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THE LINK VOLUME OF 3-MANIFOLDS IS NOT MULTIPLICATIVE UNDER COVERINGS

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Abstract. We obtain an infinite family of 3−manifolds $\{M_n\}_{n\in\mathbb{N}}$ and an infinite family of coverings $\{\varphi_n : \tilde{M}_n \to M_n\}_{n\in\mathbb{N}}$ with covering degrees unbounded and satisfying that $\mathrm{LinkVol}[\tilde{M}] = \mathrm{LinkVol}[M]$. This shows that link volume of 3-manifolds is not well behaved under covering maps, in particular, it is not multiplicative, and gives a negative answer to a question posed in a work of Rieck and Yamashita, namely, how good is the bound $\mathrm{LinkVol}[\tilde{M}] \leq q\mathrm{LinkVol}[M]$, when $\tilde{M}$ is a q-fold covering of M?

1. Introduction

It is well known that each orientable closed 3-manifold M is a covering of the 3-sphere, S^3, branched along some link $L \subset S^3$ (see [1], [2]). We write $\varphi : M \xrightarrow{t:1} (S^3, L)$ to denote a t-fold covering map $\varphi : M \to S^3$ branched along the link $L \subset S^3$.

If $L \subset S^3$ is a hyperbolic link and $\mathcal{N}(L)$ is a closed regular neighbourhood of L, then $C_L := S^3 - \mathcal{N}(L)$ admits a unique Riemannian metric of curvature -1 which induces a volume form on C_L. Integrating this form we obtain the *volume* of C_L, denoted by $\mathrm{Vol}(C_L)$. Moreover, if $\varphi : M \xrightarrow{t:1} (S^3, L)$ is a t-fold covering of S^3 branched along L, we can pull-back the Riemaniann metric on C_L to induce a unique Riemannian metric of curvature -1 in $M-\varphi^{-1}(\mathcal{N}(L))$ and $\mathrm{Vol}(M-\varphi^{-1}(\mathcal{N}(L))) = t\,\mathrm{Vol}(S^3 - \mathcal{N}(L))$. If L is a hyperbolic link, the *complexity* of $\varphi : M \xrightarrow{t:1} (S^3, L)$ is the number $t\mathrm{Vol}(S^3 - L)$; that is, it is the degree of the cover times the volume of the complement of the branching.

On the other hand, an infinite number of hyperbolic links have been shown to be universal. A link $L \subset S^3$ is *universal*, if any orientable closed 3-manifold M can be obtained as a cover of S^3 with branching L (see, for example, [9], [3] and [4]). As a consequence, given a closed 3-manifold M, it covers S^3 in many different ways (with different degrees) and the branching set of those coverings are hyperbolic links L. So for each closed 3-manifold M we have an infinite set of complexities $\{t\,\mathrm{Vol}(S^3 - \mathcal{N}(L))\}$.

Y. Rieck and Y. Yamashita ([8]) defined the link volume of a closed orientable 3-manifold M, $\mathrm{LinkVol}(M)$, as the infimum of the complexities of all possible coverings $\varphi : M \xrightarrow{t:1} (S^3, L)$ (of all possible degrees); it is a measure of how efficient is the branched covering representation of a 3-manifold M. They also proved that

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many of the basic properties of link volume are similar to the corresponding properties of the hyperbolic volume, and noted that, if $\psi : \tilde{M} \to M$ is a q-fold covering of M, then $\mathrm{LinkVol}(\tilde{M}) \leq q\mathrm{LinkVol}(M)$ for, $\mathrm{LinkVol}(M)$ can be realized by some $M \xrightarrow{t:1} (S^3, L_M)$, where L_M is a hyperbolic link in S^3. They also asked how good this upper bound is, but also suggested that even in the case $q = 2$, a possible answer of 'goodness' might not be simple. In this paper, we show that that bound is not good. Our main result is the following:

Theorem 1.1. *Let k, p be a pair of positive integers, and set $n = k(4p+1) - p$. Then the prism manifold*

$$M_n = (Oo, 0; 1/2, -1/2, -2/(4n-1)),$$

is covered by another prism manifold

$$M_k = (Oo, 0; 1/2, -1/2, -2/(4k-1)).$$

The number of sheets of the covering $M_k \to M_n$ is $4p+1$.

In [7], it is proved that $\mathrm{LinkVol}(M_n) = 2 \cdot (3.666\ldots)$, for all but finitely many n. Combining this result with Theorem 1.1 we obtain:

Corollary 1.2. *There exists an infinite collection of coverings $\psi : \tilde{M} \to M$ of 3-manifolds such that $\mathrm{LinkVol}(\tilde{M}) = \mathrm{LinkVol}(M)$. Also, the number of sheets of the covering can be chosen to be very large and is unbounded.*

Proof. In the context of the previous theorem, let p be a positive integer and set $k = p$. Then there exists an infinite family of coverings $\varphi_k : M_k \to M_n$, with $n = 4k^2$. Since $\mathrm{LinkVol}(M_k) = \mathrm{LinkVol}(M_n) = 2(3.66\ldots)$, for all but finitely many $k \in \mathbb{N}$ or finitely many $n \in \mathbb{N}$, and also it is clear that the number of sheets, $t = 4k+1$, of the coverings is very large and unbounded, for p is arbitrary, the corollary follows. $\square$

This result shows that the bound $\mathrm{LinkVol}(\tilde{M}) \leq q\mathrm{LinkVol}(M)$ is not good at all, for, as in the corollary, sometimes $q\mathrm{LinkVol}(M)$ is much larger than $\mathrm{LinkVol}(\tilde{M})$; and also we see that the link volume does not behave well under coverings.

It would be interesting to find out if the inequality above is sharp. That is, if there are covering spaces $\psi : \tilde{M} \to M$ such that $\mathrm{LinkVol}(\tilde{M}) = q\mathrm{LinkVol}(M)$.

In Section 2 we establish the background material. In Section 3, we prove Lemma 3.1 which computes the Seifert symbol of $\tilde{M}$, the covering of a Seifert manifold $M = \left(Oo, 0; \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3}\right)$ given by a representation $\omega : \pi_1(M) \to S_t$ that sends a regular fiber h of M into the identity permutation. Finally, in Section 4, we prove Theorem 1.1.

2. Preliminaries

Throughout this work, unless explicitly stated, all 3-manifolds and surfaces will be assumed to be orientable, connected and compact.

2.1. **Branched coverings.** Let N be an m-manifold. A finite-to-one open map $f : \tilde{N} \to N$ is a *branched covering of* N if there exists a subcomplex B_f (usually a submanifold) of N of codimension two such that $f| : \tilde{N} - f^{-1}(B_f) \to N - B_f$ is a t-fold covering space. This subcomplex B_f is called *the branch set of* f and $f^{-1}(B_f)$ is called *the singular set of* f. It is well known that a t-fold covering

$f : \tilde{N} \to N$ with branch set B_f determines and it is determined by a homomorphism $\omega : \pi_1(N - B_f, x_0) \to S_t$, where S_t denotes the symmetric group on t symbols. This homomorphism is defined as follows: Given a loop α at x_0, there exists a unique lift $\tilde{\alpha}_i$ which starts in $\tilde{x}_i \in \varphi^{-1}(x_0)$ and ends in some $\tilde{x}_j \in \varphi^{-1}(x_0)$; then $\omega([\alpha])$ is the permutation that sends i to j. Equivalence classes of t-fold branched coverings of (N, B) are in one-to-one correspondence with conjugacy classes of homomorphisms $\omega : \pi_1(N - B) \to S_t$.

2.2. **Seifert Fibered Spaces** $M = (Oo, g; \beta_1/\alpha_1, \dots, \beta_r/\alpha_r)$**.** Let $F_{g,r}$ denote the r-times punctured orientable surface of genus g and q_i be the boundary components of $F_{g,r}$, for $i = 1, \dots, r$. Let $M_0 = F_{g,r} \times S^1$. Then the boundary of M_0 consists of r tori $T_1, \dots, T_r$, where $T_i = q_i \times S^1$. Now glue r solid tori V_i to M_0 with homeomorphisms $f_i : \partial V_i \to T_i$ such that $f(m_i) \sim q_i^{\alpha_i} h^{\beta_i}$, where α_i and β_i are relatively prime integer numbers, for all $i = 1, \dots, r$. The manifold obtained in this way is the Seifert Fibered Space M with Seifert symbol $(Oo, g; \beta_1/\alpha_1, \dots, \beta_r/\alpha_r)$. If $h = \{x\} \times S^1$ is a *fiber* of M_0, then a presentation for $\pi_1(M)$ is

$$\pi_1(M) = \langle h, v_1, \dots, v_{2g}, q_1, \dots, q_r : [h, q_i] = 1, q_1, \cdots, q_r = \prod_{j=1}^{2g} [v_{2j-1}, v_{2j}], q_i^{\alpha_i} h^{\beta_i} \rangle.$$

The Classification Theorem of Seifert is:

Theorem 2.1 ([5])**.** *Two Seifert symbols represent homeomorphic Seifert manifolds by a fiber preserving homeomorphism if and only if one of the symbols can be changed into the other by a finite sequence of the following moves:*

(1) *Permute the ratios.*
(2) *Add or delete* $0/1$.
(3) *Replace the pair* $\beta_i/\alpha_i, \beta_j/\alpha_j$ *by* $(\beta_i + k\alpha_i)/\alpha_i, (\beta_j - k\alpha_j)/\alpha_j$ *where* k *is an integer.*

3. Some coverings of the manifold $M = (Oo, 0; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$

Lemma 3.1. *Assume* $M = (Oo, 0; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ *. If* $\omega : \pi_1(M) \to S_t$ *is the transitive representation defined by*

$$\begin{aligned} \omega(h) &= (1)(2), \cdots, (t), \\ \omega(q_i) &= \sigma_{i,1}, \cdots, \sigma_{i,\ell_i}, \ \textit{for } i = 1, 2, 3, \end{aligned}$$

where $\sigma_{i,1}, \cdots, \sigma_{i,\ell_i}$ *is the disjoint cycle decomposition of* $\omega(q_i)$*, then the covering space of* M *determined by* ω *is*

$$\tilde{M} = (Oo, g; B_{1,1}/A_{1,1}, \dots, B_{1,l_1}/A_{1,l_1}, \dots, B_{3,1}/A_{3,1}, \dots, B_{3,l_3}/A_{3,l_3}),$$

where

$$g = 1 + \frac{t - \sum_{i=1}^{3} l_i}{2},$$

$$B_{i,k} = \frac{order(\sigma_{i,k})\beta_i}{\gcd\{\alpha_i, order(\sigma_{i,k})\}}$$

and

$$A_{i,k} = \frac{\alpha_i}{\gcd\{\alpha_i, order(\sigma_{i,k})\}},$$

for $i = 1, 2, 3$ *and* $k = 1, \dots, l_i$.

Proof. First note that ω determines a covering of M_0 with representation $\overline{\omega} := \omega|\pi_1(M_0)$. If $\varphi : \tilde{M} \to M$ and $\overline{\varphi} : \tilde{M}_0 \to M_0$ are the coverings given by ω and $\overline{\omega}$, respectively, then we can obtain $\tilde{M}$ from $\tilde{M}_0$ by gluing some solid torus $\tilde{V}_{i,k}$ to $\tilde{M}_0$ with some homeomorphisms $\tilde{f}_{i,k} : \partial\tilde{V}_{i,k} \to q_{i,k} \times \tilde{h}$, where $\tilde{h}$ is a regular fiber of $\tilde{M}_0$ and $q_{i,k} \times \tilde{h}$ are the boundary tori of $\tilde{M}_0$. Such gluing homeomorphisms $\tilde{f}_{i,k}$ are the lifts of the gluing homeomorphisms $f_i : \partial V_i \to q_i \times h$, for $i = 1, 2, 3$, and there is exactly one $\tilde{f}_{i,k}$, for each cycle in q_i. Let $\tilde{F}$ and $\tilde{F}'$ be the orbit surfaces of $\tilde{M}$ and $\tilde{M}_0$, respectively; then the number of boundaries of $\tilde{F}'$ is equal to $\sum_{i=1}^{r} l_i$ and we can obtain $\tilde{F}$ from $\tilde{F}'$ by capping off all the boundaries of $\tilde{F}'$ with disks. Also, since $\omega(h) = (1)(2), \ldots, (n)$, we have a covering $\varphi' : \tilde{F}' \to F_{0,3}$ determined by the representation $\omega' := \omega|\pi_1(F_{0,3})$, where $F_{0,3}$ is the 3-times punctured sphere S^2. Then $\tilde{F}'$ is orientable and as a consequence F is too. So, if g denotes the genus of F, we can see that

$$\begin{array}{rcl} 2 - 2g & = & \chi(F_0) + \sum_{i=1}^{3} l_i \\ & = & -t + \sum_{i=1}^{3} l_i. \end{array}$$

Thus $g = 1 + \dfrac{t - \sum_{i=1}^{3} l_i}{2}$.

Now, we compute the numbers $B_{i,k}$ and $A_{i,k}$. Let $T_{i,k}$ be the component of $\partial\tilde{M}_0$ corresponding to the cycle $\sigma_{i,k}$, for $i = 1, 2, 3$. Then ω induces a covering $\varphi_{i,k} : T_{i,k} \to T_i$ of $order(\sigma_{i,k})$ sheets with representation $\omega_{i,k} : \pi_1(T_i) \to S_{order(\sigma_{i,k})}$ such that $\omega_{i,k}(h) = (1)(2)\cdots(order(\sigma_{i,k}))$, the identity permutation in $S_{order(\sigma_{i,k})}$, and $\omega_{i,k}(q_i) = \sigma_{i,k}$.

If $\tilde{h}$ is a component of $\varphi_{i,k}^{-1}(h)$ and $\tilde{q}_{i,k} = \varphi_{i,k}^{-1}(q_i)$ (the pre-image of q_i under $\varphi_{i,k}$ in the torus $T_{i,k}$), then $\{\tilde{h}, \tilde{q}_{i,k}\}$ is a basis for $\pi_1(T_{i,k})$. Note that $\tilde{q}_{i,k}$ is the union of $order(\sigma_{i,k})$ liftings of q_i. Then $\varphi_{i,k}(\tilde{h}) = h$ and $\varphi_{i,k}(\tilde{q}_{i,k}) = q_i^{order(\sigma_{i,k})}$. If $\tilde{m}_{i,k} \subset \varphi_{i,k}^{-1}(m_i)$, then there are $A_{i,k}$ and $B_{i,k}$ integer numbers such that $\tilde{m}_{i,k} = \tilde{q}_{i,k}^{A_{i,k}}\tilde{h}^{B_{i,k}}$ (since $\{\tilde{h}, \tilde{q}_{i,k}\}$ is a basis for $\pi_1(T_{i,k})$), and

$$\varphi_{i,k}(\tilde{m}_{i,k}) = \varphi_{i,k}(\tilde{q}_{i,k}^{A_{i,k}}\tilde{h}^{B_{i,k}}) = q_i^{order(\sigma_{i,k})A_{i,k}}h^{B_{i,k}}. \tag{1}$$

On the other hand, $\omega_{i,k}(m_i) = \omega_{i,k}(q^{\alpha_i}h^{\beta_i}) = (\sigma_{i,k})^{\alpha_i}$. This implies

$$\varphi_{i,k}(\tilde{m}_{i,k}) = m_i^{order((\sigma_{i,k})^{\alpha_i})} = (q_i^{\alpha_i \cdot order((\sigma_{i,k})^{\alpha_i})})(h^{\beta_i \cdot order((\sigma_{i,k})^{\alpha_i})}). \tag{2}$$

But in fact, $order((\sigma_{i,k})^{\alpha_i}) = \dfrac{order(\sigma_{i,k})}{\gcd\{\alpha_i, order(\sigma_{i,k})\}}$; then by equations (1) and (2) we have that $B_{i,k} = \frac{order(\sigma_{i,k})\cdot\beta_i}{\gcd\{\alpha_i, order(\sigma_{i,k})\}}$, and $A_{i,k} = \frac{\alpha_i}{\gcd\{\alpha_i, order(\sigma_{i,k})\}}$ for $k = 1, \ldots, l_i$ and for all $i = 1, \ldots, r$. □

We would like to remark that a more general version of the previous lemma can be found in [6].

4. Proof of the main theorem

Proof of Theorem 1.1. Let k, p be a pair of positive integers, and set $n = k(4p + 1) - p$ and $t = 4p + 1$. Now consider the representation $\omega : \pi_1(M_n) \to S_t$, defined by

$$\begin{array}{rcl}\omega(h) & = & (1)(2)\cdots(t)\\ \omega(q_1) & = & (1,2)(3,4)\cdots(t-2,t-1)(t)\\ \omega(q_2) & = & (1)(2,3)(4,5)\cdots(t-1,t) \text{ and}\\ \omega(q_3) & = & (1,2,4,6,\cdots,t-1,t,t-2,t-3,\cdots,5,3).\end{array}$$

Then it follows from Lemma 3.1 that

$$\tilde{M}_k = (Oo, 0; \underbrace{1/1, \ldots, 1/1}_{(t-1)/2-times}, 1/2, -1/2, \underbrace{-1/1, \ldots, -1/1}_{(t-1)/2-times}, -2/(4k-1))$$

is the t-fold covering of $M_n = (Oo, 0; 1/2, -1/2, -2/(4n-1))$ determined by ω. Finally, Theorem 2.1 implies that $\tilde{M}_k$ is fiber preserving homeomorphic to the Seifert Fibered Space with Seifert symbol $(Oo, 0; 1/2, -1/2, -2/(4k-1))$. □

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References

[1] James W. Alexander, *Note on Riemann spaces*, Bull. Amer. Math. Soc. **26** (1920), no. 8, 370–372, DOI 10.1090/S0002-9904-1920-03319-7. MR1560318

[2] Mark E. Feighn, *Branched covers according to J. W. Alexander*, Collect. Math. **37** (1986), no. 1, 55–60. MR882791 (88d:57002)

[3] Hugh M. Hilden, María Teresa Lozano, and José María Montesinos, *The Whitehead link, the Borromean rings and the knot* 9_{46} *are universal*, Collect. Math. **34** (1983), no. 1, 19–28. MR747855 (86b:57001)

[4] Hugh M. Hilden, María Teresa Lozano, and José María Montesinos, *On knots that are universal*, Topology **24** (1985), no. 4, 499–504, DOI 10.1016/0040-9383(85)90019-9. MR816529 (87a:57010)

[5] Peter Orlik and Frank Raymond, *On* 3*-manifolds with local* SO(2) *action*, Quart. J. Math. Oxford Ser. (2) **20** (1969), 143–160. MR0266214 (42 #1121)

[6] Jair Remigio-Juárez, *Branched coverings of Seifert manifolds*, Ph.D. Thesis, 2008.

[7] Jair Remigio-Juárez and Yo'av Rieck, *The link volumes of some prism manifolds*, Algebr. Geom. Topol. **12** (2012), no. 3, 1649–1665, DOI 10.2140/agt.2012.12.1649. MR2966698

[8] Yo'av Rieck and Yasushi Yamashita, *The link volume of 3-manifolds*, Algebr. Geom. Topol. **13** (2013), no. 2, 927–958, DOI 10.2140/agt.2013.13.927. MR3044597

[9] William P. Thurston, Universal links, preprint.

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