THE WITTEN-RESHETIKHIN-TURAEV REPRESENTATION OF THE KAUFFMAN BRACKET SKEIN ALGEBRA

FRANCIS BONAHON AND HELEN WONG

(Communicated by Martin Scharlemann)

ABSTRACT. For A a primitive $2N$–root of unity with N odd, the Witten-Reshetikhin-Turaev topological quantum field theory provides a representation of the Kauffman bracket skein algebra of a closed surface. We show that this representation is irreducible, and we compute its classical shadow in the sense of an earlier work of the authors.

The discovery of the Jones polynomial [9], and the simplification of its construction by Kauffman [10], were quickly followed by two originally unrelated developments. The first one was the introduction of the Kauffman bracket skein module of an oriented 3–manifold by Turaev [23] and Przytycki [17]. A special case led to the Kauffman bracket skein algebra $\mathcal{S}^A(S)$ of an oriented surface S which, when S is connected, was later interpreted as a quantization of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, consisting of the characters of all group homomorphisms $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ [7,8,18,24].

Another development was Witten's interpretation [26] of the Jones polynomial within the framework of a topological quantum field theory. This topological quantum field theory point of view was formalized in mathematical terms by Reshetikhin-Turaev [19,20]. In particular, the Witten-Reshetikhin-Turaev topological quantum field theory leads, for every primitive $2N$–root of unity A, to a representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V_S)$ of the skein algebra corresponding to this parameter A.

The first result of this article is the following.

Theorem 1. *Let S be a connected closed oriented surface. For every primitive $2N$– root of unity A, the Witten-Reshetikhin-Turaev representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V_S)$ is irreducible.*

This result can be compared with Roberts' proof [22] that, when $N = 2p$ with p prime, the action of the mapping class group of S on the Witten-Reshetikhin-Turaev space V_S is irreducible. In this special case, Theorem 1 can actually be deduced from some of the proofs of [22].

Our interest in the irreducibility of the Witten-Reshetikhin-Turaev representation is motivated by [3–6], where we initiated the systematic study of finite-dimensional irreducible representations of the skein algebra $\mathcal{S}^A(S)$. In particular,

when A is a $2N$–root of unity with N odd, we associate to such an irreducible representation $\rho\colon \mathcal{S}^A(S) \to \operatorname{End}(V)$ an element r_ρ of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$; see Theorem 13 for a precise statement. If we regard $\mathcal{S}^A(S)$ as a quantization of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and a representation ρ as a point of this quantization, the character $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is the *classical shadow* of ρ. With irreducibility of the Witten-Reshetikhin-Turaev representation established, we thus may inquire about the classical shadow of the best known representation of the skein algebra.

The Witten-Reshetikhin-Turaev topological quantum field theory, and the associated representation of $\mathcal{S}^A(S)$, have slightly different features according to whether N is even or odd, respectively known as the SU_2 and SO_3 cases. The classical shadow of a representation is defined only when N is odd.

Theorem 2. *When A is a primitive $2N$–root of unity with N odd, the classical shadow of the Witten-Reshetikhin-Turaev representation $\rho\colon \mathcal{S}^A(S) \to \operatorname{End}(V_S)$ is the trivial character $\iota \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, represented by the trivial homomorphism $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$.*

The proof of Theorem 2 is relatively simple, although it uses deep connections in the quantum theory with both types of Chebyshev polynomials. The Chebyshev polynomial of the first type, $T_n(x)$, is used to define the classical shadow of a representation of $\mathcal{S}^A(S)$. On the other hand, the second type of Chebyshev polynomials, $S_n(x)$, classically plays an important rôle in the representation theory of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$ and in the Witten-Reshetikhin-Turaev topological quantum field theory. We are ultimately led to Theorem 2 by exploiting these relationships and making use of some elementary relations between the two types.

In [6] we construct, for every character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, an irreducible representation $\rho\colon \mathcal{S}^A(S) \to \operatorname{End}(V_S)$ whose classical shadow r_ρ is equal to r. In particular, this associates another irreducible representation ρ_ι to the trivial character ι. The construction of [6] is unfortunately not very explicit in the special case of the trivial character ι; however, it appears that the representation that it provides has very different features from those of the Witten-Reshetikhin-Turaev representation.

1. THE KAUFFMAN BRACKET SKEIN MODULE

The *Kauffman bracket skein module* $\mathcal{S}^A(M)$ of an oriented 3–dimensional manifold M depends on a parameter $A = \mathrm{e}^{\pi\mathrm{i}\hbar} \in \mathbb{C} - \{0\}$, and is defined as follows: one first considers the vector space freely generated by all isotopy classes of framed links in the thickened surface $S \times [0, 1]$, and then one takes the quotient of this space by two relations:

- the first relation is the *skein relation* that $[L_1] = A^{-1}[L_0] + A[L_\infty]$ whenever the three links L_1, L_0 and $L_\infty \subset S \times [0, 1]$ differ only in a little ball where they are as represented on Figure 1;
- the second relation states that $[L \cup O] = -(A^2 + A^{-2})[L]$ whenever the knot O is the boundary of a disk D endowed with framing transverse to D, and the framed link L is disjoint from the disk D.

In the special case where $M = S \times [0, 1]$ for an oriented surface S, we write $\mathcal{S}^A(S) = \mathcal{S}^A\big(S \times [0, 1]\big)$ and we note that this module now comes with a natural multiplication. Indeed, if $[L_1]$, $[L_2] \in \mathcal{S}^A(S)$ are respectively represented by the framed links L_1, L_2, we can consider their superposition

$$[L_1] \cdot [L_2] = [L_1' \cup L_2'] \in \mathcal{S}^A(S)$$
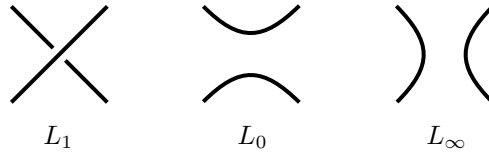
L_1 L_0 L_∞

FIGURE 1. A Kauffman triple

represented by the union of the framed link $L_1' \subset S \times [0, \frac{1}{2}]$ obtained by rescaling $L_1 \subset S \times [0,1]$ and of the framed link $L_2' \subset S \times [\frac{1}{2}, 1]$ obtained by rescaling $L_2 \subset S \times [0,1]$. This endows the skein module $\mathcal{S}^A(S)$ with the structure of an algebra, called the *Kauffman bracket skein algebra* of the oriented surface S.

2. The Witten-Reshetikhin-Turaev topological quantum field theory

We briefly review a few fundamental properties of the Witten-Reshetikhin-Turaev topological quantum field theory, and refer to [1, 2, 11, 13–15, 25] for details and proofs.

The Witten-Reshetikhin-Turaev topological quantum field theory Z_{WRT}^A depends on the choice of a primitive $2N$–root of unity A, and is defined over the category $\mathcal{C}$ constructed as follows: the objects of $\mathcal{C}$ are closed oriented surfaces S; the morphisms from S_1 to S_2 are pairs (M, L) where M is a compact oriented 3–manifold with $\partial M = (-S_1) \sqcup S_2$, where L is a framed link in the interior of M, and where M is endowed with a p_1–structure. The precise definition of a p_1–structure can be found in [2, App. B] but, for the purpose of the current article, it suffices to know that it captures certain homotopic information on the tangent bundle of the manifolds considered.

In particular, Z_{WRT}^A associates a finite-dimensional vector space $V_S = Z_{\mathrm{WRT}}^A(S)$ to each closed oriented surface S, and a linear map $Z_{\mathrm{WRT}}^A(M, L) \colon V_{S_1} \to V_{S_2}$ to each morphism (M, L) as above. In addition, the vector space $V_\varnothing$ associated to the empty surface $\varnothing$ comes equipped with a canonical identification with $\mathbb{C}$; in other words, this vector space is 1–dimensional and contains a preferred basis element that we will denote by 1.

The topological quantum field theory Z_{WRT}^A satisfies many properties, in particular those that characterize topological quantum field theories. Most of these features will play no direct rôle in the current article. However the following fact, which is grounded in properties of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$ underlying the construction of Z_{WRT}^A, is crucial for our purposes.

Lemma 3. *The linear maps $Z_{\mathrm{WRT}}^A(M, L)$ associated to the morphisms of the category $\mathcal{C}$ satisfy the skein relation that, as linear maps $V_{S_1} \to V_{S_2}$,*

$$Z_{\mathrm{WRT}}^A(M, L_1) = A^{-1} Z_{\mathrm{WRT}}^A(M, L_0) + A\, Z_{\mathrm{WRT}}^A(M, L_\infty)$$

whenever the framed links L_1, L_0 and L_∞ form a Kauffman triple in the manifold M with $\partial M = (-S_1) \sqcup S_2$, for a fixed p_1–structure on M.

Also,

$$Z_{\mathrm{WRT}}^A(M, L \cup O) = -(A^2 + A^{-2}) Z_{\mathrm{WRT}}^A(M, L)$$

whenever the knot O is the boundary of a disk D endowed with framing transverse to D, and the framed link L is disjoint from the disk D. $\qquad\square$

A first consequence of Lemma 3 is that the skein algebra $\mathcal{S}^A(S)$ acts on the space V_S, by considering the special case $M = S \times [0, 1]$. Indeed, for any framed link $L \subset S \times [0, 1]$, the pair $(S \times [0, 1], L)$ can be seen as a morphism from S to S, and therefore induces a linear map $Z_{\mathrm{WRT}}^A(S \times [0, 1], L) \colon V_S \to V_S$.

Lemma 4. *There exists a unique algebra homomorphism*

$$\rho \colon \mathcal{S}^A(S) \to \mathrm{End}(V_S)$$

such that, for every framed link L in $S \times [0, 1]$,

$$\rho\big([L]\big) = Z_{\mathrm{WRT}}^A(S \times [0, 1], L)$$

when $S \times [0, 1]$ is endowed with the product p_1–structure.

Proof. Lemma 3 shows that the rule $L \mapsto Z_{\mathrm{WRT}}^A(S \times [0, 1], L)$ is compatible with the skein relation, and therefore induces a linear map $\rho \colon \mathcal{S}^A(S) \to \mathrm{End}(V_S)$. The multiplication law of $\mathcal{S}^A(S)$ is defined by the superposition operation, which itself corresponds to the composition of morphisms $(S \times [0, 1], L)$ in the category $\mathcal{C}$. It follows that ρ is an algebra homomorphism. $\qquad\square$

This homomorphism $\rho \colon \mathcal{S}^A(S) \to \mathrm{End}(V_S)$ is the *Witten-Reshetikhin-Turaev representation* of the skein algebra $\mathcal{S}^A(S)$.

Another application of Lemma 3 will enable us to perform computations in the space V_S. Consider a 3-manifold M with boundary $\partial M = S$, endowed with a p_1–structure. A framed link $L \subset M$ provides a morphism (M, L) from the empty surface $\varnothing$ to S. This provides a linear map $Z_{\mathrm{WRT}}^A(M, L)$ from $V_\varnothing = \mathbb{C}$ to V_S, and in particular specifies an element $Z_{\mathrm{WRT}}^A(M, L)(1) \in V_S$. As above, Lemma 3 shows that the map $L \mapsto Z_{\mathrm{WRT}}^A(M, L)(1)$ defines a linear map $\Phi_M \colon \mathcal{S}^A(M) \to V_S$, from the skein module of M to the space V_S.

Lemma 5. *If M is an oriented 3–manifold with boundary $\partial M = S$, the linear map $\Phi_M \colon \mathcal{S}^A(M) \to V_S$ defined by $\Phi_M\big([L]\big) = Z_{\mathrm{WRT}}^A(M, L)(1)$ is surjective.* $\qquad\square$

We will use Lemma 5 in the special case where S is connected and where M is a handlebody H with boundary $\partial H = S$. Choose an identification $H \cong \Sigma \times [0, 1]$ of this handlebody with the product of the interval with a compact oriented surface Σ with boundary. Select also a trivalent spine for Σ, namely a trivalent graph Γ embedded in the interior of Σ such that Σ deformation retracts to Γ. We will use this data to describe a basis for V_S.

An *N–admissible weight system* assigns to each edge e of Γ a non-negative integer weight $w(e)$ such that the following conditions hold:

(1) at every vertex of Γ, the weights of the edges e_1, e_2, e_3 adjacent to this vertex satisfy the triangle inequalities $w(e_1) \leqslant w(e_2) + w(e_3)$, $w(e_2) \leqslant w(e_1) + w(e_3)$ and $w(e_3) \leqslant w(e_1) + w(e_2)$;

(2) if N is odd, the weight $w(e)$ of each edge e is even and bounded by $N - 2$; in addition, at each vertex, the sum of the weights of the adjacent edges is bounded by $2N - 4$;

(3) if N is even, the weight $w(e)$ of each edge e is bounded by $\frac{N}{2} - 2$; in addition, at each vertex, the sum of the weights of the adjacent edges is even and bounded by $N - 4$.

Let $\mathcal{W}_\Gamma$ denote the (finite) set of all N–admissible weight systems for Γ.

An N–admissible edge weight system $w \in \mathcal{W}_\Gamma$ specifies an element β_w of the skein module $\mathcal{S}^A(H)$, by replacing each edge e of Γ weighted by $w(e) \leqslant N - 2$ by a copy of the $w(e)$–th Jones-Wenzl idempotent, represented by a box $\,\rule[0.4ex]{1.5em}{0.4pt}\!\boxed{w(e)}\!\rule[0.4ex]{1.5em}{0.4pt}\,$. The precise definition of Jones-Wenzl idempotents and of the element $\beta_w \in \mathcal{S}^A(H)$ associated to the weight system $w \in \mathcal{W}_\Gamma$ can be found for instance in [14] or [25]. To give a flavor of the construction, we will just say that the Jones-Wenzl idempotent $\rule[0.4ex]{1.5em}{0.4pt}\!\boxed{\,a\,}\!\rule[0.4ex]{1.5em}{0.4pt}\,$ is defined for every $a \geqslant 0$ such that $A^{4k} \neq 1$ for every k with $0 < k < a$ (namely, for $0 \leqslant a < N$ is N is odd, and for $0 \leqslant a < \frac{N}{2}$ if N is even), and is a certain formal linear combination of families of disjoint arcs, each with a strands emanating from each end of the box lying on Σ. The strands emanating from the Jones-Wenzl idempotents are then connected by disjoint arcs near the vertices of Γ, using the fact that at each vertex the $w(e)$ add up to an even number and satisfy the triangle inequalities; see Figure 2, where $w(e_1) = 6$, $w(e_2) = 4$ and $w(e_3) = 4$. Finally, the element $\beta_w \in \mathcal{S}^A(H)$ is defined by a multiple sum of the contributions of the Jones-Wenzl idempotents associated to the edges of Γ.
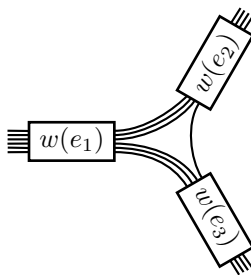

FIGURE 2

For an N–admissible weight system $w \in \mathcal{W}_\Gamma$, let $\beta_w \in \mathcal{S}^A(H)$ be associated to w as above, and let $b_w \in V_S$ be the image of β_w under the map $\Phi_H \colon \mathcal{S}^A(H) \to V_S$ of Lemma 5.

Lemma 6. *The subset $\mathcal{B}_\Gamma = \{b_w\}_{w \in \mathcal{W}_\Gamma}$ is a basis for the vector space V_S.* $\square$

The weight system space $\mathcal{W}_\Gamma$ contains a special element 0, assigning weight 0 to each edge of Γ. This provides a preferred element $b_0 \in V_S$, represented by the empty skein $[\varnothing] \in \mathcal{S}^A(H)$. By definition, b_0 is the *vacuum element* of V_S.

3. The Witten-Reshetikhin-Turaev representation is irreducible

In the rest of the article, S will always denote a connected closed oriented surface. Consider the Witten-Reshetikhin-Turaev representation $\rho \colon \mathcal{S}^A(S) \to \text{End}(V_S)$ of Lemma 4. In practice, for any oriented 3–manifold M bounding S, the map $\Phi_M \colon \mathcal{S}^A(M) \to V_S$ of Lemma 5 enables us to describe ρ by the property that

$$\rho\big([K]\big)\big(\Phi_M\big([L]\big)\big) = \Phi_M\big([K \cup L]\big)$$

for any skeins $[K] \in \mathcal{S}^A(S)$ and $[L] \in \mathcal{S}^A(M)$, where the union $K \cup L \subset M$ is defined by identifying to $S \times [0,1]$ a small neighborhood of the boundary $\partial M = S$ in M.

Theorem 7. *The Witten-Reshetikhin-Turaev representation $\rho \colon \mathcal{S}^A(S) \to \text{End}(V_S)$ is irreducible.*

Proof. We will split the proof into several steps. Let $W \subset V_S$ be a non-trivial linear subspace that is invariant under the image $\rho(\mathcal{S}^A(S))$. We want to show that W is equal to the whole space V_S. For this, we will use a handlebody $H \cong \Sigma \times [0, 1]$ bounding the surface S, a trivalent spine Γ for the surface Σ, and the basis $\mathcal{B}_\Gamma = \{b_w\}_{w \in \mathcal{W}_\Gamma}$ of Lemma 6. Our general strategy will be to show by successive simplifications that W contains the vacuum element $b_0 \in \mathcal{B}_\Gamma$ associated to the empty skein $[\varnothing] \in \mathcal{S}^A(H)$, and then to use the description of the Witten-Reshetikhin-Turaev representation by Lemmas 4 and 5 to conclude that the invariant subspace W is equal to V_S.

By hypothesis, W is non-trivial, and therefore contains a non-trivial element $\sum_{w \in \mathcal{W}_\Gamma} \alpha_w b_w$ with $\alpha_w \in \mathbb{C}$. Our first step is borrowed from [22] and [21].

Lemma 8. *Let Γ be a trivalent spine for the surface Σ. If $\sum_{w \in \mathcal{W}_\Gamma} \alpha_w b_w$ is in the invariant subspace W, then every basis element $b_w \in \mathcal{B}_\Gamma$ with non-zero coefficient $\alpha_w \neq 0$ also belongs to W.*

Proof of Lemma 8. For every edge e of Γ, there exists a disk $D_e \subset H$ such that $D_e \cap \partial H$ is equal to the boundary ∂D_e, and such that $D \cap \Gamma = D \cap e$ consists of a single point; this is an immediate consequence of the fact that the handlebody H deformation retracts to the graph Γ. Consider $[\partial D_e] \in \mathcal{S}^A(S)$ and its image $\rho([\partial D_e]) \in \mathrm{End}(V_S)$. For every basis element $b_w \in \mathcal{B}_\Gamma$ associated to the weight system $w \in \mathcal{W}_\Gamma$, a computation as in [14, Lemma 14.2] shows that

$$\rho([\partial D_e])(b_w) = -(A^{2(w(e)+1)} + A^{-2(w(e)+1)})b_w.$$

Since A is a primitive $2N$–root of unity, the numbers $A^{2(i+1)} + A^{-2(i+1)}$ are non-zero and distinct as i ranges over all admissible weights, that is, over all $i \in \{0, 2, 4, \ldots, N-3\}$ when N is odd and over all $i \in \{0, 1, 2, \ldots, \frac{N}{2} - 2\}$ when N is even.

After these observations, the lemma is just a matter of elementary linear algebra. Consider an element $v = \sum_{w \in \mathcal{W}_\Gamma} \alpha_w b_w$ of W such that $\alpha_{w_1} \neq 0$. We want to show that b_{w_1} also belongs to W.

If all the other coefficients α_w where $w \neq w_1$ are equal to 0, then $b_{w_1} = \frac{1}{\alpha_{w_1}} v \in W$ and we are done.

Otherwise, there exists another weight system $w_2 \neq w_1$ with $\alpha_{w_2} \neq 0$. The fact that $w_2 \neq w_1$ in $\mathcal{W}_\Gamma$ means that there exists an edge e of Γ such that $w_2(e) \neq w_1(e)$. Then the invariant subspace W also contains the element

$$\begin{aligned}
v' &= \left(A^{2(w_2(e)+1)} + A^{-2(w_2(e)+1)}\right)v + \rho([\partial D_e])(v) \\
&= \sum_{w \in \mathcal{W}_\Gamma} \left(A^{2(w_2(e)+1)} + A^{-2(w_2(e)+1)} - A^{2(w(e)+1)} - A^{-2(w(e)+1)}\right)\alpha_w b_w.
\end{aligned}$$

Note that, in the basis $\mathcal{B}_\Gamma = \{b_w\}_{w \in \mathcal{W}_\Gamma}$, the coordinate of v' corresponding to b_{w_1} is still non-zero since $w_2(e) \neq w_1(e)$, but that v' now has one fewer non-zero coordinate than v (because the coordinate of v' corresponding to b_{w_2} is equal to 0). We can therefore replace v by $v' \in W$, which is simpler.

Iterating this construction, we eventually reach an element of W that has exactly one non-zero coordinate, corresponding to b_{w_1}. This proves that b_{w_1} belongs to W, as required. $\qquad\square$
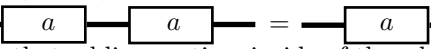
We will need to extend Lemma 8 to a slightly more general framework. Let a *partial spine* for the surface Σ be the union Γ of a finite family of disjoint trivalent

graphs and simple closed curves in Σ, such that each component of $\Sigma - \Gamma$ contains at least one component of the boundary $\partial\Sigma$. This condition guarantees that Γ can be enlarged to a trivalent spine $\hat{\Gamma}$ for Σ, by adding a few vertices and edges.

The notion of an N–admissible weight system straightforwardly extends to partial spines: such a weight system consists of an N–admissible edge weight system on each trivalent graph component of Γ; and it assigns to each closed curve component C of Γ an even weight $w(C) \in \{0, 2, 4, \ldots, N-3\}$ if N is odd, or a weight $w(C) \in \{0, 1, 2, \ldots, \frac{N}{2} - 2\}$ if N is even. Again, plugging Jones-Wenzl idempotents into the edges and closed curve components of Γ associates an element $\beta_w \in \mathcal{S}^A(H)$ to each N–admissible weight system $w \in \mathcal{W}_\Gamma$. As before, we denote by $b_w = \Phi_H(\beta_w) \in V_S$ the image of β_w under the map $\Phi_H \colon \mathcal{S}^A(H) \to V_S$ of Lemma 5, and we define $\mathcal{B}_\Gamma = \{b_w\}_{w \in \mathcal{W}_\Gamma}$. The only major difference is that $\mathcal{B}_\Gamma$ does not necessarily generate the Witten-Reshetikhin-Turaev space V_S.

Lemma 9. *The statement of Lemma 8 also holds when Γ is only a partial spine for Σ. Namely, if Γ is a partial spine for Σ and if $\sum_{w \in \mathcal{W}_\Gamma} \alpha_w b_w$ belongs to the invariant subspace W, then every element $b_w \in \mathcal{B}_\Gamma$ with non-zero coefficient $\alpha_w \neq 0$ also belongs to W.*

Proof. Enlarge the partial spine Γ to a trivalent spine $\hat{\Gamma}$ for Σ, by adding vertices inside of the edges and closed curve components of Γ and then adding edges connecting these new vertices as necessary. A weight system $w \in \mathcal{W}_\Gamma$ determines an N–admissible edge weight system for $\hat{\Gamma}$ as follows: it assigns to each edge of $\hat{\Gamma}$ that is contained in Γ the w–weight of the edge or closed curve component of Γ that contains it; and it assigns weight 0 to each of the new edges of $\hat{\Gamma} - \Gamma$. This defines an inclusion $\mathcal{W}_\Gamma \subset \mathcal{W}_{\hat{\Gamma}}$, and we will use the same letter w to denote the original $w \in \mathcal{W}_\Gamma$ and the edge weight system $w \in \mathcal{W}_{\hat{\Gamma}}$ for $\hat{\Gamma}$ that it defines.

We saw that a weight system $w \in \mathcal{W}_\Gamma$ for Γ determines an element $\beta_w \in \mathcal{S}^A(H)$. Similarly, weighing the edges of $\hat{\Gamma}$ with $w \in \mathcal{W}_{\hat{\Gamma}}$ defines another element $\hat{\beta}_w \in \mathcal{S}^A(H)$. It turns out that $\hat{\beta}_w = \beta_w$. Indeed, this immediately follows from the idempotent property $\;\rule{0.3cm}{0.4pt}\boxed{\;a\;}\rule{0.3cm}{0.4pt}\boxed{\;a\;}\rule{0.3cm}{0.4pt}\; = \;\rule{0.3cm}{0.4pt}\boxed{\;a\;}\rule{0.3cm}{0.4pt}\;$ of Jones-Wenzl idempotents, which shows that adding vertices inside of the edges and closed curve components of Γ does not change the associated element of $\mathcal{S}^A(H)$; it is immediate that adding weight 0 edges also has no impact. As a consequence, the inclusion $\mathcal{W}_\Gamma \subset \mathcal{W}_{\hat{\Gamma}}$ induces an inclusion $\mathcal{B}_\Gamma \subset \mathcal{B}_{\hat{\Gamma}} \subset V_S$.

With this observation, every element $\sum_{w \in \mathcal{W}_\Gamma} \alpha_w b_w \in V_S$ can also be written as $\sum_{w \in \mathcal{W}_{\hat{\Gamma}}} \hat{\alpha}_w b_w$, by setting $\hat{\alpha}_w = \alpha_w$ when $w \in \mathcal{W}_\Gamma \subset \mathcal{W}_{\hat{\Gamma}}$ and $\hat{\alpha}_w = 0$ when $w \in \mathcal{W}_{\hat{\Gamma}} - \mathcal{W}_\Gamma$. Lemma 9 then immediately follows by applying Lemma 8 to the trivalent spine $\hat{\Gamma}$. $\qquad\square$

Lemma 8 shows that the invariant subspace W contains at least one element $b_w \in \mathcal{B}_\Gamma$ associated to a weight system $w \in \mathcal{W}_\Gamma$ for a trivalent spine Γ. Our next goal is to show that W contains the vacuum element b_0 corresponding to the zero weight system $0 \in \mathcal{W}_\Gamma$ for all partial spines Γ. The following definition is designed to measure progress in this direction.

The *complexity* of a weight system $w \in \mathcal{W}_\Gamma$ for a partial spine Γ is defined as the triple

$$|w| = \big(e(\Gamma), \max(w), n_{\max}(w)\big) \in \mathbb{N}^3$$

where $e(\Gamma)$ is the number of edges of Γ, $\max(w)$ is the largest weight assigned by w to the edges and closed curve components of Γ, and where $n_{\max}(w)$ is the number of edges and closed curve components where this maximum is attained (and where $\mathbb{N}$ denotes the set of non-negative integers). We endow $\mathbb{N}^3$ with the lexicographic order.

Lemma 10. *If the invariant subspace W contains an element $b_w \in \mathcal{B}_\Gamma$ associated to a non-zero weight system $w \in \mathcal{W}_\Gamma$ for a partial spine Γ, then W contains another element $b_{w'} \in \mathcal{B}_{\Gamma'}$, represented by a partial spine Γ' and a weight system $w' \in \mathcal{W}_{\Gamma'}$, such that $|w'| < |w|$.*

Proof. We distinguish cases.

Case 1 (The weight system w assigns weight 0 to an edge e of Γ). The admissibility properties of w imply that, if the endpoints of e are not distinct and correspond to the same vertex of Γ, the third edge e' emanating from this vertex has weight $w(e')$ equal to 0. Replacing e by e' if necessary, we can therefore assume that the endpoints of e are distinct. The admissibility condition then shows that, at each of these endpoints, the two other adjacent edges have the same w–weight. Let Γ' be the partial spine obtained from Γ by removing e, and combining the edges that meet at each of its end vertices. By the above observation, w induces a weight system $w' \in \mathcal{W}_{\Gamma'}$, and as in the proof of Lemma 9, they are represented by the same basis element in V_S. By construction, Γ' has one fewer edge than Γ, so that $|w'| < |w|$.

$$\sum_{w'} \left\{ \begin{matrix} w(e_1) & w(e_2) & w'(e_0') \\ w(e_3) & w(e_4) & w(e_0) \end{matrix} \right\}$$
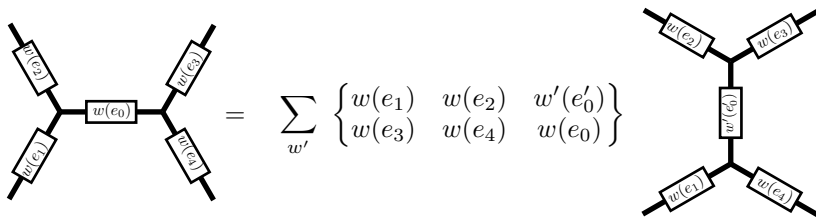
FIGURE 3. The Flip Relation

Case 2 (No edge weight is 0, and the maximum weight $\max(w)$ is attained on an edge e_0 of Γ whose endpoints are distinct). We apply to Γ the classical Flip Relation represented in Figure 3. The Flip Move replaces Γ by a new partial spine Γ' that differs from Γ only in the edge e_0, and replaces e_0 by an edge e_0' that connects differently the four edges meeting e_0. The Flip Relation

$$\beta_w = \sum_{w'} \left\{ \begin{matrix} w(e_1) & w(e_2) & w'(e_0') \\ w(e_3) & w(e_4) & w(e_0) \end{matrix} \right\} \beta_{w'}$$

expresses the element $\beta_w \in \mathcal{S}^A(H)$ represented by $w \in \mathcal{W}_\Gamma$ as a linear combination of elements $\beta_{w'} \in \mathcal{S}^A(H)$ where w' ranges over all N–admissible weight systems for Γ' that coincide with w over all edges that are common to Γ and Γ'.

A key feature of this relation are the coefficients $\left\{ \begin{matrix} w(e_1) & w(e_2) & w'(e_0') \\ w(e_3) & w(e_4) & w(e_0) \end{matrix} \right\} \in \mathbb{C}$, known as $6j$–symbols. A precise computation of these $6j$–symbols can be found

in [11] or [16]. The corresponding formula is usually complicated, and expresses a $6j$–symbol as a sum of several terms, each of which is a product of quantum integers and their inverses. However, it is somewhat simpler for the "smallest" of the N–admissible weight systems w' that are compatible with w.

Consider the weight system w'_1 that coincides with w on $\Gamma - e_0 = \Gamma' - e'_0$ and assigns weight

$$w'_1(e'_0) = \max\{|w(e_1) - w(e_4)|, |w(e_2) - w(e_3)|\}$$

to the edge e'_0. The formula is specially designed so that w'_1 is N–admissible. In fact, w'_1 is the edge weight that minimizes the weight $w'(e'_0)$ among all N–admissible weight systems $w' \in \mathcal{W}_{\Gamma'}$ that coincide with w outside of e'_0. We will not need this minimizing property, but the following other feature of w' is critical for our purposes: for this specific weight system $w'_1 \in \mathcal{W}_{\Gamma'}$, the sum occurring in the formula of [16] consists of a single term, and expresses the $6j$–symbol $\begin{Bmatrix} w(e_1) & w(e_2) & w'_1(e'_0) \\ w(e_3) & w(e_4) & w(e_0) \end{Bmatrix}$ as a product of non-zero quantum integers and their inverses. In particular, this $6j$–symbol is different from 0.

Remembering that $b_w = \Phi_H(\beta_w) \in \mathcal{B}_\Gamma$ and $b_{w'} = \Phi_H(\beta_{w'}) \in \mathcal{B}_{\Gamma'}$ for the map $\Phi \colon \mathcal{S}^A(H) \to V_S$ of Lemma 5,

$$b_w = \sum_{w'} \begin{Bmatrix} w(e_1) & w(e_2) & w'(e'_0) \\ w(e_3) & w(e_4) & w(e_0) \end{Bmatrix} b_{w'}$$

in the Witten-Reshetikhin-Turaev space V_S. By hypothesis, b_w belongs to the invariant subspace $W \subset V_S$. Lemma 9 then shows that W also contains the element $b_{w'_1} \in \mathcal{B}_{\Gamma'}$ corresponding to $w'_1 \in \mathcal{W}_{\Gamma'}$, since its coefficient in the above sum is different from 0.

By our hypothesis that the weights assigned by w to the edges of Γ are non-zero and bounded by $w(e_0) = \max(w)$, the weight $w'_1(e'_0)$ defined above is strictly less than $\max(w)$. It follows that w'_1 has lower complexity $|w'_1| < |w|$ than w.

Case 3 (No edge weight is 0, and the maximum weight $\max(w)$ is attained on an edge e_0 of Γ whose endpoints are identified). Because the endpoints of e_0 are identified, we cannot apply a Flip Move at e_0. Instead, we will apply such a move at the remaining edge e_1 that is adjacent to the vertex corresponding to the two ends of e_0. This gives a new partial spine Γ', obtained from Γ by replacing the edge e_1 by an edge e'_1 as in Figure 4.
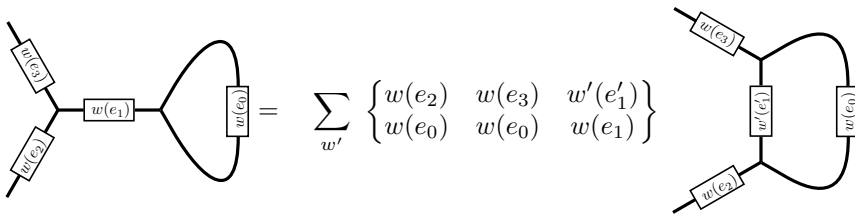

FIGURE 4. A special case of the Flip Relation

We then consider for Γ' the N–admissible weight system w'_1 that assigns weight

$$w'_1(e'_1) = \max\{w(e_0) - w(e_2), w(e_0) - w(e_3)\}$$

to the edge e'_1 and coincides with w outside of e'_1. As in Case 2, the formula of [16] shows that the $6j$–symbol occurring as coefficient of $\beta_{w'_1}$ in the Flip Relation of Figure 4 is non-zero. An application of Lemma 9 again proves that the invariant subspace $W \subset V_S$ contains the element $b_{w'_1} \in \mathcal{B}_{\Gamma'}$ associated to $w'_1 \in \mathcal{W}_{\Gamma'}$.

Note that $w'_1(e'_1) < w(e_0) = \max(w)$, so that $|w'_1| \leqslant |w|$. Because the inequality is not necessarily strict, we are not quite done yet. However, we can now apply a Flip Move to Γ' at the edge e_0, and use Case 2 to conclude.

Case 4 (N is odd, and the maximum weight $\max(w)$ is attained on a closed curve component C of Γ). Push this simple closed curve $C \subset \Sigma$ to $\Sigma \times \{1\} \subset \partial H = S$ to consider it as a partial spine in a thickening of S, and let $[C^{S_2}] \in \mathcal{S}^A(S)$ be defined by assigning weight 2 to this partial spine, namely by plugging a Jones-Wenzl idempotent ——□ 2 □—— in C; see Remark 11 to explain the notation.
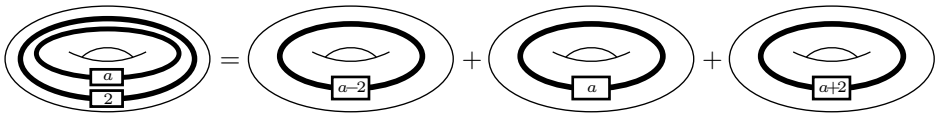

FIGURE 5. A multiplication property for Jones-Wenzl idempotents

If the subspace $W \subset V_S$ contains $b_w \in \mathcal{B}_\Gamma$, it also contains $\rho([C^{S_2}])(b_w)$ by invariance of W under the action of $\rho(\mathcal{S}^A(S)) \subset \operatorname{End}(V_S)$. The relation of Figure 5, valid in a solid torus neighborhood of C, enables us to compute this element and gives

$$\rho([C^{S_2}])(b_w) = b_{w'} + b_w + b_{w''}$$

where the weight systems w' and w'' for Γ coincide with w outside of the closed curve component C, and respectively assign weight $w(C) - 2$ and $w(C) + 2$ to C. See [14, Lemma 14.11] for a proof of this relation.

There is a little caveat needed here when $w(C)$ is equal to its maximum possible value $N - 3$. Then, w'' assigns weight $N - 1$ to C, and consequently is no longer N–admissible. The associated element $\beta_{w''} \in \mathcal{S}^A(H)$ still makes sense, but its image $\Phi_H(\beta_w) \in V_S$ under the map of Lemma 5 is equal to 0. We consequently set $b_{w''} = 0$ in this case.

In all cases, we can apply Lemma 9 to $\rho([C^{S_2}])(b_w) \in W$, and we conclude that $b_{w'}$ belongs to W. By construction, $w'(C) < w(C) = \max(w)$, so that $w' \in \mathcal{W}_\gamma$ has lower complexity $|w'| < |w|$. This concludes the proof in this case.

Case 5 (N is even, and the maximum weight $\max(w)$ is attained on a closed curve component C of Γ). The proof is almost identical to that of Case 4, except that we do not have to worry about keeping all weights even. Because of this, it suffices to consider the action of the element $[C] \in \mathcal{S}^A(S)$ represented by the closed curve C. Then, a computation similar to that of Figure 5 (see again [14, Lemma 14.11]) gives that

$$\rho([C])(b_w) = b_{w'} + b_w + b_{w''}$$

where the weight systems w' and w'' for Γ coincide with w outside of the closed curve component C, and respectively assign weight $w(C) - 1$ and $w(C) + 1$ to C (with $b_{w''} = 0$ when $w(C) = \frac{N}{2} - 2$). An application of Lemma 9 then shows that W contains the basis element $b_{w'} \in \mathcal{B}_\Gamma$. Again, $w' \in \mathcal{W}_\gamma$ has lower complexity $|w'| < |w|$, and this concludes the proof in this case.

Since the five cases considered exhaust all possibilities, the proof of Lemma 10 is now complete. □

We are now almost done with the proof of Theorem 7.

Recursively applying Lemma 10, we eventually reach a partial spine Γ' such that the invariant subspace W contains the element $b_0 \in \mathcal{B}_{\Gamma'}$ associated to the trivial weight system $0 \in \mathcal{W}_{\Gamma'}$. By definition b_0 is also the image of the empty skein $[\varnothing] \in \mathcal{S}^A(H)$ under the map Φ_H of Lemma 5.

Let L be a framed link in the handlebody H. Push L into a tubular neighborhood of the boundary $\partial H = S$, so that L defines a skein $[L] \in \mathcal{S}^A(S)$.

By invariance of W under the action of $\rho(\mathcal{S}^A(S)) \subset \mathrm{End}(V_S)$, it contains the element $\rho([L])(b_0)$. From the definition of the Witten-Reshetikhin-Turaev homomorphism ρ by Lemma 4,

$$\rho([L])(b_0) = \rho([L])\big(\Phi_H([\varnothing])\big) = \Phi_H\big([L \cup \varnothing]\big) = \Phi_H\big([L]\big)$$

where $[L]$ denotes the element of $\mathcal{S}^A(S)$ represented by L in the first two terms, whereas $[L]$ is the element of $\mathcal{S}^A(H)$ represented by L in the last two terms.

This proves that the invariant subspace $W \subset V_S$ contains the image $\Phi_H\big([L]\big)$ of every skein $[L] \in \mathcal{S}^A(H)$. Since these skeins generate $\mathcal{S}^A(H)$ and since $\Phi_H \colon \mathcal{S}^A(H) \to V_S$ is surjective by Lemma 5, this proves that W is equal to the whole space V_S.

This completes the proof of Theorem 7. □

4. THE CLASSICAL SHADOW OF THE WITTEN-RESHETIKHIN-TURAEV REPRESENTATION

4.1. **Threading polynomials along a framed link.**

Given a framed (connected) knot K in a 3–dimensional manifold M and a polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$, we can consider the linear combination

$$[K^P] = \sum_{i=0}^{n} a_i [K^{(i)}] \in \mathcal{S}^A(M)$$

where, for each i, $K^{(i)}$ is the framed link obtained by taking i parallel copies of K in the direction indicated by the framing. More generally, if $L \subset M$ is a framed link with components $K_1, K_2, \ldots, K_l$, define

$$[L^P] = \sum_{0 \leqslant i_1,\, i_2, \ldots,\, i_l \leqslant n} a_{i_1} a_{i_2} \ldots a_{i_l} [K_1^{(i_1)} \cup K_2^{(i_2)} \cup \cdots \cup K_l^{(i_l)}] \in \mathcal{S}^A(M).$$

By definition, $[L^P] \in \mathcal{S}^A(M)$ is obtained by *threading the polynomial P along the framed link L.*

We will apply this construction to the (normalized) Chebyshev polynomials of the first and second type.

The *n–th Chebyshev polynomial of the first type $T_n(x)$* is defined by the properties that $T_n(x) = xT_{n-1}(x) - T_{n-2}(x)$, $T_0(x) = 2$ and $T_1(x) = x$. The *n–th Chebyshev polynomial of the second type $S_n(x)$* is defined by the same recurrence relation $S_n(x) = xS_{n-1}(x) - S_{n-2}(x)$, the same initial condition $S_1(x) = x$, but differs in the other initial condition $S_0(x) = 1$. The two types of Chebyshev polynomials are related by the property that $T_n(x) = S_n(x) - S_{n-2}(x)$ for every n.

Remark 11. The Chebyshev polynomials of the second type $S_n(x)$ are ubiquitous in the Witten-Reshetikhin-Turaev theory and, more generally, in the representation

theory of the quantum group $U_q(\mathfrak{sl}_2)$. In particular, for each framed link L in a 3–manifold M, the element of $\mathcal{S}^A(M)$ obtained by plugging the n–th Jones-Wenzl idempotent in each component of L is equal to the element $[L^{S_n}] \in \mathcal{S}^A(M)$. See [14, §13] or [15, p. 715].

The following facts, which can for instance be found in Lemma 6.3 of [1], are crucial for our computations.

Lemma 12. *Suppose that A is a primitive $2N$–root of unity with N odd, and let V_S be the Witten-Reshetikhin-Turaev space of the surface S. Let K and L be two disjoint framed links in a 3–manifold M bounded by S. Then, for the homomorphism $\Phi_M \colon \mathcal{S}^A(M) \to V_S$ of Lemma 5,*

(1) $\Phi_M\big([K^{S_{N-1}} \cup L]\big) = 0$;
(2) $\Phi_M\big([K^{S_{N-2-n}} \cup L]\big) = \Phi_M\big([K^{S_n} \cup L]\big)$ *for every integer n.* □

4.2. The classical shadow of the Witten-Reshetikhin-Turaev representation.
As usual, S is a connected closed oriented surface. Consider the character variety

$$\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) = \{\text{homomorphisms } r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})\}/\!\!/\mathrm{SL}_2(\mathbb{C})$$

where $\mathrm{SL}_2(\mathbb{C})$ acts on homomorphisms $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ by conjugation, and where the double bar indicates that one takes the quotient in the sense of geometric invariant theory. In practice, this means that two homomorphisms r, $r' \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ represent the same point of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if and only if they induce the same trace functions, namely if and only if $\mathrm{Tr}\, r(\gamma) = \mathrm{Tr}\, r'(\gamma)$ for every $\gamma \in \pi_1(S)$.

Theorem 13 ([4]). *Suppose that A is a primitive $2N$–root of unity with N odd. If $\rho \colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ is an irreducible representation of the skein algebra $\mathcal{S}^A(S)$, then there exists a unique character $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ such that*

$$\rho\big([K^{T_N}]\big) = -\mathrm{Tr}\, r_\rho(K)\,\mathrm{Id}_V$$

for every framed knot $K \subset S \times [0, 1]$. □

By definition, $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is the *classical shadow* of the representation $\rho \colon \mathcal{S}^A(S) \to \mathrm{End}(V)$. See also [12] for an alternative approach to the key properties underlying this statement.

Theorem 14. *When A is a primitive $2N$–root of unity with N odd, the classical shadow of the Witten-Reshetikhin-Turaev representation $\rho \colon \mathcal{S}^A(S) \to \mathrm{End}(V_S)$ is the trivial character $\iota \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, represented by the trivial homomorphism $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$.*

Proof. This is a relatively simple consequence of Lemma 12. Identify $S \times [0, 1]$ to a tubular neighborhood of the boundary $S = \partial M$ in the 3–manifold M. To compute $\rho\big([K^{T_N}]\big) \in \mathrm{End}(V_S)$ for a framed knot $K \subset S \times [0, 1]$, Lemma 5 shows that it suffices to consider its action on those elements of V_S of the form $v = \Phi_M\big([L]\big)$ for a framed link $L \subset M$. Pushing L away from the neighborhood $S \times [0, 1]$ of $S = \partial M$ in M,

$$\rho\big([K^{T_N}]\big)(v) = \Phi_M\big([K^{T_N} \cup L]\big) = \Phi_M\big([K^{xS_{N-1}-2S_{N-2}} \cup L]\big)$$
$$= \Phi_M\big([K^{xS_{N-1}} \cup L]\big) - 2\Phi_M\big([K^{S_{N-2}} \cup L]\big),$$

using the property that $T_n(x) = S_n(x) - S_{n-2}(x) = xS_{n-1}(x) - 2S_{n-2}(x)$ for every n.

The term $[K^{xS_{N-1}} \cup L] \in \mathcal{S}^A(M)$ is also equal to $[K^{S_{N-1}} \cup K' \cup L]$ where K' is a push-off of K in the direction given by the framing. Its image $\Phi_M([K^{S_{N-1}} \cup K' \cup L])$ in V_S is therefore equal to 0 by Part (1) of Lemma 12. Similarly, Part (2) of Lemma 12 shows that

$$\Phi_M([K^{S_{N-2}} \cup L]) = \Phi_M([K^{S_0} \cup L]) = \Phi_M([K^1 \cup L]) = \Phi_M([L]) = v.$$

(Note that, since $1 = x^0$, the skein $[K^1] = [K^{(0)}]$ is represented by 0 copies of the knot K, and is therefore trivial.)

Therefore, $\rho([K^{T_N}])(v) = -2v$ for every $v = \Phi_M([L]) \in V_S$ represented by a framed link $L \subset M$. Since these elements generate V_S by Lemma 5, it follows that $\rho([K^{T_N}]) = -2\,\mathrm{Id}_{V_S}$.

If r_ρ is the classical shadow of the Witten-Reshetikhin-Turaev representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V_S)$, this proves that $\mathrm{Tr}\, r_\rho(K) = 2$ for every knot $K \subset S \times [0,1]$. This means that $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is the character represented by the trivial homomorphism. $\qquad\square$

References

[1] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Three-manifold invariants derived from the Kauffman bracket*, Topology **31** (1992), no. 4, 685–699, DOI 10.1016/0040-9383(92)90002-Y. MR1191373 (94a:57010)

[2] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology **34** (1995), no. 4, 883–927, DOI 10.1016/0040-9383(94)00051-4. MR1362791 (96i:57015)

[3] F. Bonahon and H. Wong, *Kauffman brackets, character varieties and triangulations of surfaces*, Topology and geometry in dimension three, Contemp. Math., vol. 560, Amer. Math. Soc., Providence, RI, 2011, pp. 179–194, DOI 10.1090/conm/560/11099. MR2866931

[4] F. Bonahon and H. Wong, *Representations of the Kauffman bracket skein algebra I: invariants and punctured surfaces*, to appear in Inventiones Mathematicæ, `arXiv:1206.1638`, 2012.

[5] F. Bonahon and H. Wong, *Representations of the Kauffman bracket skein algebra II: punctured surfaces*, preprint, `arXiv:1206.1639`, 2012.

[6] F. Bonahon and H. Wong, *Representations of the Kauffman bracket skein algebra III: closed surfaces and canonicity*, preprint, `arXiv:1505.01522`, 2015.

[7] D. Bullock, C. Frohman, and J. Kania-Bartoszyńska, *Understanding the Kauffman bracket skein module*, J. Knot Theory Ramifications **8** (1999), no. 3, 265–277, DOI 10.1142/S0218216599000183. MR1691437 (2000d:57012)

[8] D. Bullock, C. Frohman, and J. Kania-Bartoszyńska, *The Kauffman bracket skein as an algebra of observables*, Proc. Amer. Math. Soc. **130** (2002), no. 8, 2479–2485 (electronic), DOI 10.1090/S0002-9939-02-06323-2. MR1897475 (2003e:57016)

[9] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), no. 2, 335–388, DOI 10.2307/1971403. MR908150 (89c:46092)

[10] L. H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), no. 3, 395–407, DOI 10.1016/0040-9383(87)90009-7. MR899057 (88f:57006)

[11] L. H. Kauffman and S. L. Lins, *Temperley-Lieb recoupling theory and invariants of 3-manifolds*, Annals of Mathematics Studies, vol. 134, Princeton University Press, Princeton, NJ, 1994. MR1280463 (95c:57027)

[12] T. T. Q. Lê, *On Kauffman bracket skein modules at roots of unity*, Algebr. Geom. Topol. **15** (2015), no. 2, 1093–1117, DOI 10.2140/agt.2015.15.1093. MR3342686

[13] W. B. R. Lickorish, *The skein method for three-manifold invariants*, J. Knot Theory Ramifications **2** (1993), no. 2, 171–194, DOI 10.1142/S0218216593000118. MR1227009 (94g:57006)

[14] W. B. R. Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR1472978 (98f:57015)

[15] W. B. R. Lickorish, *Quantum invariants of 3-manifolds*, Handbook of geometric topology, North-Holland, Amsterdam, 2002, pp. 707–734. MR1886680 (2003d:57026)

[16] G. Masbaum and P. Vogel, *3-valent graphs and the Kauffman bracket*, Pacific J. Math. **164** (1994), no. 2, 361–381. MR1272656 (95e:57003)

[17] J. H. Przytycki, *Skein modules of 3-manifolds*, Bull. Polish Acad. Sci. Math. **39** (1991), no. 1-2, 91–100. MR1194712 (94g:57011)

[18] J. H. Przytycki and A. S. Sikora, *On skein algebras and $\mathrm{Sl}_2(\mathbf{C})$-character varieties*, Topology **39** (2000), no. 1, 115–148, DOI 10.1016/S0040-9383(98)00062-7. MR1710996 (2000g:57026)

[19] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), no. 1, 1–26. MR1036112 (91c:57016)

[20] N. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), no. 3, 547–597, DOI 10.1007/BF01239527. MR1091619 (92b:57024)

[21] J. Roberts, *Skeins and mapping class groups*, Math. Proc. Cambridge Philos. Soc. **115** (1994), no. 1, 53–77, DOI 10.1017/S0305004100071917. MR1253282 (94m:57035)

[22] J. Roberts, *Irreducibility of some quantum representations of mapping class groups*, J. Knot Theory Ramifications **10** (2001), no. 5, 763–767, DOI 10.1142/S021821650100113X. Knots in Hellas '98, Vol. 3 (Delphi). MR1839700 (2002f:57065)

[23] V. G. Turaev, *The Conway and Kauffman modules of a solid torus* (Russian, with English summary), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **167** (1988), no. Issled. Topol. 6, 79–89, 190, DOI 10.1007/BF01099241; English transl., J. Soviet Math. **52** (1990), no. 1, 2799–2805. MR964255 (90f:57012)

[24] V. G. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. École Norm. Sup. (4) **24** (1991), no. 6, 635–704. MR1142906 (94a:57023)

[25] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter & Co., Berlin, 1994. MR1292673 (95k:57014)

[26] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), no. 3, 351–399. MR990772 (90h:57009)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90089-2532

E-mail address: `fbonahon@math.usc.edu`

DEPARTMENT OF MATHEMATICS, CARLETON COLLEGE, NORTHFIELD, MINNESOTA 55057

E-mail address: `hwong@carleton.edu`