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THE CENTRALIZER OF C^r-GENERIC DIFFEOMORPHISMS AT HYPERBOLIC BASIC SETS IS TRIVIAL

JORGE ROCHA AND PAULO VARANDAS

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Abstract. In the late nineties, Smale proposed a list of problems for the next century, and, among these, it was conjectured that for every $r \geq 1$ a C^r-generic diffeomorphism has trivial centralizer. Our contribution here is to prove the triviality of C^r-centralizers on hyperbolic basic sets. In particular, C^r-generic transitive Anosov diffeomorphisms have a trivial C^1-centralizer. These results follow from a more general criterium for expansive homeomorphisms with the gluing orbit property. We also construct a linear Anosov diffeomorphism on $\mathbb{T}^3$ with discrete, non-trivial centralizer and with elements that are not roots. Finally, we prove that all elements in the centralizer of an Anosov diffeomorphism preserve some of its maximal entropy measures, and use this to characterize the centralizer of linear Anosov diffeomorphisms on tori.

1. Introduction and statement of the main results

Introduction. In the late nineties Smale proposed a list of problems for the 21^{st} century, and, among them, it is asked how typical are diffeomorphisms with trivial centralizer [30]? The centralizer $\mathcal{Z}^r(f)$ of a C^r-diffeomorphism f, defined as the set of C^r-diffeomorphisms that commute with f, contains much information about possible symmetries of the dynamics. For instance, it may be used to determine when a diffeomorphism embeds as a time-1 map of a flow, as done by Palis [18], or to study the existence of smooth conjugacies for topologically conjugate circle diffeomorphisms, a problem initiated by Herman in [13]. Thus, the problems and conjectures established by Smale have a strong motivation from the interplay between dynamical systems and the algebraic properties of the linear models for hyperbolic automorphisms. Important contributions to the conjecture have been given since the seventies. In [32], Walters proved that the C^0-centralizer of expansive dynamics is discrete. Anderson [1] proved that for a C^∞-open and dense set of Morse-Smale diffeomorphisms the centralizer is also discrete. Kopell [14] studied the triviality of the centralizer of C^r circle diffeomorphisms ($r \geq 2$) and linear transformations. Palis and Yoccoz proved Smale's conjecture for Axiom A diffeomorphisms with the strong transversality condition in the C^∞ topology [20, 21]. In

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the real analytic setting, the first author proved that diffeomorphisms with trivial centralizer contain a residual subset of C^ω Axiom A diffeomorphisms with the strong transversality condition [24]. More recently, Bonatti, Crovisier and Wilkinson [7] proved that C^1-generic diffeomorphisms have trivial centralizer, providing a solution of Smale's conjecture on the space of C^1-diffeomorphisms. In that paper, the authors introduced a notion of unbounded distortion and show that it is C^1-generic and use it to prove that C^1-generic diffeomorphisms have trivial centralizer. We also refer to [8, 15] and references therein for some partial answers to Smale's conjecture in the context of hyperbolic flows and singular-hyperbolic flows.

The problem of the triviality of the centralizer in more regular topologies is still very much an open question. A classical strategy for obtaining discreteness for C^r-diffeomorphisms ($r > 1$) is to use linearization at some hyperbolic periodic orbit and to reduce the problem of the centralizer back to the analysis of algebraic ingredients. The drawback of this strategy is that C^r-linearization is only guaranteed once non-resonance conditions are satisfied by the eigenvalues of the derivative at that periodic point and provided the diffeomorphism is sufficiently smooth (the required smoothness is given explicitly from the non-resonance conditions of the eigenvalues and, for that reason, C^∞-smoothness is assumed). Indeed, there exists a C^∞ open and dense subset of Anosov diffeomorphisms on tori and a C^∞ open and dense subset of Axiom A diffeomorphisms with strong transversality which have trivial centralizer (see [20, 21] respectively). The previous results were extended by the first author in the case of surfaces, and if $2 \leq r \leq \infty$, there exists a C^r-open and dense subset of Axiom A diffeomorphisms with the no cycles condition whose elements have trivial centralizer [11]. Later, Fisher [12] proved that the elements of a C^r ($2 \leq r \leq \infty$) open and dense set of diffeomorphisms that exhibit a codimension-one hyperbolic and non-Anosov attractor have trivial centralizer on their basin of attraction. Hence, the triviality of the centralizer at hyperbolic basic pieces is well known both in the case of C^∞ diffeomorphisms and in the case of codimension-one hyperbolic attractors.

Our main purpose in this paper is to contribute to the description of the centralizer of C^r-diffeomorphisms at hyperbolic basic pieces ($r \geq 1$) removing both the C^∞-smoothness and codimension-one assumptions, establishing the triviality of the centralizer in the case of C^r-generic diffeomorphisms that exhibit non-trivial hyperbolic basic pieces ($2 \leq r < \infty$). First, we prove a criterium for an element in the C^0-centralizer of expansive homeomorphisms with the periodic gluing orbit property to be a power of it (see Theorem A below). Then, we prove that the assumptions in the criterium are satisfied by C^1 diffeomorphisms in the centralizer of C^r-generic diffeomorphisms and, consequently, C^r-generic hyperbolic basic pieces (including transitive Anosov diffeomorphisms) have trivial centralizer (cf. Corollaries A and B). As Anosov diffeomorphisms may have non-trivial centralizers and may have positive entropy (even Anosov) diffeomorphisms in their centralizer, then it is important to provide a general characterization for elements in the centralizer of Anosov diffeomorphisms. For that purpose, we prove in Theorem B that if the measure theoretical entropy function of a finite entropy homeomorphism is lower semi-continuous, then all elements of the centralizer preserve some of its measures of maximal entropy. In consequence, the centralizer of Anosov automorphisms on tori is formed by volume preserving diffeomorphisms (Corollary D) and any C^2-partially hyperbolic diffeomorphism in its centralizer has positive entropy (cf. Corollary F). Finally, we finish this article with some questions.

Preliminaries. Let $f \in \text{Diff}^r(M)$, $r \geq 1$. Let $\text{Per}(f)$ denote the set of periodic points for f and $\Omega(f) \subset M$ denote the non-wandering set of f. An f-invariant set Λ is *transitive* if there exists $x \in \Lambda$ so that $\overline{\mathcal{O}_f(x)} := \overline{\{f^n(x) : n \in \mathbb{N}\}} = \Lambda$. We say that a compact f-invariant set $\Lambda \subset M$ is a *hyperbolic set* for f if it admits a hyperbolic splitting: there is a Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ and constants $C > 0$ and $\lambda \in (0,1)$ so that $\|Df^n(x) \mid_{E^s_x} \| \leq C\lambda^n$ and $\|(Df^n(x) \mid_{E^u_x})^{-1}\| \leq C\lambda^n$ for every $x \in \Lambda$ and $n \geq 1$. A periodic point $p \in \text{Per}(f)$ is hyperbolic if its orbit is a hyperbolic set. A compact f-invariant set Λ is called *locally maximal* if there exists an open neighborhood $U \subset M$ of Λ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$. An f-invariant set $\Lambda \subset M$ is called a *hyperbolic basic set* for f if it is a transitive locally maximal hyperbolic set.

We say that $f \in \text{Diff}^r(M)$, $r \geq 1$, is *Axiom A* if (i) $\overline{\text{Per}(f)} = \Omega(f)$ and (ii) $\Omega(f)$ is a uniformly hyperbolic set. Clearly all periodic points of Axiom A diffeomorphisms are hyperbolic. We say that f is an *Anosov diffeomorphism* if the whole manifold M is a hyperbolic set for f. One should mention that not all manifolds admit Anosov diffeomorphisms. It follows from the spectral decomposition theorem that for any Axiom A diffeomorphism f there are finitely many pairwise disjoint hyperbolic basic sets $(\Lambda_i)_{i=1,\dots,k}$ so that $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k$. We refer the reader e.g. to [29] for more details.

Given $f \in \text{Diff}^r(M)$, $r \in \mathbb{N} \cup \{\infty\}$ and $0 \leq k \leq r$, the *C^k-centralizer* for f is the subgroup of $\text{Diff}^k(M)$ defined as

$$Z^k(f) = \{g \in \text{Diff}^k(M) \colon g \circ f = f \circ g\},$$

where $\text{Diff}^0(M)$ stands for the space $\text{Homeo}(M)$ of homeomorphisms on M. For every $1 \leq k \leq n$, it is clear that $Z^k(f)$ is a subgroup of $(\text{Diff}^k(M), \circ)$ which always contains the subgroup $\{f^n \colon n \in \mathbb{Z}\}$. Clearly, $\mathcal{Z}^0(f) \supset \mathcal{Z}^1(f) \supset \mathcal{Z}^2(f) \supset \cdots \supset \mathcal{Z}^r(f) \supset \{f^n \colon n \in \mathbb{Z}\}$, and we say that $f \in \text{Diff}^r(M)$ has *trivial C^k-centralizer* if $Z^k(f) = \{f^n \colon n \in \mathbb{Z}\}$. For simplicity, will say that $f \in \text{Diff}^r(M)$ has *trivial centralizer* if its C^r-centralizer is trivial. Note that the C^0-triviality of the C^0-centralizer of $f \in \text{Diff}^r(M)$, $r \geq 1$, implies the triviality of $\mathcal{Z}^k(f)$ for all $1 \leq k \leq r$.

We now recall some ingredients from the thermodynamic formalism. Let $\mathcal{M}(M)$ denote the space of Borelian probability measures on M, endowed with the weak* topology. The space of f-invariant probability measures $\mathcal{M}_f(M) \subset \mathcal{M}(M)$ is a compact subset, and for any $\nu \in \mathcal{M}_f(M)$ the *entropy of* ν is defined as $h_\nu(f) = \sup\{h_\nu(f, \mathcal{P}) \colon \mathcal{P} \text{ is a finite partition in } M\}$, where

$$h_\nu(f, \mathcal{P}) = \inf_{n \geq 1} \frac{1}{n} H\Big(\bigvee_{j=0}^{n-1} f^{-j}(\mathcal{P})\Big) \quad \text{and} \quad H(\mathcal{P}) = \sum_{P \in \mathcal{P}} -\mu(P) \log \mu(P).$$

An f-invariant probability measure μ is called an *equilibrium state* for f with respect to a continuous potential $\phi : M \to \mathbb{R}$ if it attains the supremum in the variational principle for the topological pressure

$$P_{\text{top}}(f, \phi) = \sup_{\nu \in \mathcal{M}_f(M)} \Big\{ h_\nu(f) + \int \varphi \, d\nu \Big\}. \tag{1.1}$$

In the case that $\phi \equiv 0$ the previous expression becomes the variational principle for the topological entropy, and any measure that attains the supremum is called a *maximal entropy measure*. In the case that Λ is a hyperbolic basic set for f it is well known that for every Hölder continuous potential ϕ there exists a unique equilibrium state for $f\mid_\Lambda$ with respect to ϕ. We refer the reader to Bowen's monograph [6] for more details.

Statement of the main results. This section is devoted to the statement of our main results. Our first main result is inspired by [15] and provides a criterium for the triviality of the C^0-centralizer for expansive homeomorphisms on compact metric spaces. We need two preliminary notions that we now recall. Recall that a homeomorphism f is *expansive* if there exists $\varepsilon > 0$ so that for any $x, y \in M$ there exists $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > \varepsilon$. Any such constant ε is called an expansivity constant for f.

Given a compact metric space Λ and a homeomorphism $f \in \text{Homeo}(\Lambda)$, we say that f has the *periodic gluing orbit property* if for any $\varepsilon > 0$ there exists $K = K(\varepsilon) > 0$ such that for any points $x_0, x_1, \dots, x_k \in \Lambda$ and positive integers $n_0, n_1, \dots, n_k \geq 1$ there are positive integers $p_0, p_1, \dots, p_{k-1}, p_k \leq K(\varepsilon)$ and $x \in \Lambda$ so that $d(f^k(x), f^k(x_0)) < \varepsilon$ for all $1 \leq k \leq n_0$, $d(f^{k+\sum_{j=0}^{i-1}(p_j+n_j)}(x), f^k(x_i)) < \varepsilon$ for all $1 \leq k \leq n_i$ and $1 \leq i \leq k$, and $f^{\sum_{j=0}^{k}(p_j+n_j)}(x) = x$. This notion is weaker than specification and holds e.g. for minimal isometries on tori and transitive uniformly hyperbolic dynamics. In fact, the periodic gluing orbit property implies strong transitivity and denseness of periodic orbits. Moreover, this property holds for homeomorphisms with the periodic shadowing property on each chain recurrence class of the non-wandering set. We refer the reader to [3–5, 31] for more details.

Finally, given homeomorphisms $f, h : M \to M$, we say that *h preserves the periodic orbits of f* if $h(\mathcal{O}_f(x)) = \mathcal{O}_f(x)$ for every $x \in \text{Per}(f)$. In other words, for each $x \in \text{Per}(f)$ there exists $n(x) \in \mathbb{Z}$ so that $h(x) = f^{n(x)}(x)$. We are now in a position to state our first main result.

Theorem A. *Let Λ be a compact metric space and let $f : \Lambda \to \Lambda$ be an expansive homeomorphism. Assume that f satisfies the periodic gluing orbit property. If $h \in \mathcal{Z}^0(f)$ preserves periodic orbits of f, then $h = f^k$ for some $k \in \mathbb{Z}$.*

In what follows we deduce some consequences. The first one, which is a consequence of Theorem A together with Lemma 3.1, shows that the centralizer at hyperbolic basic pieces is typically trivial.

Corollary A. *Let $1 \leq r \leq \infty$, $f_0 \in \mathit{Diff}^r(M)$ and $\Lambda_{f_0} \subset M$ be a hyperbolic basic set for f_0. Let $\mathcal{U} \subset \mathit{Diff}^r(M)$ be an open neighborhood of f_0 and let $U \subset M$ be an open set such that the analytic continuation of the hyperbolic basic set defined by*

$$\mathcal{U} \ni f \mapsto \Lambda_f := \bigcap_{n\in\mathbb{Z}} f^n(U) \tag{1.2}$$

is well defined. There exists a C^r-open neighborhood $\mathcal{U}$ of f and a residual subset $\mathcal{R} \subset \mathcal{U}$ so that $\mathcal{Z}^1(g\mid_{\Lambda_g})$ is trivial for every $g \in \mathcal{R}$.

It follows from the previous corollary that C^r-generic Axiom A diffeomorphisms, restricted to their non-wandering set, have trivial centralizer. One should also note that the first author constructed an open set of transitive Anosov diffeomorphisms on $\mathbb{T}^2$ with a non-trivial C^0-centralizer [26], and consequently Corollary A is no longer true when considering the C^0-centralizer. Thus, there exists an element in the centralizer that permutes periodic orbits (of the same period). In what follows we are also interested in the C^r-centralizer of Anosov diffeomorphisms, $r \geq 1$. By structural stability, the set $\mathcal{A}^r(M)$ of transitive Anosov diffeomorphisms is an open set in $\text{Diff}^r(M)$, $r \geq 1$. Among the classes of transitive diffeomorphisms we should refer to the set $\mathcal{A}^r_m(M)$ of volume preserving Anosov diffeomorphisms.

Corollary B. *Let M be a compact Riemannian manifold that supports Anosov diffeomorphisms and $1 \leq r \leq \infty$. There are C^r-residual subsets $\mathcal{R}_1 \subset \mathcal{A}^r(M)$ and $\mathcal{R}_2 \subset \mathcal{A}^r_m(M)$ of Anosov diffeomorphisms so that the C^1-centralizer of every $f \in \mathcal{R}_i$ is trivial, $i = 1, 2$.*

We observe that the previous result implies an explosion of differentiability. Indeed, given $1 \leq r \leq \infty$ any C^1 diffeomorphism that commutes with a C^r-generic Anosov diffeomorphism is itself C^r-smooth. As previously mentioned, whenever $r = 1$ and $r = \infty$ the previous result is a direct consequence of [7] and [20], respectively. For intermediate regularity $1 < r < \infty$, in order to use the criterium established by Theorem A we avoid a countable number of conditions (each of which determines a closed set with empty interior in $\text{Diff}^r(M)$). We now observe that the centralizer of a linear Anosov diffeomorphism may have rationally independent Anosov diffeomorphisms. Indeed, there exists a linear Anosov automorphism f on $\mathbb{T}^3$ whose C^∞-centralizer is $Z(f) \simeq \mathbb{Z}^2 \times \mathbb{Z}_2$ (hence it is discrete and non-trivial) and contains an Anosov diffeomorphism h that does not satisfy any of the equations $h^n = f^m$ for $m, n \in \mathbb{Z} \setminus \{0\}$ (see e.g. [22]). Such an example does not belong to the residual subsets described in Corollary A.

The main purpose now is to describe the elements in the centralizer of *every* Anosov diffeomorphism on tori. In what follows we discuss the entropy of common invariant measures.

Theorem B. *Let M be a compact metric space and assume that $f \in Homeo(M)$ have finite topological entropy. If the entropy map $h : \mathcal{M}_f(M) \to \mathbb{R}_+$ given by $\mu \mapsto h_\mu(f)$ is upper semicontinuous, then every $g \in \mathcal{Z}^0(f)$ preserves a maximal entropy measure of f. In addition, if $\mathcal{Z}^0(f)$ is finitely generated, then there exists a maximal entropy measure that is preserved by* all *elements in $\mathcal{Z}^0(f)$.*

It is well known that the measure theoretical entropy function of expansive homeomorphisms, including Anosov diffeomorphisms, is upper semicontinuous (see e.g. [33]). Thus the following is a direct consequence of Theorem B:

Corollary C. *Assume that $f \in Diff^1(M)$ is a transitive Anosov diffeomorphism. Then every $g \in Z^0(f)$ preserves all the equilibrium states of f associated to Hölder continuous potentials. In particular, the unique maximal entropy measure for f is preserved by all $g \in \mathcal{Z}^0(f)$ and, consequently, $h_{\text{top}}(f) = \sup_{\mu \in \mathcal{M}_f(M) \cap \mathcal{M}_1(g)} h_\mu(f)$.*

The previous result should be compared with [28]: if $f, g \in \text{Homeo}(M)$ are commuting homeomorphisms, $h_{\text{top}}(f) > 0$ and g is expansive, then $h_{\text{top}}(g) > 0$. In the case of linear Anosov automorphisms the maximal entropy measure coincides with the Lebesgue measure. Therefore, we deduce that all diffeomorphisms commuting with volume preserving Anosov diffeomorphisms are themselves volume preserving. More precisely:

Corollary D. *Assume that $f = f_A$ is a linear Anosov automorphism on $\mathbb{T}^n$. Then, for every $r \geq 1$,*

$$\mathcal{Z}^r_m(f) := \mathcal{Z}^r(f) \cap Diff^r_m(M) = \mathcal{Z}^r(f).$$

Every Anosov diffeomorphism f on $\mathbb{T}^n$ is topologically conjugated to a linear Anosov automorphism f_A induced by a hyperbolic matrix $A \in GL(n,\mathbb{Z})$ with $|\det A| = 1$ (cf. [17]). Clearly f_A is volume preserving and the Lebesgue measure is the unique maximal entropy measure. Moreover, it is not hard to check that the C^0-centralizer for f is homeomorphic to the centralizer of f_A (see e.g. [27, Theorem 2]). Now, we relate our results to the ones due to Katok [16] on the centralizer of diffeomorphisms preserving an invariant measure. For every $n \geq 2$ there exists a unique full supported maximal entropy measure for an Anosov diffeomorphism f on $\mathbb{T}^n$. Hence, the following is a direct consequence of Corollary 3.1 and Theorem 4.1 in [16]:

Corollary E. *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a linear Anosov diffeomorphism, $n \geq 2$. The following hold:*

(1) *if $n = 2$ and $g \in \mathcal{Z}^1(f)$, then there are $k, \ell \in \mathbb{Z}$ so that $f^k g^\ell = id$; and*
(2) *if $n = 3$ and $g, h \in \mathcal{Z}^1(f)$, then there are $k, \ell, m \in \mathbb{Z}$ so that $f^k g^\ell h^m = id$.*

Relations of the form $f^k g^\ell h^m = id$ are often associated to the existence of *roots* on the centralizer (e.g. if $g^2 = f$, then g is a root of f, and if $h^5 = id$, then h is a root of the identity), and to remove them constitutes an important step for establishing the trivial centralizer (see for instance [20]). Plykin's example (see [22]) implies that Corollary E (1) above is no longer true for Anosov diffeomorphisms on $\mathbb{T}^3$. It also implies that considering two elements in the centralizer at Corollary E (2) is optimal: there exist commuting Anosov C^∞ diffeomorphisms f and g such that $f^k g^\ell \neq id$ for all $k, \ell \in \mathbb{Z}$.

Finally, we discuss some rigidity phenomenon relating elements in the centralizer with their topological entropy. First we observe the following:

Proposition A. *If $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ is a linear Anosov diffeomorphism and $g \in \mathcal{Z}^0(f_A)$ has positive entropy, then g is a root of an Anosov diffeomorphism.*

Our next result concerns the entropy of partially hyperbolic diffeomorphisms in the centralizer of an Anosov diffeomorphism. We say that $g \in \text{Diff}^1(M)$ is *partially hyperbolic* if there exists a Dg-invariant splitting $TM = E^u \oplus E^c$ and constants $C > 0$ and $\lambda \in (0,1)$ so that $\|(Df^n(x) \mid_{E^u_x})^{-1}\| \leq C\lambda^n$ and $\|(Df^n(x) \mid_{E^u_x})^{-1}\| \, \|(Df^n(x) \mid_{E^c_x}\| \leq C\lambda^n$ for every $x \in M$ and every $n \geq 1$. While Corollary C asserts that measures of maximal entropy are preserved by any element in the centralizer of a transitive Anosov diffeomorphism, it is not clear how to compute their entropy. The following result shows that all partial hyperbolic diffeomorphisms commuting with an Anosov diffeomorphism have positive topological entropy. More precisely:

Corollary F. *If $1 \leq r \leq \infty$ and $f \in Diff^1(\mathbb{T}^n)$ is a linear Anosov diffeomorphism, then every $g \in \mathcal{Z}^2(f)$ is volume preserving and $h_{\text{top}}(g) \geq \int \sum_i \lambda_i(g,x)^+ \, dLeb(x)$ where $\sum_i \lambda_i(g,\cdot)^+$ denotes the sum of positive Lyapunov exponents of g with respect to Lebesgue. In particular, if $g \in \mathcal{Z}^2(f)$ is partially hyperbolic, then $h_{\text{top}}(g) > 0$.*

2. C^0-trivial centralizers

In what follows we will prove Theorem A, whose proof is inspired by [15]. Let Λ be a compact metric space and let $f : \Lambda \to \Lambda$ be an expansive homeomorphism satisfying the periodic gluing orbit property and let $h \in \mathcal{Z}^0(f)$ be fixed. Assume throughout that $h \in \mathcal{Z}^0(f)$ preserves the periodic orbits of f.

Lemma 2.1. *For every $\varepsilon > 0$ there exists a periodic point $q \in Per(f)$ whose orbit is ε-dense in Λ. In particular $\overline{Per(f)} = \Lambda$.*

Proof. The proof is a direct consequence of the periodic gluing orbit property. Indeed, by compactness of Λ, for every $\varepsilon > 0$ there exists a periodic point that is ε-dense. Indeed, given $\varepsilon > 0$ pick any finite set $\{x_1, x_2, \dots, x_k\} \subset \Lambda$ that is $\varepsilon/2$-dense in Λ and takes $n_i = 1$ for every $1 \le i \le k$. The periodic gluing orbit property assures the existence of a positive integer $K = K(\frac{\varepsilon}{2}) \ge 1$ of positive integers $p_1, \dots, p_k \le K$ and of a periodic point q, of period at most $(1 + K)k$, so that $d(q, x_1) < \varepsilon$, $d(f^{\sum_{j=0}^{i-1}(p_j+1)}(q), x_i) < \varepsilon$ for all $2 \le i \le k$, and $f^{\sum_{j=0}^{k}(p_j+1)}(q) = q$. In consequence $\mathcal{O}_f(q)$ is ε-dense in Λ. Since $\varepsilon > 0$ was chosen arbitrary this also proves the second claim in the lemma. □

We proceed to prove that $h = f^k$ for some $k \in \mathbb{Z}$. As h preserves periodic orbits, given $p \in \mathrm{Per}(f)$ there exists a unique $n(p) \in \mathbb{Z} \cap [-\frac{\pi(p)}{2}, \frac{\pi(p)}{2}]$ so that $h(p) = f^{n(p)}(p)$. Throughout, let $n : \mathrm{Per}(f) \to \mathbb{Z}$ be as defined above and note that $n(q) = n(f^j(q))$ for every $q \in \mathrm{Per}(f)$ and $0 \le j \le \pi(q)$. We first prove that if $n(\cdot)$ is bounded, then the theorem follows.

Lemma 2.2. *If $n(\cdot) : Per(f) \to \mathbb{Z}$ is bounded, then there exists $k \in \mathbb{Z}$ so that $h = f^k$.*

Proof. Assume there exists $N_0 \ge 1$ so that $|n(p)| \le N_0$ for every $p \in \mathrm{Per}(f)$. Then it makes sense to consider the decomposition

$$\mathrm{Per}(f) = \bigsqcup_{|j| \le N_0} \big\{ p \in \mathrm{Per}(f \mid_\Lambda) \colon n(p) = j \big\}.$$

By Lemma 2.1, for every $\ell \ge 1$ there exists a periodic point $p_\ell \in \mathrm{Per}(f)$ that is $\frac{1}{\ell}$-dense in Λ. Using the previous decomposition on the space of periodic points and the pigeonhole principle, there exists $k \in \{-N_0, \dots, N_0\}$ so that the set $P_k := \big\{ p \in \mathrm{Per}(f) \colon n(p) = k \big\}$ contains infinitely many periodic points of the family $(p_\ell)_{\ell \ge 1}$. Hence P_k is dense in Λ and $h \mid_{P_k} = f^k$. This implies that $h = f^k$ and proves the lemma. □

The remainder of the proof is to assure that the hypothesis of the previous lemma is satisfied. We need the following estimate on $n(\cdot)$.

Lemma 2.3. *Given $p \in Per(f)$ of prime period $\pi(p) \ge 1$ there exists $\eta = \eta_p > 0$ (depending on p, f and h) so that for every $q \in Per(f) \cap B(p, \eta)$ either $n(q) = n(p)$ or $|n(q)| > \frac{\pi(p)}{2}$.*

Proof. Given $p \in \mathrm{Per}(f)$ pick $\zeta > 0$ small enough such that the collection of balls $\{B(f^j(p), \zeta)\}_{j=0,\dots,\pi(p)-1}$ is pairwise disjoint and

$$f^k(B(f^j(p), \zeta)) \cap \Big(\bigcup_{s \ne j+k} B(f^s(p), \zeta) \Big) = \emptyset \tag{2.1}$$

for every $j, k \in \{0, \dots, \pi(p) - 1\}$. Clearly, the homeomorphism $\tilde{h} := h \circ f^{-n(p)}$ belongs to $\mathcal{Z}^0(f)$ and $\tilde{h}(p) = p$. Using that $\tilde{h}$ is uniformly continuous, there exists $0 < \eta < \zeta/2$ so that $d(q, p) < \eta$ implies $d(\tilde{h}(q), \tilde{h}(p)) = d(f^{n(q)-n(p)}(q), p) < \frac{\zeta}{2}$. Since q and $\tilde{h}(q) = f^{n(q)-n(p)}(q)$ belong to $B(p, \eta)$ and h preserves the periodic orbits of f (hence the same holds for $\tilde{h}$) we conclude that either $\tilde{h}(q) = q$ or

$\tilde{h}(q) \in \mathcal{O}_f(q) \setminus \{q\}$. These correspond to the cases that $n(q) = n(p)$ or, using (2.1), that $|n(q) - n(p)| \geq \pi(p)$, respectively. This proves the dichotomy that $n(q) = n(p)$ or $|n(q)| \geq \pi(p)/2$ as claimed in the lemma. □

Theorem A will follow as a consequence of Lemma 2.2 together with the following proposition.

Proposition 2.1. *$n(\cdot) : Per(f) \to \mathbb{Z}$ is bounded.*

Proof. Assume by contradiction that $n(\cdot) : \mathrm{Per}(f) \to \mathbb{Z}$ is unbounded. As f is expansive let $\varepsilon > 0$ be an expansivity constant for f. Given $p_1, p_2 \in \mathrm{Per}(f)$ we have that $h(\mathcal{O}_f(p_i)) = \mathcal{O}_f(p_i)$ $(i = 1, 2)$. Hence, diminishing $\varepsilon > 0$ if necessary, we can assume the ε-neighborhood V_i of the orbit $\mathcal{O}_f(p_i)$ $(i = 1, 2)$ to satisfy

$$\mathrm{dist}_H(V_1 \cup h(V_1) \cup h^{-1}(V_1), V_2 \cup h(V_2) \cup h^{-1}(V_2)) > 0, \tag{2.2}$$

where dist_H denotes the Hausdorff distance. Let $K(\varepsilon) > 0$ be given by the periodic gluing orbit property.

Let $p_3 \in \mathrm{Per}(f)$ such that $|n(p_3)| > 2K(\varepsilon)$; such a point does exist because $n(\cdot)$ is unbounded. By definition of $n(\cdot)$ we get that $2K(\varepsilon) \leq |n(p_3)| \leq \frac{\pi(p_3)}{2}$.

Since ε is assumed to be an expansivity constant for f we conclude that the diameter of the dynamic ball

$$B(p_3, k, \varepsilon) := \{x \in M : d(f^j(x), f^j(p_3)) < \varepsilon \text{ for every } -k \leq j \leq k\}$$

tends to zero as $k \to \infty$. Let $\eta = \eta_{p_3} > 0$ be given by Lemma 2.3 and $k_3 \geq 1$ be such that $\mathrm{diam}(B(p_3, k_3, \varepsilon)) < \eta$. Given $k \geq 1$ arbitrary, the periodic gluing orbit property assures the existence of a periodic point $p \in \mathrm{Per}(f)$ and times $0 \leq t_1, t_2, t_3 \leq K(\varepsilon)$ so that:

(1) $d(f^j(p), f^j(p_1)) \leq \varepsilon$ for every $0 \leq j \leq k\pi(p_1)$,
(2) $d(f^{j+t_1+k\pi(p_1)}(p), f^j(p_2)) \leq \varepsilon$ for every $0 \leq j \leq \pi(p_2)$,
(3) $d(f^{j+t_2+\pi(p_2)+t_1+k\pi(p_1)}(p), f^j(p_3)) \leq \varepsilon$ for every $0 \leq j \leq k_3\pi(p_3)$, and
(4) $\pi(p) = t_3 + \pi(p_3) + t_2 + \pi(p_2) + t_1 + k\pi(p_1)$.

In particular, one can choose $k \gg 1$ so that

$$k\pi(p_1) \geq \max\{3K(\varepsilon) + \pi(p_2) + k_3\pi(p_3), |n(p_3)|\}.$$

Since the orbit of p intersects the η-neighborhood of the orbit of p_3 and $n(f^j(p_3)) = n(p_3)$ for every $0 \leq j \leq \pi(p_3) - 1$, Lemma 2.3 assures that $|n(p)| > K(\varepsilon)$. By items (1) and (2), the point $z = f^{t_1+k\pi(p_1)}(p)$ belongs to V_2 and $f^{-j}(z) \in V_1$ for every $t_1 \leq j \leq k\pi(p_1)$. Moreover, by the choice of $k \geq 1$, we have that $f^{-|n(z)|}(z) = f^{-|n(p)|}(z) \in V_1$. As either $h(z) = f^{-|n(z)|}(z)$ or $h^{-1}(z) = f^{-|n(z)|}(z)$ this leads to a contradiction with (2.2). Thus $n(\cdot) : \mathrm{Per}(f) \to \mathbb{Z}$ is bounded, which proves the proposition. □

3. Trivial centralizers on hyperbolic basic pieces of C^r-generic diffeomorphisms

Using that hyperbolic basic pieces admit analytic continuations $g \mapsto \Lambda_g$, Corollaries A and B on the triviality of the centralizer of C^r-generic diffeomorphisms on hyperbolic basic pieces are consequences of Lemma 3.1 below. First we establish some notation. Given a C^1-diffeomorphism f on a compact Riemannian manifold

M and $p \in \mathrm{Per}(f)$, let $\sigma(Df^{\pi(p)}(p)) \subset \mathbb{C}$ denote the spectrum of the linear transformation $Df^{\pi(p)}(p) : T_pM \to T_pM$, where $\pi(p) \geq 1$ is the prime period of p. We need the following:

Lemma 3.1. *Let $1 \leq r \leq \infty$, $f_0 \in \mathit{Diff}^r(M)$ and $\Lambda_{f_0} \subset M$ be a locally maximal hyperbolic set for f_0. Let $\mathcal{U} \subset \mathit{Diff}^r(M)$ be an open neighborhood of f_0 and let $U \subset M$ be an open set such that the analytic continuation of the hyperbolic basic sets* (1.2) *is well defined. Then, there exists a C^r-residual subset $\mathcal{R} \subset \mathcal{U}$ such that for any $f \in \mathcal{R}$ the following holds: $Df^{\pi(p)}(p)$ is not linearly conjugated to $Df^{\pi(q)}(q)$ for any periodic points p, q so that $p \notin \mathcal{O}_f(q)$.*

Proof. The proof of this lemma is well known, and its proof relies on perturbations at periodic points similar to the ones in the Kupka-Smale theorem. For that reason we just sketch the underlying ideas, referring the reader e.g. to [19] for the details on these classical perturbations in the case of the Kupka-Smale theorem. Given $r \geq 1$ and $n \geq 1$ it is clear that the set $\mathcal{R}_n$ formed by all diffeomorphisms $f \in \mathcal{U}$ so that all periodic points of period smaller than or equal to n are hyperbolic and $\sigma(Df^{\pi(p)}(p)) \cap \sigma(Df^{\pi(q)}(q)) = \emptyset$ forms a C^r-open subset of $\mathcal{U}$. The proof that $\mathcal{R}_n$ is C^r-dense relies on a finite number of disjointly supported perturbations of the form

$$(1 + \chi\beta)\tilde{f} \tag{3.1}$$

in the neighborhood of periodic points of period smaller than or equal to n, where $\tilde{f}$ corresponds to f in local coordinates, β is a smooth bump function and $\chi > 0$ is small to guarantee that the perturbed diffeomorphism $g \in \mathcal{R}_n$ is indeed C^r-close to f. Hence $\mathcal{R} = \bigcap_{n \geq 1} \mathcal{R}_n$ is the desired residual subset of $\mathcal{U}$. □

Remark 3.1. Given $1 \leq r \leq \infty$, the classical methods used in the proof of the Kupka-Smale theorem for C^r volume preserving diffeomorphisms (see e.g. Robinson [23, Lemma 14]) yield that C^r-generically the eigenvalues of different periodic points have all distinct absolute values in the case that $\dim M \geq 3$. In the case that $\dim M = 2$ the previous strategy can be used to deduce a similar result among the classes using perturbations C^∞ close to the identity map obtained as a finite composition of small rotations instead of homothetic perturbations in (3.1).

3.1. **Proof of Corollary B.** Fix $1 \leq r \leq \infty$. We will prove the corollary in the case of Anosov diffeomorphisms in $\mathcal{A}^r(M)$ (the proof for volume preserving diffeomorphisms $\mathcal{A}^r_m(M)$ is completely analogous). Note that M is a hyperbolic basic piece for any Anosov diffeomorphism in $\mathcal{A}^r(M)$ (due to transitivity).

Let $\mathcal{D}$ be a countable C^r-dense set of Anosov diffeomorphisms in $\mathcal{A}^r(M)$. By structural stability, given $f \in \mathcal{D}$ the topological class of f contains a C^r open neighborhood $\mathcal{U}_f \subset \mathrm{Diff}^r(M)$ of f. By Lemma 3.1, there exists a C^r-residual subset $\mathcal{R}_f \subset \mathcal{U}_f$ of Anosov diffeomorphisms so that $Dg^{\pi(p)}(p)$ is not linearly conjugated to $Dg^{\pi(q)}(q)$ for any $g \in \mathcal{R}_f$ and any periodic points $p, q \in \mathrm{Per}(g)$ so that $p \notin \mathcal{O}_g(q)$.

We claim that every $g \in \mathcal{R}_f$ has trivial C^1-centralizer. Indeed, given $g \in \mathcal{R}_f$ and $h \in \mathcal{Z}^1(g)$ it is enough to show that h preserves periodic orbits (cf. Theorem A). But, using $h \circ g^n = g^n \circ h$ for every $n \in \mathbb{Z}$, if p is any periodic point of prime period $n = \pi(p)$, then $Dg^{\pi(p)}(h(p)) = Dh(p) \cdot Dg^{\pi(p)}(p) \cdot [Dh(p)]^{-1}$. This shows that $Dg^{\pi(p)}(h(p))$ and $Dg^{\pi(p)}(p)$ are linearly conjugated. By the construction of the residual subset $\mathcal{R}_f$ we conclude that h preserves the periodic orbits of g and, consequently, the C^1-centralizer of every $g \in \mathcal{R}_f$ is trivial. Then, $\mathcal{R} := \bigcup_{f \in \mathcal{D}} \mathcal{R}_f$

is a C^r-residual subset of $\mathcal{A}^r(M)$ that satisfies the requirements of the corollary. This proves Corollary B.

4. Centralizers of Anosov diffeomorphisms and positive entropy elements

4.1. **Proof of Theorem B.** Assume that the homeomorphism $f \in \text{Homeo}(M)$ has finite topological entropy and that the entropy map $\mu \mapsto h_\mu(f)$ is upper semicontinuous: given invariant measures so that $\lim_{n\to\infty} \mu_n = \mu$ (in the weak* topology) it holds that $\limsup_{n\to\infty} h_{\mu_n}(f) \le h_\mu(f)$. Under these assumptions f has at least one equilibrium state for every continuous potential ϕ (see e.g. [33]).

Fix an arbitrary $g \in \mathcal{Z}^0(f)$. As $g \circ f = f \circ g$ then $g_* \circ f_* = f_* \circ g_*$ where the push-forward dynamics $f_* : \mathcal{M}(M) \to \mathcal{M}(M)$ is given by $(f_*\eta)(A) = \eta(f^{-1}A)$ for every Borelian A in M (and g_* is defined similarly). Let μ be a maximal entropy measure for f. It is not hard to check from the commutativity of f and g that $g_*\mu$ is an f-invariant probability measure and that $h_{g_*\mu}(f) = h_\mu(f)$. The first is a direct consequence of the fact that $g_* \circ f_* = f_* \circ g_*$. The second follows as, for every finite partition $\mathcal{P}$ on M,

$$
\begin{aligned}
h_{g_*\mu}(f, \mathcal{P}) &= \inf_{n\ge 1} \frac{1}{n} H_{g_*\mu}\big(\bigvee_{\ell=0}^{n-1} f^{-\ell}(\mathcal{P})\big) = \inf_{n\ge 1} \frac{1}{n} H_\mu\big(g^{-1}\big(\bigvee_{\ell=0}^{n-1} f^{-\ell}(\mathcal{P})\big)\big) \\
&= \inf_{n\ge 1} \frac{1}{n} H_\mu\big(\bigvee_{\ell=0}^{n-1} f^{-\ell}(g^{-1}(\mathcal{P}))\big) = h_\mu(\, f, g^{-j}(\mathcal{P})\,).
\end{aligned}
$$

Since the partitions $\mathcal{P}$ (resp. $g^{-j}(\mathcal{P})$) can be taken with arbitrarily small diameter we conclude that $h_{g_*\mu}(f) = h_\mu(f)$, as claimed. Recursively, $h_{g_*^j\mu}(f) = h_\mu(f)$ for every $j \ge 0$. As the measure theoretic entropy function is affine we get that the f-invariant probability measures

$$
\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} g_*^j \mu \tag{4.1}
$$

lie in the closed simplex $\mathcal{S} := \{\eta \in \mathcal{M}_f(M) \colon h_\eta(f) = h_\mu(f)\}$ for every $n \ge 1$. Since both f_* and g_* are continuous, if μ_∞ is an accumulation point of the sequence $(\mu_n)_n$, then $\mu_\infty \in \mathcal{M}_f(M) \cap \mathcal{M}_1(g)$. Moreover, $h_{\mu_\infty}(f) \ge \limsup_{n\to\infty} h_{\mu_n}(f) = h_\mu(f)$ by the upper semicontinuity of the entropy map $\mathcal{M}_f(M) \ni \eta \mapsto h_\eta(f)$. Consequently μ_∞ is a maximal entropy measure for f that is preserved by both f and g; consequently $H_{\text{top}}(f) = \sup_{\mu\in\mathcal{M}_f(M)\cap\mathcal{M}_1(g)} h_\mu(f) = h_{\mu_\infty}(f)$. This proves the first statement in the theorem.

Finally, if $\mathcal{Z}^0(f)$ is finitely generated and $G = \{g_1, \dots, g_k\}$ is a set of generators, then the previous argument shows that every push-forward $(g_{i+1})_*$ preserves the simplex $\mathcal{S}_+ = \{\eta \in \mathcal{M}_f(M) \colon h_\eta(f) = h_{\text{top}}(f)\}$ for every $1 \le i \le k$. Then, if μ is a maximal entropy measure for f, then any accumulation point of the probability measures

$$
\frac{1}{n^k} \sum_{j_1=0}^{n-1} \cdots \sum_{j_k=0}^{n-1} (g_1)_*^{j_1} \dots (g_k)_*^{j_k} \mu
$$

is a maximal entropy measure for f that is preserved by all elements in $\mathcal{Z}^0(f)$. This finishes the proof of the theorem.

Remark 4.1. If $g \in \mathcal{Z}^0(f)$ and $\mathcal{E}_\phi \subset \mathcal{M}_f(M)$ denotes the space of equilibrium states for f with respect to a continuous potential ϕ, similar computations as above yield that any accumulation point μ_∞ of the sequence of invariant probability measures defined by the Cesaro averages (4.1) (taking $\mu \in \mathcal{E}_\phi$ instead of a maximal entropy measure) satisfies $h_{\mu_\infty}(f) + \int \phi \, d\mu_\infty \geq h_\mu(f) + \int \phi \, d\mu_\infty$, where $\int \phi \, d\mu_\infty = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \int \phi \circ g^j \, d\mu$ for some subsequence $(n_k)_k$. Therefore, one can only expect $g_*(\mathcal{E}_\phi) \subset \mathcal{E}_\phi$ in the very special case that g_* preserves the level set $\{\eta \in \mathcal{M}_f(M)\colon \int \phi \, d\eta = \int \phi \, d\mu\}$. This is true in the case that ϕ is constant when the equilibrium states are maximal entropy measures.

4.2. **Proof of Proposition A.** Assume that $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ is a linear Anosov diffeomorphism and $g \in \mathcal{Z}^0(f_A)$ has positive entropy. The proof makes use of this crucial fact: given the hyperbolic automorphism f_A on $\mathbb{T}^2$ every element in the C^0-centralizer of f_A that fixes the origin is linear [21, page 100]. Therefore, for any $g \in \mathcal{Z}^0(f_A)$ there exists $k \in \mathbb{N}$ and $B \in GL(2,\mathbb{Z})$ with $|\det B| = 1$ so that $g^k = f_B$. As f_B is C^∞ then the Pesin formula implies $h_{\text{top}}(f_B) = \sum_i \lambda_i^+(f_B)$ where $\lambda_i^+(f_B)$ denotes the positive Lyapunov exponents of f_B with respect to Lebesgue. Thus $h_{\text{top}}(f_B) = k \, h_{\text{top}}(g) > 0$ if and only if there exists at least one eigenvalue of B with absolute value larger than one. Since f_B is volume preserving the latter implies that $B \in SL(2,\mathbb{Z})$ is a hyperbolic matrix and, equivalently, f_B is an Anosov automorphism. This proves the proposition.

Remark 4.2. The centralizer of linear Anosov diffeomorphisms on $\mathbb{T}^2$ can indeed contain roots (e.g. $f_B \in Z(f_A)$ where $A, B \in SL(2,\mathbb{Z})$ are given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and verify $B^2 = A$). Nevertheless, the existence of roots in the centralizer is rare for more regular diffeomorphisms. Indeed, for every $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$ there exists a C^k-open and dense set of positive entropy real analytic diffeomorphisms on surfaces that have trivial centralizer [25].

5. Further questions

Given a C^r-diffeomorphism $f \in \text{Diff}^r(M)$ and $0 \leq k \leq r$, the centralizer $Z^k(f)$ is a subgroup of $\text{Diff}^k(M)$. In the case where there exists an open subset $\mathcal{U} \subset \text{Diff}^r(M)$ so that $Z^k(f)$ is finitely generated for every $f \in \mathcal{U}$ it makes sense to study the continuity points of the map $\mathcal{U} \ni f \mapsto N^k(f)$ where $N^k(f) \in \mathbb{N} \cup \{\infty\}$ denotes the minimum number of generators for $Z_0^k(f)$. Some results can be deduced in the case $r = k = 1$. Since C^1-generically in $\text{Diff}^1(M)$ the centralizer is trivial (cf. [7]) there exists a residual subset $\mathcal{R} \subset \text{Diff}^1(M)$ so that $\mathcal{R} \ni f \mapsto N^1(f)$ is constant and equal to one. By [2], there exists an open subset of C^1-Anosov diffeomorphisms with trivial centralizer, hence formed by continuity points for the function $f \mapsto N^1(f)$. The following questions remain open:

1. *Given $r \geq 1$, does the set of C^r-Anosov diffeomorphisms (resp. C^r Axiom A diffeomorphisms with the no cycles condition) that have trivial centralizer form a C^r open and dense set on the space of the space of C^r-Anosov diffeomorphisms (resp. C^r-Axiom A diffeomorphisms with the no cycles condition)?*

2. *If $1 \leq k \leq r$, what are the continuity points of the functions N^k? Are all Anosov diffeomorphisms points of (semi)continuity? Are there Anosov diffeomorphisms whose centralizers are not finitely generated?*

Related to the last question, one should mention that the centralizer of every Anosov diffeomorphism on $\mathbb{T}^n$ is finitely generated (cf. [10]). Moreover, not all Anosov diffeomorphisms are continuity points of the functions N^k, even if we restrict to linear hyperbolic diffeomorphisms. It is definitely interesting to discuss the triviality of the centralizer with the set of entropies associated to each element of the centralizer. Indeed, observe that the triviality of the centralizer $Z^0(f)$ seldom implies that

$$H_f := \{h_{\text{top}}(g) : g \in \mathcal{Z}^0(f)\} \tag{5.1}$$

coincides with the arithmetic progression $\{nh_{\text{top}}(f) : n \in \mathbb{N}_0\}$. However, it is not true that this condition implies the triviality of the centralizer, as can be easily observed in the examples constructed by the first author in [26]. The following question is suggested by this class of examples:

3. *Assume that $f : \mathbb{T}^n \to \mathbb{T}^n$ is a C^1-Anosov diffeomorphism and that $H_f = \{nh_{\text{top}}(f) : n \in \mathbb{N}_0\}$. Is it true that all elements in the centralizer are either of the form f^k, $k \in \mathbb{Z}$, roots of the identity or compositions of these?*

Finally, we believe that positive entropy elements in the centralizer of an Anosov diffeomorphism should have some rigidity condition. So, we ask:

4. *Let f be an Anosov diffeomorphism on $\mathbb{T}^n$ $(n \geq 2)$. Are all positive entropy elements in the C^1-centralizer of f partially hyperbolic?*

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Departamento de Matemática, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal

E-mail address: jrocha@fc.up.pt

Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil

E-mail address: paulo.varandas@ufba.br