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TORSION SECTIONS OF ABELIAN FIBRATIONS

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Abstract. Let K be a number field, and let B be a smooth projective curve over K of genus ≤ 1 such that $B(K)$ is infinite. Let A be an Abelian variety defined over the function field $K(B)$. Suppose there are infinitely many non-trivial, pairwise disjoint extensions L/K of bounded degree such that $B(L)$ is infinite and that $A_P(L)_{\text{tor}} \neq 0$ for every point $P \in B(L)$ at which the specialization A_P is smooth. We show that $A(K(B))_{\text{tor}} \neq 0$. If $A/K(B)$ is not a constant Abelian variety over K, the extensions L/K need not have bounded degree, and we can replace $B(K)$ being infinite by $B(K) \neq \emptyset$. This provides evidence in support of a question of Graber-Harris-Mazur-Starr on rational pseudo-sections of arithmetic surjective morphisms.

1. Introduction

Let $X \subset \mathbf{P}^n$ be a variety of positive dimension defined over a field K. In [4], Graber-Harris-Mazur-Starr investigate various scenario under which the geometry of X and the algebra of K would guarantee that X has a K-rational point. One of the questions they raise is the following [4, Question 7, p. 540]:

> Let K be a number field, and let C be a smooth curve of genus ≥ 1 over the function field $K(t)$. Suppose that for every non-trivial algebraic extension L/K and every element $t_0 \in L$ such that C_t has a smooth specialization at t_0, the curve C_{t_0} has an L-rational point. Does C have a $K(t)$-rational point?

In this paper we provide evidence in support of this question. Before we state our result we first set up some notation.

For the rest of this paper, let K be a number field, and let L/K be a finite extension. Let B be a smooth projective curve over K of genus ≤ 1. Let D be a fixed ample, K-rational Cartier divisor of B. Fix a multiplicative height H_D on B with respect to D. For any Zariski dense subset S of B, define

$$S(L) = \{P \in S : P \text{ is a closed point defined over } L\},$$
$$S(L, x, D) = \{P \in S(L) : H_D(P) \leq x\}.$$

Let A be an Abelian variety of dimension > 0 defined over the function field $K(B)$. Denote by $\pi : \mathcal{A} \to B$ the associated Néron model over K. For any closed point $P \in B$, denote by $\mathcal{A}_P$ the fiber of π above P, and by $U \subset B$ the Zariski open set whose closed points are those for which $\mathcal{A}_P$ is smooth. We say that π is isotrivial over K (equivalently: A is a constant Abelian variety over K) if there

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exists an Abelian variety A_0 defined over K such that A and A_0 are isomorphic over $\overline{K}(B)$.

Theorem 1.1. *With the notation as above, suppose that*

(a) *there exist infinitely many non-trivial finite extensions L/K which are pairwise disjoint over K, such that $B(L)$ is infinite and*[1]

$$\#\{P \in U(L,x,D) : \mathcal{A}_P(L)_{\mathrm{tor}} \neq 0\} \geq \#U(L,x,D)^{\lambda_L} + O_L(1) \tag{1.1}$$

for some constant $\lambda_L > 1/2$;

(b) *the extensions L/K in* (a) *have bounded degree; and*

(c) *$B(K)$ is infinite.*

Then $A(K(B))_{\mathrm{tor}} \neq 0$. Furthermore, if π is not isotrivial over K, we can drop condition (b) *and replace* (c) *by*

(c') *If B has genus* 0*, then $B(K) \neq \emptyset$.*

To prove this theorem, first we use an effective variant of Artin's conjecture on primitive roots (Lemma 2.1) to show that condition (1.1) for a given non-trivial finite extension L/K implies that $A(L(B))_{\mathrm{tor}} \neq 0$ (cf. Lemma 2.4). By varying L we deduce that $A(K(B))_{\mathrm{tor}} \neq 0$; a separate argument is needed to handle the case of constant Abelian varieties. In connection with the first step, note that the hypothesis $\lambda_L > 1/2$ is optimal; cf. Remark 2.5.

Remark 1.2. After we completed our manuscript, we learned of the papers [2], [3] which contain significant results concerning variation of Galois representations in one-parameter families. Some of these results have a similar flavor to ours; we now discuss these results in the context of our paper.

Using monodromy techniques, Ellenberg-Hall-Kowalski [3, Thm. 7] show that for every $d \geq 1$ there exists an integer $\ell(d)$ such that, for every prime $\ell > \ell(d)$, the set

$$\bigcup_{[L:K]=d} \{P \in U(L) : \mathcal{A}_P(L) \text{ has a non-zero } \ell\text{-torsion point}\} \tag{1.2}$$

is finite. This result, coupled with conditions (a) and (b), has a similar flavor to Corollary 2.3 below, which guarantees the existence of a prime ℓ and a positive portion of points $P \in B(L)$ at which $\mathcal{A}_P(L)$ has non-trivial ℓ-torsions. However, it does not replace Corollary 2.3 for two reasons: Condition (a) does not preclude the possibility that for any given finite extension L/K:

(i) for any finite set of primes Σ, only finitely many fibers $\mathcal{A}_P(L)$ have non-trivial ℓ-torsions for some $\ell \in \Sigma$; or

(ii) for every $P \in U(L)$, every $\#\mathcal{A}_P(L)_{\mathrm{tor}}$ is a q-power order for some prime $q < \ell([L:K])$.

Under either one of these scenarios, we cannot apply (1.2) in conjunction with conditions (a) and (b) to deduce Corollary 2.3.

Cadoret-Tamagawa [2, Thm. 1.1] recently proved the ℓ-primary torsion conjecture for Abelian fibrations over a curve. Specifically, for any prime ℓ, they show that there exists a constant $N_1(L,\ell) > 0$ such that for all $P \in U(L)$, the ℓ-power torsions of $\mathcal{A}_P(L)$ have order $\leq \ell^{N_1(L,\ell)}$. It is very likely that (1.2) can be extended

[1] Every constant in this paper (asymptotic or otherwise) depends on $\mathcal{A}, B, D, K, \pi$ and those quantities (if any) adorning the corresponding sign. The *asymptotic* constants could change from one line to another, as usual.

to all primes as follows: For any prime ℓ there exists a constant $N_2(\ell, d)$ such that if $m > N_2(\ell, d)$, then the set

$$\bigcup_{[L:K]=d} \{P \in U(L) : \mathcal{A}_P(L) \text{ a point of order } \ell^m\}$$

is finite. Combine such a result for each $\ell \leq \ell(d)$ with the work of Cadoret-Tamagawa and we can resolve issue (ii). However, issue (i) remains. Furthermore, if π is not isotrivial, then we no longer impose condition (b), in which case (1.2) is not applicable.

The issues raised above should not be interpreted as a limitation of [2], [3]: These are significant works concerning a different set of problems for which the restriction on ℓ and $[L : K]$ are natural and necessary, and their connection with the questions of Graber-Harris-Mazur-Starr (for which it would be natural to remove these restrictions) are only tangential. Also, to streamline this discussion we have simplified the statements of results we cited from [2], [3] (but without affecting the conclusion). With that in mind, this analysis shows that these powerful results are not directly applicable to our situation. Note that these works are based on deep monodromy techniques, while ours relies on basic theory of heights; it would be interesting to see if diophantine techniques can be applied to other monodromy problems.

2. The absolute case

Denote by $\mathcal{O}_L$ the ring of integers of L. Given a maximal ideal $\mathfrak{q} \subset \mathcal{O}_L$, denote by $\mathbf{F}_\mathfrak{q}$, $N_\mathfrak{q}$ and $e_\mathfrak{q}$ its residue field, absolute norm and absolute ramification index, respectively.

Fix a K-rational model of $\pi : \mathcal{A} \to B$. Then there exists an integer M_π depending only on π such that for any finite extension L/K, we can reduce π modulo any $\mathfrak{q} \subset \mathcal{O}_L$ with $\mathfrak{q} \nmid M_\pi$. Any two such models behave differently modulo $\mathfrak{q}$ for at most finitely many $\mathfrak{q}$. That does not affect our argument or conclusion. We can assume that $M_\pi \geq 2$. Let $Z = B - U$; it is a proper Zariski closed set.

A variant of Artin's conjecture on primitive roots ([10], [1]) says that given a curve C/L of genus ≤ 1 and a finite generated subgroup $\Gamma \subset C(L)$ of positive rank, the image of Γ modulo $\mathfrak{q}$ is large. The proof of the following lemma comes down to a weak but effective analog of such results; it is weaker in that our reduction has small image, but the proof gives an effective bound for the norm of the smallest such $\mathfrak{q}$ (needed for subsequent applications). Unlike [10], [1] we rely entirely on diophantine techniques.

Lemma 2.1. *Let $\pi : \mathcal{A} \to B$ be as in the theorem. For any finite extension L/K where $B(L)$ is infinite, there exists a constant $C_3(L) > 0$ such that for any $\mathfrak{p} \subset \mathcal{O}_L$ with $N_\mathfrak{q} > C_3(L)$,*

(2.1) $$\#\{P \in U(L, x, D) : \mathcal{A}_P \text{ has good reduction at } \mathfrak{q}\} \gg_{L,\mathfrak{q}} \#U(L, x, D).$$

In particular, there exist smooth fibers over L with good reduction at $\mathfrak{q}$. Furthermore, if $B(K)$ is infinite, then we can take $C_3(L) = C_2(K)^{[L:\mathbf{Q}]}$ for some constant $C_2(K) > 0$.

Proof. First, suppose B has genus 0. The ample divisor D gives rise to a K-embedding $\psi : B \to \mathbf{P}^N$, and we can take H_D to be the restriction to $\psi(B)$ of the usual exponential height on $\mathbf{P}^N$. Regardless of whether π is isotrivial over K, the hypothesis in the theorem guarantees that $B(K)$ contains at least one point, say P_0. Using this point we can define a K-isomorphism $\phi : \psi(B)\to\mathbf{P}^1$. Since $\deg\phi = 1$ and $\deg\psi = \deg D$, by [6, Thm. B.2.5(b)] we have[2]

$$H_{B,D}(P) \gg\ll H_{\mathbf{P}^1,\infty}((\phi\psi)(P))^{\deg D}. \tag{2.2}$$

Since $\phi\psi$ is bijective on L-rational points of B and $\mathbf{P}^1$, it follows that

$$\begin{aligned} &\#\{P \in U(L, x, D) : \mathcal{A}_P \text{ has good reduction at } \mathfrak{q}\} \\ \gg\ll \quad &\#\{Q \in \mathbf{P}^1(L, x^{1/\deg D}, \infty) : \mathcal{A}_Q \text{ has good reduction at } \mathfrak{q}\}. \end{aligned} \tag{2.3}$$

Recall that our chosen model for π is $\mathfrak{q}$-integral for any $\mathfrak{q} \nmid M_\pi$. For such a $\mathfrak{q}$ and for any point $Q \in B(L)$, the fiber $\mathcal{A}_Q$ has good reduction at $\mathfrak{q}$ if and only if $Q \pmod{\mathfrak{q}} \notin Z(\mathbf{F}_q)$. From the proof of Schanual's theorem over number fields [8, Thm. 3.5.3] we readily see that

$$\#\left\{Q \in \mathbf{P}^1(L, X, \infty) : \begin{array}{c} Q \text{ is } \mathfrak{q}\text{-integral and} \\ Q \pmod{\mathfrak{q}} \notin Z(\mathbf{F}_q) \end{array}\right\} \gg \left(1 - \frac{\deg Z}{N_\mathfrak{q}}\right)\cdot\#\mathbf{P}^1(L, X, \infty).$$

Recall (2.3), and the lemma follows if B has genus 0. Note that in the argument above, the only restriction on $\mathfrak{q}$ is that $\mathfrak{q} \nmid M_\pi$; in particular, any $\mathfrak{q}$ with norm $> M_\pi^{[L:\mathbf{Q}]}$ will do.

Next, suppose B has genus 1. Let L/K be a finite extension for which $B(L)$ is infinite, so B/L is an elliptic curve. Néron's theorem then [6, Thm. B.6.3] says that

$$\#B(L, x, D) \gg\ll_{H_D} \left(\frac{V_{\mathrm{rank}(B/L)}}{\mathrm{Reg}_{B/L}} + o_{B,D,L}(1)\right)(\log x)^{\mathrm{rank}(B/L)/2}, \tag{2.4}$$

where V_n denotes the volume of the n-dimensional unit ball, and $\mathrm{Reg}_{B/L}$ denotes the regulator of $B(L)$. In general, not every point of $B(\mathbf{F}_\mathfrak{q})$ comes from a point of $B(L)$, so the argument for $\mathbf{P}^1$ does not apply; instead we mimic the proof of (2.4). Any two multiplicative heights on an elliptic curve with respect to a given ample divisor D differ by multiplicative $O(1)$-constant depending on B, D and the choice of height functions, so it suffices to consider the case where H_D is the exponential[3] of the canonical height on $B(L)$ with respect to D.

Given a finite extension L/K for which $B(L)$ is infinite, fix a set of points $P_1, \ldots, P_r \in B(L)$ which are linearly independent in $B(L)/B(L)_{\mathrm{tor}}$. The canonical height defines a quadratic form on $B(L)/B(L)_{\mathrm{tor}} \otimes \mathbf{R}$; denote by $|| \; ||$ the associated

[2]We include the curves in the notation for heights to avoid confusion. Also, strictly speaking, the asymptotic constant depends also on ϕ, which is fixed and depends on B and $P_0 \in B(K)$. So by our convention for asymptotic constants, we will drop the reference to ϕ to simplify the notation. Finally, by [6, Thm. B.2.5(b)] any two choices for $H_{B,D}$ differ by an $O(1)$-multiple that depends only on the heights, so (2.2) remains valid.

[3]Néron's theorem is naturally stated in terms of (logarithmic) canonical height, while Schanual's theorem is commonly stated in terms of exponential height. To get a uniform statement and *proof* of our theorem we need to fix at the outset H_D to be either exponential or logarithmic, resulting in a height function that will always be unnatural regardless of the genus of B.

norm. Denote by $[n]$ the multiplication-by-n isogeny on B. Define

$$\begin{array}{rcl} \mathcal{L} & := & \text{sublattice of } B(L)/B(L)_{\mathrm{tor}} \otimes \mathbf{R} \text{ generated by the } P_i, \\ \mathcal{B}_r(R) & := & \{x \in \mathcal{L} \otimes \mathbf{R} : ||x|| \le R\}, \\ \alpha & := & \text{a fixed integer} > \deg Z, \\ F_\alpha & := & \{\sum_i a_i P_i : a_i \in \mathbf{Z}, 0 \le a_i < \alpha\}, \\ \tau_\alpha & := & \max\{||P|| : P \in F_\alpha\}, \\ \Lambda_\alpha & := & \text{sublattice in } \mathcal{L} \otimes \mathbf{R} \text{ generated by } \alpha P_1, \cdots, \alpha P_r, \\ F_{\alpha,\lambda} & := & \text{translate of } F_\alpha \text{ by } \lambda \in \Lambda_\alpha, \\ F_{\alpha,\lambda}^\bullet & := & \text{convex hull of } F_{\alpha,\lambda}. \end{array}$$

For $M > \tau_\alpha$, the triangle inequality gives the inclusions

$$\mathcal{B}_r(M - \tau_\alpha) \quad \subseteq \quad \bigcup_{\substack{\lambda \in \Lambda_\alpha \\ ||\lambda|| \le M - \tau_\alpha}} F_{\alpha,\lambda}^\bullet \quad \subseteq \quad \mathcal{B}_r(M). \tag{2.5}$$

Take volume with respect to $||\,||$ on both sides of the first inclusion, note that different $F_{\alpha,\lambda}^\bullet$ have the same volume as $F_\alpha^\bullet$, and thus get

$$\mathrm{vol}(\mathcal{B}_r(M - \tau_\alpha)) \le \mathrm{vol}(F_\alpha^\bullet) \cdot \#\{\lambda \in \Lambda_\alpha : \ ||\lambda|| \le M - \tau_\alpha\}.$$

$F_\alpha^\bullet$ is the α-multiple of (the closure of) a fundamental domain for the lattice $\mathcal{L}$, so $\mathrm{vol}(F_\alpha^\bullet)$ is equal to α^r times the regulator of $B(L)$. Also, $\mathrm{vol}(\mathcal{B}_r(x))$ is x^r times V_r, the volume of the unit r-sphere. Thus

$$\#\{\lambda \in \Lambda_\alpha : ||\lambda|| \le M - \tau_\alpha\} \ge \frac{V_r}{\mathrm{Reg}_{B/L}} \Big(\frac{M - \tau_\alpha}{\alpha}\Big)^r. \tag{2.6}$$

From the second inclusion in (2.5) we get, for $M > \tau_\alpha$,

$$\begin{aligned} &\#\{P \in B(L) : \ ||P|| \le M \text{ and } \mathcal{A}_P \text{ has good reduction at } \mathfrak{q}\} \\ \ge &\sum_{\substack{\lambda \in \Lambda_\alpha \\ ||\lambda|| \le M - \tau_\alpha}} \#\{P \in B(L) \cap F_{\alpha,\lambda} : \ \mathcal{A}_P \text{ has good reduction at } \mathfrak{q}\}. \end{aligned}$$

Recall that the K-rational model for π we fixed at the outset is $\mathfrak{q}$-integral for all $\mathfrak{q} \subset \mathcal{O}_L$ with $\mathfrak{q} \nmid M_\pi$. For any fixed $\alpha > \deg Z$, the points in the finite set F_α are pairwise distinct, so for $N_\mathfrak{q} \gg_{\alpha,L} 1$, the reduction modulo $\mathfrak{q}$ of the points in F_α remain pairwise distinct. Fix such a $\mathfrak{q}$. Reduction is a group homomorphism (the proof in [11, Prop. 7.2.1] applies verbatium to Abelian varieties), so for any $\lambda \in \Lambda_\alpha$ the points in $B(L) \cap F_{\alpha,\lambda}$ have pairwise distinct reduction modulo $\mathfrak{q}$, and hence $B(L) \cap F_{\alpha,\lambda}$ gives rise to at least $\alpha^r - \deg Z$ distinct points in $U(\mathbf{F}_\mathfrak{q})$ upon reduction. Consequently, for such $\mathfrak{q}$ and for $M > \tau_\alpha$,

$$\begin{aligned} &\#\{P \in B(L) : ||P|| \le M \text{ and } \mathcal{A}_P \text{ has good reduction at } \mathfrak{q}\} \\ &\ge \sum_{\substack{\lambda \in \Lambda_\alpha \\ ||\lambda|| \le M - \alpha}} (\alpha^r - \deg Z) \ \gg \ \frac{V_r}{\mathrm{Reg}_{B/L}} M^r \Big(1 - \frac{\tau_\alpha}{M}\Big)^r \Big(1 - \frac{\deg Z}{\alpha^r}\Big) \qquad \text{by (2.6).} \end{aligned}$$

Since α is a fixed integer $> \deg Z$, the estimate (2.1) for genus one B now follows from (2.4) and the remark at the end of that paragraph.

Finally, suppose $B(K)$ is infinite. Then we can take the points $P_1, \ldots, P_r$ to be generators of $B(K)/B(K)_{\mathrm{tor}}$, in which case there exists a constant $C_1(B, K, \alpha) > 0$ such that for all primes $\mathfrak{p} \subset \mathcal{O}_K$ with $N_\mathfrak{p} > C_1(B, K, \alpha)$, the points in F_α are all

$\mathfrak{p}$-integral and their reduction modulo $\mathfrak{p}$ are pairwise distinct. The same conclusion then holds for $F_\alpha \pmod{\mathfrak{q}}$ for any $\mathfrak{q} \subset \mathcal{O}_L$ of norm $> C_1(B,K,\alpha)^{[L:K]}$, and we obtain the last part of the lemma for genus one B. $\square$

For the proof of the next lemma we will make use of the standard correspondence between geometric points of the generic fiber of π and *multisections* of $\pi : \mathcal{A} \to B$, i.e. geometrically irreducible (but possibly singular) curves $\mathcal{T} \subset \mathcal{A}$ such that $\pi|_{\mathcal{T}}$ is a finite cover of B. Specifically, for any finite extension L/K, the $\mathrm{Gal}(\overline{L(B)}/L(B))$-orbits of points $Q \in A(\overline{L(B)})$ are in bijective correspondence with the L-rational, L-irreducible multisections $\mathcal{T}(Q)$ of π, and $\deg \pi|_{\mathcal{T}(Q)} = [L(B)(T(Q)) : L(B)]$. For any integer n, denote by $[n]$ the group of n-torsion points of $A(\overline{L(B)})$.

Lemma 2.2. *Let $\pi : \mathcal{A} \to B$ be as in the theorem. Let L/K be a finite extension for which $B(L)$ is infinite. Then for all but finitely many $P \in U(L)$, if $t_P \in \mathcal{A}_P(\overline{L})$ has prime order ℓ, then there exists a point $Q \in A[\ell](\overline{L(B)})$ such that $t_P \in \mathcal{T}(Q)$.*

Proof. Suppose $Q \in A[\ell](\overline{L(B)})$ and $Q \neq 0$. Then for every $P \in U(L)$, every $\overline{L}$-rational point in $\mathcal{T}(Q)_P := \mathcal{T}(Q) \cap \mathcal{A}_P$ has order dividing ℓ. If this order is not ℓ, then $\mathcal{T}(Q)$ would intersect the zero section. Distinct L-irreducible multisections intersect at most finitely many points, so since $Q \neq 0$, for all but finitely many $P \in U(L)$, every $\overline{L}$-rational point of $\mathcal{T}(Q)_P$ has order ℓ. Denote by $\mathcal{E}_\ell$ the set of exceptional P for at least one non-zero point $Q \in A[\ell](\overline{L(B)})$; it is a finite set.

$\mathcal{T}(Q)_P$ is precisely the fiber of the finite cover $\pi|_{\mathcal{T}(Q)} : \mathcal{T}(Q) \to B$ above P, and the degree of this cover is the degree of the field extension $K(B)(Q)/K(B)$. Since we are in characteristic zero, for all $P \in U(L) \backslash \mathcal{E}_\ell$ this fiber consists of $\deg(\pi|_{\mathcal{T}(Q)})$ distinct points in $\mathcal{A}_P(\overline{L})$. So for all $P \in U(L) \backslash \mathcal{E}_\ell$,

$$\begin{aligned}
& \#\{x \in \mathcal{A}_P[\ell](\overline{L}) : x \text{ is contained in an } L\text{-rational torsion multisection}\} \\
= & \sum_{Q \in A[\ell](\overline{K(B)})} \deg(\pi|_{\mathcal{T}(Q)}) \\
= & \deg[\ell]|_A \\
= & \deg[\ell]|_{\mathcal{A}_P}
\end{aligned}$$

and the lemma follows. $\square$

Corollary 2.3. *Let $\pi : \mathcal{A} \to B$ and L/K be as in Lemma 2.2. Fix a prime ℓ, and let S be an infinite subset of $U(L)$ such that $\mathcal{A}_P(L)$ contains a point t_P of order ℓ for every $P \in S$. Then there exists a torsion point $Q \in A(\overline{K(L)})$ such that the corresponding multisection $\mathcal{T}(Q)$ contains t_P for $\gg_{\ell, \dim A} \#S(L,x,D)$ of the points $P \in S(L,x,D)$.*

Proof. As P runs through elements of $S \subset U(L)$, Lemma 2.2 says that each t_P is contained in exactly one of the multisections of π corresponding to an ℓ-torsion point of $A(\overline{K(B)})$. There are $\ell^{2\dim A} - 1$ such torsion points over $\overline{K(B)}$; apply the pigeon-hole principle and we are done. $\square$

Lemma 2.4. *Let $\pi : \mathcal{A} \to B$ be as in the theorem. Let L/K be a finite extension such that $B(L)$ is infinite and there exists a constant $\lambda_L > 1/2$ with*

$$\#\{P \in U(L,x,D) : \mathcal{A}_P(L)_{\mathrm{tor}} \neq 0\} \gg_L \big(\#U(L,x,D)\big)^{\lambda_L}. \tag{2.7}$$

Then $A(L(B))_{\mathrm{tor}} \neq 0$.

Proof. With $C_3(L) > 0$ as in Lemma 2.1, fix $\mathfrak{q} \subset \mathcal{O}_L$ such that

$$N_{\mathfrak{q}} > C_4(\pi, L) := \max\{1 + [L : \mathbf{Q}], C_3(L), M_\pi^{[L:\mathbf{Q}]}\}. \tag{2.8}$$

From $N_{\mathfrak{q}} > M_\pi^{[L:\mathbf{Q}]}$ we see that $\mathfrak{q} \nmid M_\pi$, so we can reduce π modulo $\mathfrak{q}$. By [9, p. 502] (which applies to the *full* torsion, not just the prime-to-$N_{\mathfrak{q}}$ part), the requirement $N_{\mathfrak{q}} - 1 > e_{\mathfrak{q}}$ implies that for any $P \in U(L)$ at which $\mathcal{A}_P$ has good reduction at $\mathfrak{q}$, we have

$$\#\mathcal{A}_P(L)_{\mathrm{tor}} \le \#\mathcal{A}_P(\mathbf{F}_{\mathfrak{q}}) \le (1 + 2\dim A \cdot N_{\mathfrak{q}}^{1/2})^{2\dim A}. \tag{2.9}$$

Since $N_{\mathfrak{q}} > C_3(L)$, apply Lemma 2.1 and get that $\#\mathcal{A}_P(L)_{\mathrm{tor}}$ is bounded for $\gg_L \#U(L, x, D)$ of the points P in $U(L, x, D)$. Condition (2.7) then implies that there exists a *prime* ℓ with

$$\#\{P \in U(L, x, D) : \mathcal{A}_P(L)[\ell] \neq 0\} \gg_{L,\ell} \big(\#U(L, x, D)\big)^{\lambda_L}.$$

By Corollary 2.3, we can find a non-zero point $Q \in A[\ell](\overline{K(B)})$ such that the L-rational, L-irreducible multisection $\mathcal{T} := \mathcal{T}(Q)$ satisfies

$$\#\{P \in U(L, x, D) : \mathcal{T} \cap \mathcal{A}_P(L)_{\mathrm{tor}} \neq 0\} \gg_{L,\ell} \big(\#U(L, x, D)\big)^{\lambda_L}. \tag{2.10}$$

In particular, $\mathcal{T}(L)$ is infinite, so $\mathcal{T}$, being L-irreducible, is absolutely irreducible, whence a (possibly singular) curve over L of geometric genus $p_g(\mathcal{T}) \le 1$. Note that $\pi|_{\mathcal{T}} : \mathcal{T} \to B$ is a morphism and the image of $\mathcal{T}(L)$ is contained in

$$\{P \in U(L) : \mathcal{T} \cap \mathcal{A}_P(L)_{\mathrm{tor}} \neq 0\} \cup Z(L).$$

So if $\deg \pi|_{\mathcal{T}} > 1$, then this image must be a thin subset of $B(U)$, contradicting (2.10) since $\lambda_L > 1/2$. Thus $\deg \pi|_{\mathcal{T}} = 1$, whence $\mathcal{T}$ is an actual section, as desired. $\square$

Remark 2.5. For any integer $d \neq 0, 1, -432$, the torsion subgroup of $y^2 = x^3 + d$ over $\mathbf{Q}$ is $\mathbf{Z}/3, \mathbf{Z}/2$ or trivial, depending on whether D is a square, a cube, or neither [7, p. 34]. Thus the curve $y^2 = x^3 + t$ has no non-trivial torsion point over $\mathbf{Q}(t)$, while $\gg\ll [\#\mathbf{P}^1(\mathbf{Q}, x, \infty)]^{1/2}$ of its specializations above rational points on $\mathbf{P}^1(\mathbf{Q})$ of height $< x$ do. In particular, the condition $\lambda_L > 1/2$ is optimal.

3. Proof of Theorem 1.1

For every non-trivial finite extension L/K furnished by condition (a) in the theorem, Lemma 2.4 produces a non-trivial torsion section $\mathcal{T}_L$ of π over L. If A is not a constant Abelian variety over $\overline{K}$, then the Mordell-Weil group of A over $\overline{K}(B)$ is finitely generated [8, Thm. 6.2]; in particular, $\#A(L(B))_{\mathrm{tor}}$ is bounded independently of the finite extension L/K. By hypothesis there are infinitely many such L/K, so $\mathcal{T}_{L_1} = \mathcal{T}_{L_2}$ for at least two such extensions L_i/K. This common, non-trivial torsion section is then defined over $L_1 \cap L_2$, which by hypothesis is K, and we are done.

Now, suppose A is a constant Abelian variety over $\overline{K}$. Then $A(\overline{K}(B))$ is no longer finitely generated and we need to proceed differently. We now give an alternative, arithmetic argument that is applicable to all $A/K(B)$ under the additional conditions (b) and (c).

By condition (c), we can find a point $P_0 \in U(K)$. Fix a K-rational affine open set $W \subset B$ containing P_0 such that $\pi|_W$ is smooth. Denote by R_W the corresponding affine coordinate ring. The point P_0 is still contained in $W(L)$ for any finite extension L/K; denote by $\hat{R}_{W,P}$ the completion of $R_W \otimes_K L$ at the maximal ideal $\mathfrak{m}_{P_0}$ of $R_W \otimes_K L$ corresponding to P_0, and denote by $\hat{L}_{P_0}$ its field of fractions. Then the formal group argument in [11, Prop. VII.3.1], which generalizes readily to Abelian varieties and does *not* require $\hat{R}_{W,P_0}$ to have a finite residue field (only that the residue field be perfect), implies that $A(\hat{L}_{P_0})_{\text{tor}}$ injects into $\mathcal{A}_P(\hat{R}_{W,P_0}/\mathfrak{m}_{P_0}) \simeq \mathcal{A}_{P_0}(L)$ under the specialization map, whence

$$\#A(L(B))_{\text{tor}} \le \#A(\hat{L}_{P_0})_{\text{tor}} \le \#\mathcal{A}_{P_0}(L)_{\text{tor}}. \tag{3.1}$$

Since $B(K)$ is infinite, the constant $C_3(L)$ in the proof of Lemma 2.4 is equal to $C_2(K)^{[L:\mathbf{Q}]}$, whence the constant $C_4(\pi, L)$ in (2.8) depends only on π and $[L:\mathbf{Q}]$ and not on L. Thus we can write this constant as $C_4(\pi, [L:\mathbf{Q}])$. Fix a prime $\mathfrak{q} \subset \mathcal{O}_L$ such that

(i) P_0 is $\mathfrak{q}$-integral,
(ii) $N_\mathfrak{q} > C_4(\pi, [L:\mathbf{Q}])$, and
(iii) $\mathcal{A}_{P_0}$ has good reduction at $\mathfrak{q}$.

Then the argument leading to (2.9) plus (3.1) now gives

$$\#A(L(B))_{\text{tor}} \le (1 + 2\dim A \cdot N_\mathfrak{q}^{1/2})^{2\dim A}. \tag{3.2}$$

The right side is bounded from above in terms of $P_0, \mathfrak{q}, \dim A$ and $[L:\mathbf{Q}]$. But we can do better.

Claim. $\#A(L(B))_{\text{tor}}$ is bounded from the above in terms of $P_0, \pi, \dim A$ and $[L:\mathbf{Q}]$ only.

It follows from the claim that $\#A(L(B))_{\text{tor}}$ is uniformly bounded as L/K runs through all non-trivial finite extensions of bounded degree. By conditions (a) and (b) of the theorem and Lemma 2.4, we can find two distinct finite extensions $L_1/K, L_2/K$ of bounded degree with $\mathcal{T}_{L_1} = \mathcal{T}_{L_2}$ and $L_1 \cap L_2 = K$, whence this common non-trivial torsion section is defined over K, and the theorem follows.

It remains to verify the claim; we do that by bounding the norm of $\mathfrak{q} \subset \mathcal{O}_L$ that satisfy (i)-(iii). Since $P_0 \in B(K)$, condition (i) holds provided $N_\mathfrak{q} > C_5(K)^{[L:\mathbf{Q}]}$ for some constant $C_5(K) > 0$. Given such a $\mathfrak{q}$, for any finite extension L/K the fiber $\mathcal{A}_{P_0}$ has bad reduction at $\mathfrak{q}$ if and only if $P_0 \in U(K)$ is congruent to a point $P' \in Z(L)$ modulo a prime $\mathfrak{q}$. Thus there are at most finitely many choices (independent of L) for the residual characteristic of exceptional $\mathfrak{q}$, since P_0 and Z are fixed. Let $C_6(P_0, \pi)$ be the largest of these finitely many exceptional residual characteristics. Set $C_7(\pi, [L:\mathbf{Q}], P_0) = C_4(\pi, [L:\mathbf{Q}]) + C_5(K)^{[L:\mathbf{Q}]} + C_6(P_0, \pi)$. Bertrand's postulate [5, p. 343] then furnishes a rational prime q in the interval

$$[\, C_7(\pi, [L:\mathbf{Q}], P_0),\ 2 \cdot C_7(\pi, [L:\mathbf{Q}], P_0)\,]; \tag{3.3}$$

then any prime $\mathfrak{q} \subset \mathcal{O}_L$ lying above q would satisfy the conditions (i)-(iii). Note that $N_\mathfrak{q} \le q^{[L:\mathbf{Q}]}$ depends only on $[L:\mathbf{Q}], \pi$ and P_0. Recall (3.2) and the claim follows. □

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