipliers and weights cannot be weakened at the same time.

The following is our main result which can be understood as a limiting case of (1.4), namely, $s_j = n/2$, $1 \le j \le N$ (or $s = Nn/2$ in (1.5)).

Theorem 1.1. *Let $2 < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. Assume $w_j \in A_{p_j/2}$, $j = 1, \dots, N$, and set $w = w_1^{p/p_1} \dots w_N^{p/p_N}$. Then*

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B_{2,1}^{(n/2,\dots,n/2)}((\mathbb{R}^n)^N)}.$$

It should be remarked that

$$H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N) \hookrightarrow B_{2,1}^{(n/2,\dots,n/2)}((\mathbb{R}^n)^N) \hookrightarrow L^\infty(\mathbb{R}^{Nn}), \quad s_1, \dots, s_N > n/2,$$

and the second embedding yields

$$\|m\|_{L^\infty(\mathbb{R}^{Nn})} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B_{2,1}^{(n/2,\dots,n/2)}((\mathbb{R}^n)^N)}. \tag{1.6}$$

In the linear case, Seeger [19] proved

$$\|T_m\|_{\mathcal{H}^1(\mathbb{R}^n)\to L^{1,2}(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B_{2,1}^{n/2}(\mathbb{R}^n)},$$

where $\mathcal{H}^1(\mathbb{R}^n)$ is the Hardy space and $L^{1,2}(\mathbb{R}^n)$ is the Lorentz space. Then, by (1.6) with $N = 1$, interpolation and duality, we see that

$$\|T_m\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B^{n/2}_{2,1}(\mathbb{R}^n)}$$

for $1 < p < \infty$. The following is a multilinear version of this boundedness which can be obtained as a corollary of Theorem 1.1.

Corollary 1.2. *Let $1 < p_1, \dots, p_N, p < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. Then*

$$\|T_m\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_N}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B^{Nn/2}_{2,1}(\mathbb{R}^{Nn})}.$$

The boundedness of T_m under the condition $\sup_{j\in\mathbb{Z}} \|m_j\|_{B^{Nn/r}_{r,1}(\mathbb{R}^{Nn})} < \infty$, $1 \le r < 2$, was discussed in [22], but the case $r = 2$ could not be treated there.

2. Preliminaries

For two nonnegative quantities A and B, the notation $A \lesssim B$ means that $A \le CB$ for some unspecified constant $C > 0$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi.$$

To distinguish linear and multilinear operators, for $m \in L^\infty(\mathbb{R}^n)$, we denote by $m(D)$ the linear Fourier multiplier operator defined by

$$m(D)f(x) = \mathcal{F}^{-1}[m\widehat{f}](x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} m(\xi)\widehat{f}(\xi)\,d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{r^n}\int_{|x-y|<r} |f(y)|\,dy$$

for locally integrable functions f on $\mathbb{R}^n$. Let $0 < p < \infty$ and $w \ge 0$. The weighted Lebesgue space $L^p(w)$ consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty.$$

We say that a weight w belongs to the Muckenhoupt class A_p, $1 < p < \infty$, if

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$, $|B|$ is the Lebesgue measure of B and p' is the conjugate exponent of p (that is, $1/p + 1/p' = 1$). It is well known that the Hardy-Littlewood maximal operator M is bounded on $L^p(w)$ if and only if $w \in A_p$ ([5, Theorem 7.3]).

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left|(1 + |\xi|^2)^{s/2}\widehat{f}(\xi)\right|^2 d\xi\right)^{1/2} < \infty.$$

The norm of the Sobolev space of product type $H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)$, $s_1,\dots,s_N \in \mathbb{R}$, for $F \in \mathcal{S}'((\mathbb{R}^n)^N)$ is also defined by

$$\|F\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)} = \left(\int_{(\mathbb{R}^n)^N} \left| (1+|\xi_1|^2)^{s_1/2} \dots (1+|\xi_N|^2)^{s_N/2} \widehat{F}(\xi) \right|^2 d\xi \right)^{1/2},$$

where $\xi = (\xi_1,\dots,\xi_N) \in (\mathbb{R}^n)^N$.

We recall the definition of Besov spaces of usual and product types, respectively. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (1.1) with $d = n$, and set $\psi_0(\eta) = 1 - \sum_{k=1}^{\infty} \psi(\eta/2^k)$, $\psi_k(\eta) = \psi(\eta/2^k)$, $k \ge 1$. Note that $\operatorname{supp}\psi_0 \subset \{\eta \in \mathbb{R}^n : |\eta| \le 2\}$, $\operatorname{supp}\psi_k \subset \{\eta \in \mathbb{R}^n : 2^{k-1} \le |\eta| \le 2^{k+1}\}$, $k \ge 1$, and $\sum_{k=0}^{\infty} \psi_k(\eta) = 1$, $\eta \in \mathbb{R}^n$. For $1 \le p,q \le \infty$ and $s \in \mathbb{R}$, the Besov space $B^s_{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{k=0}^{\infty} 2^{ksq} \|\psi_k(D)f\|_{L^p}^q \right)^{1/q} < \infty.$$

The norm of the Besov space of product type $B^{(s_1,\dots,s_N)}_{p,q}((\mathbb{R}^n)^N)$, $s_1,\dots,s_N \in \mathbb{R}$, for $F \in \mathcal{S}'((\mathbb{R}^n)^N)$ is also defined by

$$\|F\|_{B^{(s_1,\dots,s_N)}_{p,q}((\mathbb{R}^n)^N)} = \left(\sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1 s_1+\dots+k_N s_N)q} \left\| \Psi_{(k_1,\dots,k_N)}(D)F \right\|_{L^p}^q \right)^{1/q},$$

where

$$\begin{aligned} \text{(2.1)} \qquad \Psi_{(k_1,\dots,k_N)}(\xi) &= (\psi_{k_1} \otimes \dots \otimes \psi_{k_N})(\xi) \\ &= \psi_{k_1}(\xi_1) \times \dots \times \psi_{k_N}(\xi_N), \quad \xi = (\xi_1,\dots \xi_N) \in (\mathbb{R}^n)^N. \end{aligned}$$

The following lemmas will be used in the proof of Theorem 1.1.

Lemma 2.1 ([5, Proposition 2.7]). *Let ϕ be a function which is positive, radial, decreasing (as a function on $(0,\infty)$) and integrable. Then*

$$\sup_{t>0} |[t^{-n}\phi(t^{-1}\cdot)] * f(x)| \lesssim Mf(x).$$

Lemma 2.2 ([13]). *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp}\psi \subset \{\xi \in \mathbb{R}^n : 1/r \le |\xi| \le r\}$ for some $r > 1$. If $1 < p < \infty$ and $w \in A_p$, then*

$$\left\| \left\{ \sum_{j\in\mathbb{Z}} |\psi(D/2^j)f|^2 \right\}^{1/2} \right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)},$$

where $\psi(D/2^j)f = \mathcal{F}^{-1}[\psi(\cdot/2^j)\widehat{f}]$.

Lemma 2.3 ([20, Theorem 1.4, Proposition 1.1]). *Let $s > 0$. Then the following inequalities hold:*

$$\text{(2.2)} \quad \|fg\|_{B^{(s,\dots,s)}_{2,1}((\mathbb{R}^n)^N)} \lesssim \|f\|_{B^{(s,\dots,s)}_{2,1}((\mathbb{R}^n)^N)} \|g\|_{B^{(s,\dots,s)}_{\infty,1}((\mathbb{R}^n)^N)},$$

$$\text{(2.3)} \quad \|f(2^\ell\cdot)\|_{B^{(s,\dots,s)}_{2,1}((\mathbb{R}^n)^N)} \lesssim \left(\max\{1, 2^{\ell s}\} 2^{-\ell n/2}\right)^N \|f\|_{B^{(s,\dots,s)}_{2,1}((\mathbb{R}^n)^N)}, \quad \ell \in \mathbb{Z}.$$

Lemma 2.3 with $N = 1$ can be found in [23, Remark 2.8.2/1, Proposition 3.4.1/1]. Since the proof of Lemma 2.3 was omitted in [20], we shall give it in Appendix A for the reader's convenience.

Let N be a natural number, and let ϕ_0 be a C^∞-function on $[0, \infty)$ satisfying

$$\phi_0(t) = 1 \quad \text{on} \quad [0, 1/(4N)], \qquad \operatorname{supp} \phi_0 \subset [0, 1/(2N)].$$

We also set $\phi_1(t) = 1 - \phi_0(t)$. For $(i_1, \dots, i_N) \in \{0, 1\}^N$, we define a function $\Phi_{(i_1,\dots,i_N)}$ on $\mathbb{R}^{Nn} \setminus \{0\}$ by

$$\Phi_{(i_1,\dots,i_N)}(\xi) = \phi_{i_1}(|\xi_1|/|\xi|) \dots \phi_{i_N}(|\xi_N|/|\xi|), \quad \xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N. \tag{2.4}$$

Note that $\Phi_{(0,0,\dots,0)} = 0$. Then we have

Lemma 2.4 ([6, Lemma 3.1])**.** *Let $\Phi_{(i_1,\dots,i_N)}$ be the same as in (2.4). Then the following are true:*

(1) *For $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \setminus \{(0, \dots, 0)\}$,*

$$\sum_{\substack{(i_1,i_2,\dots,i_N)\in\{0,1\}^N \\ (i_1,i_2,\dots,i_N)\neq(0,0,\dots,0)}} \Phi_{(i_1,i_2,\dots,i_N)}(\xi) = 1.$$

(2) *For $(i_1, \dots, i_N) \in \{0, 1\}^N$ and $(\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^n \times \dots \times \mathbb{N}_0^n$,*

$$|\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \Phi_{(i_1,\dots,i_N)}(\xi)| \le C_{(i_1,\dots,i_N)}^{\alpha_1,\dots,\alpha_N} (|\xi_1| + \dots + |\xi_N|)^{-(|\alpha_1|+\dots+|\alpha_N|)}$$

for all $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \setminus \{(0, \dots, 0)\}$.

(3) *If $i_j = 1$ for some $1 \le j \le N$ and $i_k = 0$ for all $1 \le k \le N$ with $k \ne j$, then $\operatorname{supp} \Phi_{(i_1,\dots,i_N)} \subset \{(\xi_1, \dots, \xi_N) : |\xi_k| \le |\xi_j|/N$ for $k \ne j\}$. If $i_j = i_{j'} = 1$ for some $1 \le j, j' \le N$ with $j \ne j'$, then $\operatorname{supp} \Phi_{(i_1,\dots,i_N)} \subset \{(\xi_1, \dots, \xi_N) : |\xi_j|/(4N) \le |\xi_{j'}| \le 4N|\xi_j|,\ |\xi_k| \le 4N|\xi_j|$ for $k \ne j, j'\}$.*

3. Key estimate

In this section, we prove the following lemma which plays an important role in the proof of Theorem 1.1.

Lemma 3.1. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\phi(x) = \phi(-x)$, $x \in \mathbb{R}^n$, and $\phi(x) = 1$ on $\{x \in \mathbb{R}^n : |x| \le 2\}$. Then*

$$\begin{aligned} &\left|T_{m(\cdot/2^j)}(f_1, \dots, f_N)(x)\right| \\ &\lesssim \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\dots+k_N)n/2} \left\|\Psi_{(k_1,\dots,k_N)}(D)m\right\|_{L^2} \prod_{i=1}^{N} (|\phi|^2)_{(k_i-j)} * |f_i|^2(x)^{1/2} \end{aligned}$$

for $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, where $(|\phi|^2)_{(j)}(x) = 2^{-jn}|\phi(2^{-j}x)|^2$ and $\Psi_{(k_1,\dots,k_N)}$ is defined by (2.1).

Proof. Let $\{\psi_{k_i}\}_{k_i=0}^{\infty}$ be the partition of unity appearing in the definition of Besov spaces of product type. Then

$$
\begin{aligned}
&T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)\\
&= (2^{jn})^N \int_{(\mathbb{R}^n)^N} \mathcal{F}^{-1}m\left(2^j(x-y_1),\dots,2^j(x-y_N)\right) f_1(y_1)\dots f_N(y_N)\,dy\\
&= (2\pi)^{-Nn}(2^{jn})^N \sum_{k_1,\dots,k_N=0}^{\infty} \int_{(\mathbb{R}^n)^N} \psi_{k_1}(2^j(y_1-x))\dots\psi_{k_N}(2^j(y_N-x))\\
&\qquad \times \widehat{m}\left(2^j(y_1-x),\dots,2^j(y_N-x)\right) f_1(y_1)\dots f_N(y_N)\,dy,
\end{aligned}
$$

where $dy = dy_1 \dots dy_N$. Let ϕ be a function appearing in the statement of Lemma 3.1. Since $\operatorname{supp}\psi_{k_i} \subset \{|y_i| \le 2^{k_i+1}\}$, we have $\psi_{k_i}(y_i) = \psi_{k_i}(y_i)\phi(y_i/2^{k_i})$. Hence, using Schwarz's inequality, a change of variables and $\phi(y_i) = \phi(-y_i)$, we have

$$
\begin{aligned}
&\left|T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)\right|\\
&\lesssim (2^{jn})^N \sum_{k_1,\dots,k_N=0}^{\infty} \Bigg| \int_{(\mathbb{R}^n)^N} [\Psi_{(k_1,\dots,k_N)}\widehat{m}](2^j(y_1-x),\dots,2^j(y_N-x))\\
&\qquad \times \phi\left(\frac{2^j(y_1-x)}{2^{k_1}}\right)\dots\phi\left(\frac{2^j(y_N-x)}{2^{k_N}}\right) f_1(y_1)\dots f_N(y_N)\,dy\Bigg|\\
&\le (2^{jn})^N \sum_{k_1\dots k_N=0}^{\infty} \left\|[\Psi_{(k_1,\dots,k_N)}\widehat{m}]\left(2^j(y_1-x),\dots,2^j(y_N-x)\right)\right\|_{L^2}\\
&\quad \times \left\|\phi\left(\frac{x-y_1}{2^{k_1-j}}\right)\dots\phi\left(\frac{x-y_N}{2^{k_N-j}}\right) f_1(y_1)\dots f_N(y_N)\right\|_{L^2}\\
&= \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\dots+k_N)n/2} \left\|\Psi_{(k_1,\dots,k_N)}\widehat{m}\right\|_{L^2} \prod_{i=1}^{N} (|\phi|^2)_{(k_i-j)} * |f_i|^2(x)^{1/2}.
\end{aligned}
$$

By Plancherel's theorem, we have the desired estimate. ☐

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The following notation will be used: $\mathcal{A}_0$ denotes the set of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ for which $\operatorname{supp}\varphi$ is compact and $\varphi = 1$ on some neighborhood of the origin; $\mathcal{A}_1$ denotes the set of $\widetilde{\psi} \in \mathcal{S}(\mathbb{R}^n)$ for which $\operatorname{supp}\widetilde{\psi}$ is a compact subset of $\mathbb{R}^n \setminus \{0\}$.

Proof of Theorem 1.1. Let $2 < p_1,\dots,p_N < \infty$, $1/p_1 + \dots + 1/p_N = 1/p$, $w_i \in A_{p_i/2}$, $i = 1,\dots,N$, and set $w = w_1^{p/p_1}\dots w_N^{p/p_N}$. We also assume that $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies $\sup_{j\in\mathbb{Z}} \|m_j\|_{B_{2,1}^{(n/2,\dots,n/2)}((\mathbb{R}^n)^N)} < \infty$, where m_j is defined by (1.2). It follows from Lemma 2.4 (1) that

$$
\begin{aligned}
m(\xi) &= \sum_{\substack{(i_1,i_2,\dots,i_N)\in\{0,1\}^N\\ (i_1,i_2,\dots,i_N)\ne(0,0,\dots,0)}} \Phi_{(i_1,i_2,\dots,i_N)}(\xi)m(\xi)\\
&= \sum_{\substack{(i_1,i_2,\dots,i_N)\in\{0,1\}^N\\ (i_1,i_2,\dots,i_N)\ne(0,0,\dots,0)}} m_{(i_1,i_2,\dots,i_N)}(\xi).
\end{aligned}
$$

Estimate for $m_{(1,0,\dots,0)}$. We first consider the case where $(i_1,\dots,i_N)$ satisfies $\sharp\{j : i_j = 1\} = 1$, and may assume, without loss of generality, that $i_1 = 1$. This means $m_{(i_1,i_2,\dots,i_N)} = m_{(1,0,\dots,0)}$, and we simply write m instead of $m_{(1,0,\dots,0)}$. By Lemma 2.4 (3),

$$\operatorname{supp} m \subset \{\xi = (\xi_1,\dots,\xi_N) \in (\mathbb{R}^n)^N : |\xi_i| \le |\xi_1|/N,\ i = 2,\dots,N\}. \tag{4.1}$$

Let ψ be as in (1.1) with $d = n$. It is known that, if $v \in A_\infty = \bigcup_{1\le p<\infty} A_p$, then the weighted Hardy space $\mathcal{H}^p(v)$ coincides with the weighted Triebel-Lizorkin space $F^0_{p,2}(v)$ ([2, Appendix]). Then $\|g\|_{L^p(v)} \lesssim \|g\|_{\mathcal{H}^p(v)} \approx \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j g|^2\right)^{1/2} \right\|_{L^p(v)}$ for appropriate functions g, where $\Delta_j g = \psi(D/2^j)g$ (see also [6, Remark 2.6]). The assumption $w_i \in A_{p_i/2} \subset A_{p_i}$, $i = 1,\dots,N$, implies that $w = w_1^{p/p_1}\dots w_N^{p/p_N} \in A_{Np}$ ([14, p.1233]). Hence,

$$\|T_m(f_1,\dots,f_N)\|_{L^p(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j T_m(f_1,\dots,f_N)|^2 \right)^{1/2} \right\|_{L^p(w)}. \tag{4.2}$$

It follows from (4.1) that if $(\xi_1,\dots,\xi_N) \in \operatorname{supp} m$, then $|\xi_1 + \dots + \xi_N| \approx |\xi_1|$ and $|\xi_i| \lesssim |\xi_1|$ for $2 \le i \le N$, and we can find functions $\varphi \in \mathcal{A}_0$ and $\widetilde{\psi} \in \mathcal{A}_1$ independent of j such that

$$\begin{aligned}
&m(\xi)\psi((\xi_1 + \dots + \xi_N)/2^j) \\
&= m(\xi)\psi((\xi_1 + \dots + \xi_N)/2^j)\widetilde{\psi}(\xi_1/2^j)^2\varphi(\xi_2/2^j)\dots\varphi(\xi_N/2^j),
\end{aligned}$$

where we have used the fact that $\operatorname{supp}\psi \subset \{\eta \in \mathbb{R}^n : 1/2 \le |\eta| \le 2\}$. Hence, setting $m^{(j)}(\xi) = m(2^j\xi)\psi(\xi_1 + \dots + \xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)$, we see that

$$\begin{aligned}
&\Delta_j T_m(f_1, f_2,\dots,f_N)(x) \\
&= \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\dots+\xi_N)} m(\xi)\psi((\xi_1+\dots+\xi_N)/2^j)\widehat{f_1}(\xi_1)\dots\widehat{f_N}(\xi_N)\, d\xi \\
&= T_{m^{(j)}(\cdot/2^j)}(\widetilde{\Delta}_j f_1, f_2,\dots,f_N)(x),
\end{aligned}$$

where $\widetilde{\Delta}_j f_1 = \widetilde{\psi}(D/2^j)f_1$. By Lemmas 3.1 and 2.1,

$$\begin{aligned}
&|T_{m^{(j)}(\cdot/2^j)}(\widetilde{\Delta}_j f_1, f_2,\dots,f_N)| \\
&\lesssim \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\dots+k_N)n/2} \left\| \Psi_{(k_1,\dots,k_N)}(D)m^{(j)} \right\|_{L^2} \\
&\qquad \times \left((|\phi|^2)_{(k_1-j)} * |\widetilde{\Delta}_j f_1|^2 \right)^{1/2} \prod_{i=2}^{N} \left((|\phi|^2)_{(k_i-j)} * |f_i|^2 \right)^{1/2} \\
&\lesssim \left\{ \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\dots+k_N)n/2} \left\| \Psi_{(k_1,\dots,k_N)}(D)m^{(j)} \right\|_{L^2} \left((|\phi|^2)_{(k_1-j)} * |\widetilde{\Delta}_j f_1|^2 \right)^{1/2} \right\} \\
&\qquad \times \prod_{i=2}^{N} M(|f_i|^2)^{1/2}.
\end{aligned}$$

It follows from Schwarz's inequality that the sum concerning f_1 in the last line is estimated by

$$\left(\sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2}\left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2}\right)^{1/2}$$
$$\times\left(\sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2}\left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2}(|\phi|^2)_{(k_1-j)}*|\widetilde{\Delta}_j f_1|^2\right)^{1/2}$$
$$= \|m^{(j)}\|_{B_{2,1}^{(n/2,\dots,n/2)}}^{1/2}$$
$$\times\left(\sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2}\left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2}(|\phi|^2)_{(k_1-j)}*|\widetilde{\Delta}_j f_1|^2\right)^{1/2}.$$

Thus, by Hölder's inequality, the $L^p(w)$-norm on the right-hand side of (4.2) is estimated by

$$\left(\sup_{j\in\mathbb{Z}}\|m^{(j)}\|_{B_{2,1}^{(n/2,\dots,n/2)}}\right)^{1/2} \times\Bigg\|\Bigg(\sum_{\substack{j\in\mathbb{Z}\\ k_1,\dots,k_N\ge 0}} 2^{(k_1+\cdots+k_N)n/2}\left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2} \times(|\phi|^2)_{(k_1-j)}*|\widetilde{\Delta}_j f_1|^2\Bigg)^{1/2}\Bigg\|_{L^{p_1}(w_1)}\left(\prod_{i=2}^{N}\left\|M(|f_i|^2)^{1/2}\right\|_{L^{p_i}(w_i)}\right). \tag{4.3}$$

Since $2<p_i<\infty$ and $w_i\in A_{p_i/2}$, $i=2,\dots,N$, we have

$$\prod_{i=2}^{N}\left\|M(|f_i|^2)^{1/2}\right\|_{L^{p_i}(w_i)}\lesssim\prod_{i=2}^{N}\|f_i\|_{L^{p_i}(w_i)}. \tag{4.4}$$

In order to estimate the $L^{p_1}(w_1)$-norm concerning f_1 in (4.3), we use a duality argument. Let $g\in\mathcal{S}$ be such that the $L^{(p_1/2)'}(w_1^{1-(p_1/2)'})$-norm of g is equal to one. Note that $w_1^{1-(p_1/2)'}\in A_{(p_1/2)'}$ ([5, Proposition 7.2]), and consequently the Hardy-Littlewood maximal operator M is bounded on $L^{(p_1/2)'}(w_1^{1-(p_1/2)'})$. Then, by Lemmas 2.1 and 2.2,

$$\left|\int_{\mathbb{R}^n}\Bigg(\sum_{\substack{j\in\mathbb{Z}\\ k_1,\dots,k_N\ge 0}} 2^{(k_1+\cdots+k_N)n/2}\left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2} \times[(|\phi|^2)_{(k_1-j)}*|\widetilde{\Delta}_j f_1|^2](x)\Bigg)g(x)\,dx\right|$$
$$\le\sum_{\substack{j\in\mathbb{Z}\\ k_1,\dots,k_N\ge 0}} 2^{(k_1+\cdots+k_N)n/2}\left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2} \times\int_{\mathbb{R}^n}|\widetilde{\Delta}_j f_1(x)|^2\,[(|\phi|^2)_{(k_1-j)}*|g|](x)\,dx$$

$$\lesssim \left\{ \sup_{j\in\mathbb{Z}} \sum_{k_1,\dots,k_N\ge 0} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\dots,k_N)}(D)m^{(j)} \right\|_{L^2} \right\} \times \left\{ \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} |\widetilde{\Delta}_j f_1(x)|^2 \right) Mg(x)\,dx \right\}$$

$$\le \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B_{2,1}^{(n/2,\dots,n/2)}} \right) \left\| \sum_{j\in\mathbb{Z}} |\widetilde{\Delta}_j f_1(x)|^2 \right\|_{L^{p_1/2}(w_1)} \|Mg\|_{L^{(p_1/2)'}(w_1^{1-(p_1/2)'})}$$

$$\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B_{2,1}^{(n/2,\dots,n/2)}} \right) \|f_1\|_{L^{p_1}(w_1)}^2 \|g\|_{L^{(p_1/2)'}(w_1^{1-(p_1/2)'})}.$$

Since $L^{(p_1/2)'}(w_1^{1-(p_1/2)'})$ is the dual space of $L^{p_1/2}(w_1)$ under the duality $\int_{\mathbb{R}^n} fg\,dx$, $f \in L^{p_1/2}(w_1)$, $g \in L^{(p_1/2)'}(w_1^{1-(p_1/2)'})$, by taking the supremum over all g as above, the $L^{p_1}(w_1)$-norm concerning f_1 in (4.3) is estimated by

$$\left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B_{2,1}^{(n/2,\dots,n/2)}} \right)^{1/2} \|f_1\|_{L^{p_1}(w_1)}.$$

Combining this with (4.2), (4.3) and (4.4), we have

$$\|T_m(f_1,\dots,f_N)\|_{L^p(w)} \lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B_{2,1}^{(n/2,\dots,n/2)}} \right) \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}.$$

We shall prove

$$\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B_{2,1}^{(n/2,\dots,n/2)}} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B_{2,1}^{(n/2,\dots,n/2)}}. \tag{4.5}$$

Once this is proved, we have the desired estimate,

$$\|T_m(f_1,\dots,f_N)\|_{L^p(w)} \lesssim \left(\sup_{j\in\mathbb{Z}} \|m_j\|_{B_{2,1}^{(n/2,\dots,n/2)}} \right) \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}.$$

Let Ψ be as in (1.1) with $d = Nn$. Since $\operatorname{supp} \Psi(\cdot/2^\ell) \subset \{2^{\ell-1} \le |\xi| \le 2^{\ell+1}\}$ and

$$\operatorname{supp} \widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N) \subset \{2^{-j_0} \le |\xi| \le 2^{j_0}\} \tag{4.6}$$

for some $j_0 \in \mathbb{N}$, it follows from (2.2) that

$$\|m^{(j)}(\xi)\|_{B_{2,1}^{(n/2,\dots,n/2)}} \le \sum_{\ell=-j_0}^{j_0} \|m^{(j)}(\xi)\Psi(\xi/2^\ell)\|_{B_{2,1}^{(n/2,\dots,n/2)}}$$

$$\lesssim \sum_{\ell=-j_0}^{j_0} \|m(2^j\xi)\Psi(\xi/2^\ell)\|_{B_{2,1}^{(n/2,\dots,n/2)}} \times \|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\|_{B_{\infty,1}^{(n/2,\dots,n/2)}}.$$

By (2.3),

$$\|m(2^j\xi)\Psi(\xi/2^\ell)\|_{B_{2,1}^{(n/2,\dots,n/2)}} = \|m_{j+\ell}(2^{-\ell}\cdot)\|_{B_{2,1}^{(n/2,\dots,n/2)}}$$

$$\lesssim (\max\{1,(2^{-\ell})^{n/2}\}(2^{-\ell})^{-n/2})^N \|m_{j+\ell}\|_{B_{2,1}^{(n/2,\dots,n/2)}} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B_{2,1}^{(n/2,\dots,n/2)}}$$

for all $|\ell| \le j_0$. On the other hand, by Lemma 2.4 (2) and (4.6),

$$\left|\partial_\xi^\alpha \left(\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\dots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\right)\right| \le C_\alpha$$

for all α and j, and consequently,

$$\sup_{j\in\mathbb{Z}} \|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\dots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\|_{B^{(n/2,\dots,n/2)}_{\infty,1}} < \infty.$$

Combining these estimates, we have (4.5).

Estimate for $m_{(1,1,i_3,\dots,i_N)}$. We next consider the case where $(i_1,\dots,i_N)$ satisfies $\sharp\{j : i_j = 1\} \ge 2$, and may assume, without loss of generality, that $i_1 = i_2 = 1$. This means $m_{(i_1,i_2,i_3,\dots,i_N)} = m_{(1,1,i_3,\dots,i_N)}$, where $i_3,\dots,i_N \in \{0,1\}$. We simply write m instead of $m_{(1,1,i_3,\dots,i_N)}$ as before. By Lemma 2.4 (3),

$$\operatorname{supp} m \subset \{|\xi_1|/(4N) \le |\xi_2| \le 4N|\xi_1|,\ |\xi_i| \le 4N|\xi_1|,\ i = 3,\dots,N\}. \tag{4.7}$$

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (1.1) with $d = n$. By (4.7), we can find $\varphi \in \mathcal{A}_0$ and $\widetilde{\psi} \in \mathcal{A}_1$ independent of j such that

$$m(\xi)\psi(\xi_1/2^j) = m(\xi)\psi(\xi_1/2^j)\widetilde{\psi}(\xi_1/2^j)\widetilde{\psi}(\xi_2/2^j)^2\varphi(\xi_3/2^j)\dots\varphi(\xi_N/2^j).$$

Hence, setting $m^{(j)}(\xi) = m(2^j\xi)\psi(\xi_1)\widetilde{\psi}(\xi_2)\varphi(\xi_3)\dots\varphi(\xi_N)$, we see that

$$\begin{aligned} &T_m(f_1,\dots,f_N)(x) \\ &= \sum_{j\in\mathbb{Z}} \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\dots+\xi_N)} m(\xi)\psi(\xi_1/2^j)\widehat{f_1}(\xi_1)\dots\widehat{f_N}(\xi_N)\,d\xi \\ &= \sum_{j\in\mathbb{Z}} T_{m^{(j)}(\cdot/2^j)}(\widetilde{\Delta}_j f_1, \widetilde{\Delta}_j f_2, f_3,\dots,f_N)(x), \end{aligned}$$

where $\widetilde{\Delta}_j f_i = \widetilde{\psi}(D/2^j) f_i$, $i = 1,2$. It follows from Lemmas 3.1, 2.1 and Schwarz's inequality that

$$\begin{aligned} &|T_{m^{(j)}}(\widetilde{\Delta}_j f_1, \widetilde{\Delta}_j f_2, f_3,\dots,f_N)| \\ &\lesssim \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\dots+k_N)n/2} \left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2} \\ &\quad\times \left(\prod_{i=1}^{2} \left((|\phi|^2)_{(k_i-j)} * |\widetilde{\Delta}_j f_i|^2\right)^{1/2}\right) \left(\prod_{i=3}^{N} \left((|\phi|^2)_{(k_i-j)} * |f_i|^2\right)^{1/2}\right) \\ &\lesssim \left\{\prod_{i=1}^{2}\left(\sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\dots+k_N)n/2}\left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2}(|\phi|^2)_{(k_i-j)} * |\widetilde{\Delta}_j f_i|^2\right)^{1/2}\right\} \\ &\quad\times \left\{\prod_{i=3}^{N} M(|f_i|^2)^{1/2}\right\}. \end{aligned}$$

Then, by Hölder's inequality, the $L^p(w)$-norm of $T_m(f_1,\dots,f_N)$ is estimated by

$$\Big\{\prod_{i=1}^{2}\Big\|\Big(\sum_{\substack{j\in\mathbb{Z}\\ k_1,\dots,k_N\ge 0}} 2^{(k_1+\cdots+k_N)n/2}\Big\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\Big\|_{L^2} \times (|\phi|^2)_{(k_i-j)} * |\widetilde{\Delta}_j f_i|^2\Big)^{1/2}\Big\|_{L^{p_i}(w_i)}\Big\}\Big\{\prod_{i=3}^{N}\Big\|M(|f_i|^2)^{1/2}\Big\|_{L^{p_i}(w_i)}\Big\}.$$

The rest of the proof is similar to that for $m_{(1,0,\dots,0)}$, and we omit it. ☐

5. Proof of Corollary 1.2

In this section, we prove Corollary 1.2. Though we can prove it as a corollary of Theorem 1.1 by a standard argument, we shall give a proof for the reader's convenience.

Proof of Corollary 1.2. In this proof, we always assume that exponents $p_1,\dots,p_N,p$ satisfy $1/p_1+\cdots+1/p_N=1/p$. To use a duality argument, we measure the smoothness of multipliers by $B^{Nn/2}_{2,1}$ instead of $B^{(n/2,\dots,n/2)}_{2,1}$.

By Theorem 1.1 with $w_j\equiv 1$, $j=1,\dots,N$, and the embedding

$$B^{Nn/2}_{2,1}(\mathbb{R}^{Nn})\hookrightarrow B^{(n/2,\dots,n/2)}_{2,1}((\mathbb{R}^n)^N),$$

we see that

$$\|T_m\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_N}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}}\|m_j\|_{B^{Nn/2}_{2,1}(\mathbb{R}^{Nn})} \tag{5.1}$$

for $2<p_1,\dots,p_N<\infty$. Let $\epsilon>0$ be sufficiently small. Then (5.1) with $(1/p_1,1/p_2,\dots,1/p_N)=(\epsilon,\epsilon,\dots,\epsilon)$ holds. Since

$$\int_{\mathbb{R}^n} T_m(f_1,f_2,\dots,f_N)(x)g(x)\,dx=\int_{\mathbb{R}^n} T_{m^{*1}}(g,f_2,\dots,f_N)(x)f_1(x)\,dx,$$

where

$$m^{*1}(\xi)=m(-(\xi_1+\xi_2+\cdots+\xi_N),\xi_2,\dots,\xi_N),\quad \xi=(\xi_1,\xi_2,\dots,\xi_N)\in(\mathbb{R}^n)^N,$$

by duality, the boundedness of $T_{m^{*1}}$ from $L^{p_1}\times L^{p_2}\times\cdots\times L^{p_N}$ to L^p implies that of T_m from $L^{p'}\times L^{p_2}\times\cdots\times L^{p_N}$ to $L^{p_1'}$. Note that if $(1/p_1,1/p_2,\dots,1/p_N,1/p)=(\epsilon,\epsilon,\dots,\epsilon,N\epsilon)$, then $(1/p',1/p_2,\dots,1/p_N,1/p_1')=(1-N\epsilon,\epsilon,\dots,\epsilon,1-\epsilon)$. Thus, (5.1) with m replaced by m^{*1} and $(1/p_1,1/p_2,\dots,1/p_N)=(\epsilon,\epsilon,\dots,\epsilon)$ yields

$$\|T_m\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_N}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}}\|(m^{*1})_j\|_{B^{Nn/2}_{2,1}(\mathbb{R}^{Nn})}$$

for $(1/p_1,1/p_2,\dots,1/p_N)=(1-N\epsilon,\epsilon,\dots,\epsilon)$, where $(m^{*1})_j(\xi)=m^{*1}(2^j\xi)\Psi(\xi)$. Since $(m^{*1})_j=(m(2^j\cdot)\Psi^{*1})^{*1}$, using the fact $B^{Nn/2}_{2,1}$ is invariant under the map $f\mapsto f^{*1}$ ([18, Proposition 2.1.3/6]), we have

$$\sup_{j\in\mathbb{Z}}\|(m^{*1})_j\|_{B^{Nn/2}_{2,1}(\mathbb{R}^{Nn})}\lesssim \sup_{j\in\mathbb{Z}}\|m(2^j\cdot)\Psi^{*1}\|_{B^{Nn/2}_{2,1}(\mathbb{R}^{Nn})}.$$

Moreover, since $\operatorname{supp}\Psi^{*1}$ is included in some annulus, the same argument as in the proof of (4.5) gives

$$\sup_{j\in\mathbb{Z}}\|m(2^j\cdot)\Psi^{*1}\|_{B^{Nn/2}_{2,1}(\mathbb{R}^{Nn})}\lesssim \sup_{j\in\mathbb{Z}}\|m_j\|_{B^{Nn/2}_{2,1}(\mathbb{R}^{Nn})}.$$

Hence, (5.1) with $(1/p_1, 1/p_2, \dots, 1/p_N) = (1-N\epsilon, \epsilon, \dots, \epsilon)$ holds for any sufficiently small $\epsilon > 0$. Similarly, by considering

$$m^{*i}(\xi) = m(\xi_1, \dots, \xi_{i-1}, -(\xi_1 + \cdots + \xi_N), \xi_{i+1}, \dots, \xi_N), \quad i = 2, \dots, N,$$

we see that (5.1) with $(1/p_1, \dots, 1/p_{i-1}, 1/p_i, 1/p_{i+1}, \dots, 1/p_N) = (\epsilon, \dots, \epsilon, 1 - N\epsilon, \epsilon, \dots, \epsilon)$ holds for any sufficiently small $\epsilon > 0$.

Let $1 < p_1, \dots, p_N, p < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. Our goal is to prove (5.1) for these $p_1, \dots, p_N, p$. To do this, we check an existence of $0 < \theta_0, \theta_1, \dots, \theta_N < 1$ and a sufficiently small $\epsilon > 0$ satisfying

$$\begin{cases} \theta_0 + \theta_1 + \cdots + \theta_N = 1 \\ \theta_0 \begin{bmatrix} \epsilon \\ \epsilon \\ \vdots \\ \epsilon \end{bmatrix} + \theta_1 \begin{bmatrix} 1 - N\epsilon \\ \epsilon \\ \vdots \\ \epsilon \end{bmatrix} + \cdots + \theta_N \begin{bmatrix} \epsilon \\ \epsilon \\ \vdots \\ 1 - N\epsilon \end{bmatrix} = \begin{bmatrix} 1/p_1 \\ 1/p_2 \\ \vdots \\ 1/p_N \end{bmatrix}. \end{cases} \tag{5.2}$$

Once this existence is assured, by multilinear interpolation ([1, Chapter 4, Theorem 2.7]), we have the desired result, because a convex hull of points satisfying (5.1) includes $(1/p_1, \dots, 1/p_N)$. We define an $(N+1) \times (N+1)$ matrix A_ϵ, $\epsilon \geq 0$, by

$$A_\epsilon = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \epsilon & 1 - N\epsilon & \epsilon & \cdots & \epsilon \\ \epsilon & \epsilon & 1 - N\epsilon & \cdots & \epsilon \\ \vdots & \vdots & \vdots & & \vdots \\ \epsilon & \epsilon & \epsilon & \cdots & 1 - N\epsilon \end{bmatrix}.$$

Then (5.2) can be written as $A_\epsilon \begin{bmatrix} \theta_0 & \theta_1 & \cdots & \theta_N \end{bmatrix}^t = \begin{bmatrix} 1 & 1/p_1 & \cdots & 1/p_N \end{bmatrix}^t$. If $\epsilon = 0$, then the determinant of A_ϵ is equal to one and the solution to the linear equations (5.2) is $\begin{bmatrix} \theta_0 & \theta_1 & \cdots & \theta_N \end{bmatrix}^t = \begin{bmatrix} 1 - 1/p & 1/p_1 & \cdots & 1/p_N \end{bmatrix}^t$. Therefore, by the continuity of the determinant and solution with respect to ϵ, the existence of $\theta_0, \theta_1, \dots, \theta_N, \epsilon$ we are looking for is assured. □

Appendix A

In this appendix, we give a proof of Lemma 2.3. We only treat the case $N = 2$ for the sake of simplicity, but our argument given here works for the others. Let $s > 0$, and let $\Psi_{(j_1,j_2)}(\xi) = \psi_{j_1}(\xi_1)\psi_{j_2}(\xi_2)$, $\xi = (\xi_1, \xi_2) \in (\mathbb{R}^n)^2$, $j_1, j_2 \geq 0$, be as in (2.1).

We first consider (2.2). Since $fg = \sum_{\substack{j_1,j_2 \geq 0 \\ k_1,k_2 \geq 0}} \Psi_{(j_1,j_2)}(D)f\, \Psi_{(k_1,k_2)}(D)g$, we have

$$\|fg\|_{B^{(s,s)}_{2,1}} \leq \sum_{\ell_1,\ell_2 \geq 0} 2^{(\ell_1+\ell_2)s} \left(\sum_{\substack{j_1,j_2 \geq 0 \\ k_1,k_2 \geq 0}} \|\Psi_{(\ell_1,\ell_2)}(D)[\Psi_{(j_1,j_2)}(D)f\, \Psi_{(k_1,k_2)}(D)g]\|_{L^2} \right).$$

By using the support properties

$$\operatorname{supp} \Psi_{(\ell_1,\ell_2)} \subset \{(\xi_1, \xi_2) : 2^{\ell_i - 1} \leq |\xi_i| \leq 2^{\ell_i + 1},\, i = 1, 2\}$$

with $2^{\ell_i-1} \leq |\xi_i| \leq 2^{\ell_i+1}$ replaced by $|\xi_i| \leq 2$ if $\ell_i = 0$ and

$$\operatorname{supp} \mathcal{F}[\Psi_{(j_1,j_2)}(D)f\, \Psi_{(k_1,k_2)}(D)g] \subset \{(\xi_1, \xi_2) : |\xi_i| \leq 2^{j_i+k_i+2},\, i = 1, 2\},$$

we see that the right-hand side of the inequality above is equal to

$$\sum_{\substack{j_1,j_2\ge 0\\ k_1,k_2\ge 0}} \left(\sum_{\substack{\ell_1\le j_1+k_1+2\\ \ell_2\le j_2+k_2+2}} 2^{(\ell_1+\ell_2)s} \|\Psi_{(\ell_1,\ell_2)}(D)[\Psi_{(j_1,j_2)}(D)f\,\Psi_{(k_1,k_2)}(D)g]\|_{L^2} \right).$$

Since $\sup_{\ell_1,\ell_2\ge 0} \left\|\mathcal{F}^{-1}\Psi_{(\ell_1,\ell_2)}\right\|_{L^1} < \infty$, by Young's inequality and Hölder's inequality, this is estimated by

$$\begin{aligned}
&\sum_{\substack{j_1,j_2\ge 0\\ k_1,k_2\ge 0}} \left(\sum_{\substack{\ell_1\le j_1+k_1+2\\ \ell_2\le j_2+k_2+2}} 2^{(\ell_1+\ell_2)s} \|\Psi_{(j_1,j_2)}(D)f\,\Psi_{(k_1,k_2)}(D)g\|_{L^2} \right)\\
&\le \sum_{\substack{j_1,j_2\ge 0\\ k_1,k_2\ge 0}} \|\Psi_{(j_1,j_2)}(D)f\|_{L^2} \|\Psi_{(k_1,k_2)}(D)g\|_{L^\infty} \left(\sum_{\substack{\ell_1\le j_1+k_1+2\\ \ell_2\le j_2+k_2+2}} 2^{(\ell_1+\ell_2)s} \right)\\
&\lesssim \left(\sum_{j_1,j_2\ge 0} 2^{(j_1+j_2)s} \|\Psi_{(j_1,j_2)}(D)f\|_{L^2} \right) \left(\sum_{k_1,k_2\ge 0} 2^{(k_1+k_2)s} \|\Psi_{(k_1,k_2)}(D)g\|_{L^\infty} \right).
\end{aligned}$$

Hence, we have (2.2).

We next consider (2.3), and set $\ell_+ = \max\{\ell, 0\}$ for $\ell \in \mathbb{Z}$. By a change of variables,

$$\begin{aligned}
\|f(2^\ell\cdot)\|_{B^{(s,s)}_{2,1}} &= 2^{-\ell n} \sum_{j,k\ge 0} 2^{(j+k)s} \|\mathcal{F}^{-1}[(\psi_j(2^\ell\cdot)\otimes\psi_k(2^\ell\cdot))\widehat{f}]\|_{L^2}\\
&= 2^{-\ell n} \left(\sum_{\substack{j\le \ell_+\\ k\le \ell_+}} + \sum_{\substack{j\le \ell_+\\ k> \ell_+}} + \sum_{\substack{j> \ell_+\\ k\le \ell_+}} + \sum_{\substack{j> \ell_+\\ k> \ell_+}} \right) 2^{(j+k)s} \|\mathcal{F}^{-1}[(\psi_j(2^\ell\cdot)\otimes\psi_k(2^\ell\cdot))\widehat{f}]\|_{L^2}.
\end{aligned}$$

We only treat the second sum, because the other sums can be handled in the same way. Since $\psi_k = \psi(\cdot/2^k)$, $k \ge 1$, we have

$$\begin{aligned}
&\sum_{k\ge \ell_++1} 2^{(j+k)s} \|\mathcal{F}^{-1}[(\psi_j(2^\ell\cdot)\otimes\psi_k(2^\ell\cdot))\widehat{f}]\|_{L^2}\\
&= \sum_{k\ge \ell_++1} 2^{(j+k)s} \|\mathcal{F}^{-1}[(\psi_j(2^\ell\cdot)\otimes\psi_{k-\ell})\widehat{f}]\|_{L^2}\\
&= \sum_{k\ge \ell_+-\ell+1} 2^{(j+k+\ell)s} \|\mathcal{F}^{-1}[(\psi_j(2^\ell\cdot)\otimes\psi_k)\widehat{f}]\|_{L^2}\\
&\le \max\{1, 2^{\ell s}\} \sum_{k\ge 0} 2^{(j+k)s} \|\mathcal{F}^{-1}[(\psi_j(2^\ell\cdot)\otimes\psi_k)\widehat{f}]\|_{L^2}.
\end{aligned}$$

By the support property $\operatorname{supp}\psi_j(2^\ell\cdot) \subset \{\xi_1 : |\xi_1| \le 2^{j-\ell+1}\}$, if $\ell \ge 0$, then

$$\begin{aligned}
&\sum_{j\le\ell_+}\left(\max\{1,2^{\ell s}\}\sum_{k\ge0}2^{(j+k)s}\|\mathcal{F}^{-1}[(\psi_j(2^\ell\cdot)\otimes\psi_k)\widehat{f}]\|_{L^2}\right)\\
&=2^{\ell s}\sum_{k\ge0}2^{ks}\left(\sum_{j\le\ell}2^{js}\|\mathcal{F}^{-1}[(\psi_j(2^\ell\cdot)(\psi_0+\psi_1)\otimes\psi_k)\widehat{f}]\|_{L^2}\right)\\
&\lesssim 2^{\ell s}\sum_{k\ge0}2^{ks}\left\{\sum_{j\le\ell}2^{js}\left(\|\mathcal{F}^{-1}[(\psi_0\otimes\psi_k)\widehat{f}]\|_{L^2}+\|\mathcal{F}^{-1}[(\psi_1\otimes\psi_k)\widehat{f}]\|_{L^2}\right)\right\}\\
&\lesssim \left(2^{\ell s}\right)^2\sum_{j,k\ge0}2^{(j+k)s}\left(\|\mathcal{F}^{-1}[(\psi_j\otimes\psi_k)\widehat{f}]\|_{L^2}\right).
\end{aligned}$$

If $\ell < 0$, then

$$\begin{aligned}
&\sum_{j\le\ell_+}\left(\max\{1,2^{\ell s}\}\sum_{k\ge0}2^{(j+k)s}\|\mathcal{F}^{-1}[(\psi_j(2^\ell\cdot)\otimes\psi_k)\widehat{f}]\|_{L^2}\right)\\
&=\sum_{k\ge0}2^{ks}\|\mathcal{F}^{-1}[(\psi_0(2^\ell\cdot)\otimes\psi_k)\widehat{f}]\|_{L^2}\le\sum_{j,k\ge0}2^{ks}\|\mathcal{F}^{-1}[(\psi_0(2^\ell\cdot)\psi_j\otimes\psi_k)\widehat{f}]\|_{L^2}\\
&\lesssim\sum_{j,k\ge0}2^{ks}\|\mathcal{F}^{-1}[\psi_j\otimes\psi_k)\widehat{f}]\|_{L^2}\le\sum_{j,k\ge0}2^{(j+k)s}\|\mathcal{F}^{-1}[\psi_j\otimes\psi_k)\widehat{f}]\|_{L^2}.
\end{aligned}$$

Hence, combining these estimates, we have the desired estimate.

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