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SOME POSITIVITIES IN CERTAIN TRIANGULAR ARRAYS

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Abstract. Let $\{T_{n,k}\}_{n,k\geq 0}$ be an array of nonnegative numbers satisfying the recurrence relation

$$T_{n,k} = (a_1k^2 + a_2k + a_3)T_{n-1,k} + (b_1k^2 + b_2k + b_3)T_{n-1,k-1}$$

with $T_{n,k} = 0$ unless $0 \leq k \leq n$. We obtain some results for the total positivity of the matrix $\left(T_{n,k}\right)_{n,k\geq 0}$, Pólya frequency properties of the row and column generating functions, and q-log-convexity of the row generating functions. This allows a unified treatment of the properties above for some triangular arrays of the second kind, including the Stirling triangle, Jacobi-Stirling triangle, Legendre-Stirling triangle, and central factorial numbers triangle.

1. Introduction

The Jacobi-Stirling numbers $\mathrm{JS}_n^k(z)$ of the second kind, which were introduced in [14], are the coefficients of the integral composite powers of the Jacobi differential operator

$$\ell_{\alpha,\beta}[y](t) = \frac{1}{(1-t)^{\alpha}(1+t)^{\beta}}\left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t)\right)',$$

with fixed real parameters $\alpha, \beta \geq -1$. They also satisfy the following recurrence relation:

$$\begin{cases} \mathrm{JS}_0^0(z) = 1, \qquad \mathrm{JS}_n^k(z) = 0, \quad \text{if } k \notin \{1,\ldots,n\}, \\ \mathrm{JS}_n^k(z) = \mathrm{JS}_{n-1}^{k-1}(z) + k(k+z)\,\mathrm{JS}_{n-1}^k(z), \quad n,k \geq 1, \end{cases} \tag{1.1}$$

where $z = \alpha + \beta + 1$. In fact, these numbers are a generalization of the Legendre-Stirling numbers of the second kind: it suffices to choose $\alpha = \beta = 0$. Recently, the Jacobi-Stirling numbers and Legendre-Stirling numbers have generated a significant amount of interest from some researchers in combinatorics. It was found that the Jacobi-Stirling numbers and the Legendre-Stirling numbers share many properties with the classical Stirling numbers, such as similar recurrence relations, generating functions and total positivity properties; see Andrews *et al.* [1–3], Egge [12], Everitt *et al.* [13, 14], Gelineau and Zeng [15], and Mongelli [20, 21] for details. Moreover, several combinatorial interpretations of the Legendre-Stirling numbers

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(see Andrews and Littlejohn [3] and Egge [12] for instance) and the Jacobi-Stirling numbers (see Andrews *et al.* [1], Gelineau and Zeng [15], and Mongelli [21] for instance) have been found.

In this paper, we consider a generalization of Jacobi-Stirling numbers. Let $\{T_{n,k}\}_{0\le k\le n}$ be an array of nonnegative numbers satisfying the recurrence relation

$$T_{n,k} = (a_1k^2 + a_2k + a_3)T_{n-1,k} + (b_1k^2 + b_2k + b_3)T_{n-1,k-1} \tag{1.2}$$

with $T_{n,k} = 0$ unless $0 \le k \le n$ and $T_{0,0} = 1$. It is natural to assume that $a_1k^2 + a_2k + a_3 \ge 0$ for $k \ge 0$ and $b_1k^2 + b_2k + b_3 \ge 0$ for $k > 0$. This implies that $a_1, b_1 \ge 0$. It is clear that $T_{n,k}$ contains the Jacobi-Stirling numbers of the second kind and Legendre-Stirling numbers of the second kind as the special cases.

The aim of this paper is to study some positivities related to the matrix $(T_{n,k})_{n,k\ge0}$. In Section 2, we prove the total positivity of the matrix $(T_{n,k})_{n,k\ge0}$ and obtain the Pólya frequency properties of the row and column generating functions. Finally, we also derive the strong q-log-convexity of the row generating functions by a lemma. In Section 3, as applications of our results, we obtain the properties above for some triangular arrays of the second kind in a unified approach, including the Stirling triangle, Jacobi-Stirling triangle, Legendre-Stirling triangle, and central factorial numbers triangle.

2. Main results

2.1. **Total positivity of the triangular array.** Recall that a matrix $M = (m_{ij})_{i,j\ge0}$ of nonnegative numbers is said to be *totally positive* (TP for short) if all its minors are nonnegative. See Karlin [16] or Brenti [7] for more details.

Theorem 2.1. *Let $\{T_{n,k}\}_{n\ge k\ge0}$ be the nonnegative array as above* (1.2)*. Then the matrix $(T_{n,k})_{n,k\ge0}$ is TP.*

Proof. Let $g(k) = a_1k^2 + a_2k + a_3$ and $f(k) = b_1k^2 + b_2k + b_3$. Assume that the matrices

$$J_n = (J_{i,j}) = \begin{pmatrix} g(0) & f(1) & 0 & \cdots & 0 \\ 0 & g(1) & f(2) & \cdots & 0 \\ 0 & 0 & g(2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & g(n-1) \end{pmatrix},$$

$$Q_n = \begin{pmatrix} T_{1,0} & T_{1,1} & 0 & \cdots & 0 \\ T_{2,0} & T_{2,1} & T_{2,2} & \cdots & 0 \\ T_{3,0} & T_{3,1} & T_{3,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ T_{n,0} & T_{n,1} & T_{n,2} & \cdots & T_{n,n-1} \end{pmatrix}$$

and

$$T_n = (T_{i,j})_{0\le i,j\le n-1} = \begin{pmatrix} T_{0,0} & 0 & 0 & \cdots & 0 \\ T_{1,0} & T_{1,1} & 0 & \cdots & 0 \\ T_{2,0} & T_{2,1} & T_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ T_{n-1,0} & T_{n-1,1} & T_{n-1,2} & \cdots & T_{n-1,n-1} \end{pmatrix},$$

where $J_{i,i} = g(i-1), J_{i,i+1} = f(i)$. It is easy to see that

$$Q_n = T_n J_n.$$

Note that J_n is totally positive. Thus, by induction on n, we derive that the nonnegative matrices Q_n and T_n are TP using the classical Cauchy-Binet formula which expresses a determinant of any submatrix of Q_n as a sum of products of determinants of submatrix of T_n and J_n; see Karlin [16] for details. This completes the proof. □

2.2. **Pólya frequency sequences.** If the matrix $(a_{n-k})_{n,k\geq 0}$ is TP, then the real sequence $\{a_n\}_{n\geq 0}$ is called the *Pólya frequency* (PF, for short) sequence. In particular, a finite sequence of nonnegative numbers is PF if and only if its generating function has only real zeros ([16, p. 399]). PF sequences and polynomials with only real zeros often arise in combinatorics and other branches of mathematics. There is a fair amount of interest in and an extensive literature on this subject; see [4, 5, 18, 25, 29] for instance.

Lemma 2.2 ([5]). *Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be two PF sequences, and let $\{c_n\}_{n\geq 0}$ be defined by $\sum_{n\geq 0} c_n x^n = \sum_{n\geq 0} a_n x^n \sum_{n\geq 0} b_n x^n$. Then $\{c_n\}_{n\geq 0}$ is also PF.*

Theorem 2.3. *Let $\{T_{n,k}\}_{n,k\geq 0}$ be the nonnegative array as above* (1.2)*. Then each column sequence of $\{T_{n,k}\}_{n,k\geq 0}$ is a PF sequence.*

Proof. Let the column generating function $f_k(x) = \sum_{n\geq k} T_{n,k}x^n$. By the recurrence relation (1.2), we have

$$\begin{aligned}
f_k(x) &= \sum_{n\geq k} T_{n,k}x^n \\
&= \sum_{n\geq k} (a_1k^2 + a_2k + a_3)T_{n-1,k}x^n + \sum_{n\geq k} (b_1k^2 + b_2k + b_3)T_{n-1,k-1}x^n \\
&= (a_1k^2 + a_2k + a_3)xf_k(x) + (b_1k^2 + b_2k + b_3)xf_{k-1}(x),
\end{aligned}$$

which implies that

$$f_k(x) = \frac{(b_1k^2 + b_2k + b_3)xf_{k-1}(x)}{1 - (a_1k^2 + a_2k + a_3)x}.$$

Thus, we derive

$$f_k(x) = f_0(x)\prod_{i=1}^{k} \frac{(b_1i^2 + b_2i + b_3)x}{1 - (a_1i^2 + a_2i + a_3)x} = \frac{1}{1 - a_3x}\prod_{i=1}^{k} \frac{(b_1i^2 + b_2i + b_3)x}{1 - (a_1i^2 + a_2i + a_3)x}.$$

So it follows from Lemma 2.2 that the column sequence $\{T_{n,k}\}_{n=k}^{\infty}$ is PF since the sequence $\{a^i\}_{i\geq 0}$ is PF. This completes the proof. □

Theorem 2.4. *Let $\{T_{n,k}\}_{n,k\geq 0}$ be the nonnegative array as above* (1.2) *and its row generating function $T_n(q) = \sum_{k=0}^{n} T_{n,k}q^k$. If $a_3 = b_1 = 0$ and $b_2 + b_3 > 0$, then* (1) *$T_n(q)$ has only simple nonpositive real zeros for $a_1 + a_2 > 0$; and* (2) *0 is a zero of $T_n(q)$ with multiplicity* 2*, and $T_n(q)$ has $n-2$ simple negative real zeros for $a_1 + a_2 = 0$.*

Proof. In the following, we will prove that (1) holds by induction on n. Similarly, we can obtain (2), whose proof is omitted for brevity.

It is easy to see that (1) is true for $n = 1, 2, 3$. Assume that (1) is valid for some $n-1$, where $n \geq 4$. For $a_3 = b_1 = 0$, by the recurrence relation (1.2), we have

$$\begin{aligned} T_n(q) &= \sum_{k=0}^{n}(a_1k^2 + a_2k)T_{n-1,k}q^k + \sum_{k=1}^{n}(b_2k + b_3)T_{n-1,k-1}q^k \\ &= (b_2 + b_3)qT_{n-1}(q) + (a_1 + a_2 + b_2q)qT'_{n-1}(q) + a_1q^2T''_{n-1}(q) \\ &= q\left[(b_2 + b_3)T_{n-1}(q) + (a_1 + a_2 + b_2q)T'_{n-1}(q) + a_1qT''_{n-1}(q)\right]. \end{aligned} \tag{2.1}$$

By the induction hypothesis, $T_{n-1}(q)$ has $n-1$ simple real nonpositive zeros. Consequently, $T'_{n-1}(q)$ has $n-2$ simple real zeros, say

$$0 > x_{n-1,1} > x_{n-1,2} > \ldots > x_{n-1,n-2},$$

which satisfies

$$sign[T_{n-1}(x_{n-1,i})] = (-1)^i \quad \text{and} \quad sign[T''_{n-1}(x_{n-1,i})] = (-1)^{i+1}$$

for $1 \leq i \leq n-2$. Thus, by the recurrence relation (2.1), we have

$$T_n(x_{n-1,i}) = x_{n-1,i}\left[(b_2 + b_3)T_{n-1}(x_{n-1,i}) + a_1x_{n-1,i}T''_{n-1}(x_{n-1,i})\right],$$

which implies

$$sign[T_n(x_{n-1,i})] = (-1)^{i+1} \tag{2.2}$$

for $1 \leq i \leq n-2$. Thus, by the Intermediate Value Theorem, we can obtain $n-3$ zeros $y_{n,i}$ for $i = 3, 4, \ldots, n-1$ of $T_n(q)$ satisfying

$$0 > x_{n-1,1} > y_{n,3} > x_{n-1,2} > y_{n,4} > \cdots > y_{n,n-2} > x_{n-1,n-3} > y_{n,n-1} > x_{n-1,n-2}.$$

Since $q = 0$ is also a zero of $T_n(q)$ by (2.1), we have obtained so far that $T_n(q)$ has $n-2$ zeros. We now claim that $T_n(q)$ must have two more zeros. By (2.2), we obtain that $sign[T_n(x_{n-1,n-2})] = (-1)^{n-1}$, and since $sign[T_n(q)] = (-1)^n$ for $q \to -\infty$, there is also a zero $y_{n,n} < x_{n-1,n-2}$ of $T_n(q)$. Finally, if $a_1 + a_2 > 0$, then, by the definition,

$$\begin{aligned} T'_n(0) = T_{n,1} &= (a_1 + a_2)T_{n-1,1} + (b_2 + b_3)T_{n-1,0} = (a_1 + a_2)T_{n-1,1} \\ &= (a_1 + a_2)^{n-1}(b_2 + b_3) > 0, \end{aligned}$$

which implies that $T_n(q) < 0$ for $q < 0$ and sufficiently close to 0. Furthermore, by $sign[T_n(x_{n-1,1})] = 1$, we derive that $T_n(q)$ has another zero $y_{n,2}$ satisfying

$$x_{n-1,1} < y_{n,2} < 0.$$

Thus, we have determined that $T_n(q)$ has n simple nonpositive zeros. The proof is complete. □

Recall that a sequence $\{a_n\}_{n\geq 0}$ is *unimodal* if there is an index $m \geq 0$ such that $a_0 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots$ and is called *log-concave* if $a_{k-1}a_{k+1} \leq a_k^2$ for all $k \geq 1$. The unimodal and log-concave sequences often arise in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated; see [6,25] for details. It is known that a PF sequence is log-concave and log-concavity implies unimodality for positive sequences. Thus, from Theorems 2.3 and 2.4 we find that the column and row sequences of $\{T_{n,k}\}_{n\geq k\geq 0}$ are log-concave for $a_3 = b_1 = 0$. The sequence of binomial coefficients along any finite transversal of Pascal's triangle is log-concave [26]. See [27,28] for some other general results. In fact, using the method in [27,28], we can obtain the following similar result.

Corollary 2.5. *Let a, b, n_0, k_0 be nonnegative integers. Assume that $\{T_{n,k}\}_{n\geq k\geq 0}$ is the nonnegative array as above* (1.2)*. If $a_3 = b_1 = 0$, then $\{T_{n_0-ai,k_0+bi}\}_{i\geq 0}$ is log-concave and therefore unimodal.*

Proof. To show that $\{T_{n_0-ai,k_0+bi}\}_{i\geq 0}$ is log-concave, it suffices to prove

$$T^2_{n_0-ai,k_0+bi} \geq T_{n_0-ai-a,k_0-bi+b}T_{n_0-ai+a,k_0-bi-b},$$

that is,

$$T^2_{n,k} \geq T_{n-a,k+b}T_{n+a,k-b},$$

where $n = n_0 - ai$ and $k = k_0 + bi$. By Theorems 2.4 and 2.3, we have

$$T^2_{n,k} \geq T_{n,k-b}T_{n,k+b}$$

and

$$T^2_{n,k} \geq T_{n-a,k}T_{n+a,k}.$$

It follows that

$$T^4_{n,k} \geq T_{n,k-b}T_{n,k+b}T_{n-a,k}T_{n+a,k}. \tag{2.3}$$

On the other hand, by Theorem 2.1,

$$\begin{aligned} T_{n+a,k}T_{n,k-b} &\geq T_{n,k}T_{n+a,k-b}, \\ T_{n-a,k}T_{n,k+b} &\geq T_{n,k}T_{n-a,k+b}, \end{aligned}$$

which imply

$$T_{n,k-b}T_{n,k+b}T_{n-a,k}T_{n+a,k} \geq T^2_{n,k}T_{n+a,k-b}T_{n-a,k+b}. \tag{2.4}$$

Thus, it follows from (2.3) and (2.4) that

$$T^4_{n,k} \geq T^2_{n,k}T_{n+a,k-b}T_{n-a,k+b},$$

that is,

$$T^2_{n,k} \geq T_{n+a,k-b}T_{n-a,k+b},$$

as desired. This completes the proof. □

2.3. **Strong q-log-convexity of row generating functions.** For a polynomial $f(q)$ with real coefficients, we write $f(q) \geq_q 0$ whenever $f(q)$ has only nonnegative coefficients. We call a polynomial sequence $\{f_n(q)\}_{n\geq 0}$ *q-log-convex* whenever

$$f_{n+1}(q)f_{n-1}(q) - f_n(q)^2 \geq_q 0 \tag{2.5}$$

for $n \geq 1$. It is called *strongly q-log-convex* if

$$f_{n+1}(q)f_{m-1}(q) - f_n(q)f_m(q) \geq_q 0 \tag{2.6}$$

for any $n \geq m \geq 1$. Similarly, if the opposite inequality in (2.5) or (2.6) holds, then it is called *q-log-concave* or *strongly q-log-concave*, respectively. Clearly, the strong q-log-convexity of polynomial sequences implies the q-log-convexity. However, the converse does not follow. The q-log-concavity and q-log-convexity of polynomials has been extensively studied; see [8–11, 17, 19, 23, 24, 27] for instance. To prove the strong q-log-convexity of the sequence of row generating functions of $\{T_{n,k}\}_{n,k\geq 0}$, we need the next result.

Lemma 2.6. *Given three nonnegative sequences* $\{a_i\}_{i=0}^n$, $\{b_i\}_{i=0}^n$ *and* $\{c_i\}_{i\geq 0}$, *if* $a_ib_j \geq a_jb_i$ *and* c_i *is increasing for* $n \geq i \geq j \geq 0$, *then*

$$\sum_{i=0}^{n} c_ia_ix^i \sum_{i=0}^{n} b_ix^i - \sum_{i=0}^{n} a_ix^i \sum_{i=0}^{n} c_ib_ix^i \geq_q 0.$$

Proof. Note that

$$\begin{aligned}
&\sum_{i=0}^{n} c_ia_ix^i \sum_{i=0}^{n} b_ix^i - \sum_{i=0}^{n} a_ix^i \sum_{i=0}^{n} c_ib_ix^i \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} c_ia_ib_jx^{i+j} - \sum_{i=0}^{n}\sum_{j=0}^{n} c_ia_jb_ix^{i+j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} x^{i+j}c_i(a_ib_j - a_jb_i) \\
&= \sum_{m=0}^{2n} x^m \sum_{j,i\geq 0, i+j=m}^{i,j\leq n} c_i(a_ib_j - a_jb_i).
\end{aligned}$$

Since each term in the inner sum with $i = j$ is clearly zero, we can write:

$$\begin{aligned}
\sum_{j,i\geq 0, i+j=m}^{i,j\leq n} c_i(a_ib_j - a_jb_i) = &\sum_{j>i\geq 0, i+j=m}^{i,j\leq n} c_i(a_ib_j - a_jb_i) \\
&+ \sum_{i>j\geq 0, i+j=m}^{i,j\leq n} c_i(a_ib_j - a_jb_i).
\end{aligned} \tag{2.7}$$

Since $a_ib_j \geq a_jb_i$ for $i \geq j \geq 0$, every term in the second sum is nonnegative. For each term in the first sum (say indexed $i = i^*, j = j^*, i^* < j^*$), there is a term in the second sum with $i = j^*, j = i^*$ and

$$c_{i^*}(a_{i^*}b_{j^*} - a_{j^*}b_{i^*}) + c_{j^*}(a_{j^*}b_{i^*} - a_{i^*}b_{j^*}) = (c_{j^*} - c_{i^*})(a_{j^*}b_{i^*} - a_{i^*}b_{j^*}) \geq 0$$

since c_n is increasing. Thus, the polynomial

$$\sum_{i=0}^{n} c_ia_ix^i \sum_{i=0}^{n} b_ix^i - \sum_{i=0}^{n} a_ix^i \sum_{i=0}^{n} c_ib_ix^i$$

has only nonnegative coefficients. □

Theorem 2.7. *Let* $\{T_{n,k}\}_{n,k\geq 0}$ *be the nonnegative array as above* (1.2) *and its row generating functions* $T_n(q) = \sum_{k=0}^{n} T_{n,k}q^k$. *If* $a_2 \geq 0$ *and* $b_2 \geq 0$, *then the polynomial sequence* $\{T_n(q)\}_{n\geq 0}$ *is strongly q-log-convex.*

Proof. To prove that $\{T_n(q)\}_{n\geq 0}$ is strongly q-log-convex, it suffices to show

$$T_{n+1}(q)T_{m-1}(q) - T_n(q)T_m(q) \geq_q 0$$

for $n \geq m$. By the recurrence relation (1.2), we have

$$\begin{aligned}
T_n(q) &= \sum_{k=0}^{n}(a_1k^2+a_2k+a_3)T_{n-1,k}q^k+\sum_{k=1}^{n}(b_1k^2+b_2k+b_3)T_{n-1,k-1}q^k \\
&= \sum_{k=0}^{n}[a_1k(k-1)+(a_2+a_1)k+a_3]T_{n-1,k}q^k \\
&\quad +\sum_{k=1}^{n}[b_1k(k-1)+(b_2+b_1)k+b_3]T_{n-1,k-1}q^k \\
&= a_1q^2T''_{n-1}(q)+(a_1+a_2)qT'_{n-1}(q)+a_3T_{n-1}(q)+b_1q^2[qT_{n-1}(q)]'' \\
&\quad +(b_2+b_1)q[qT_{n-1}(q)]'+b_3qT_{n-1}(q) \\
&= [a_3+(b_1+b_2+b_3)q]\,T_{n-1}(q)+[a_1+a_2+(3b_1+b_2)q]qT'_{n-1}(q) \\
&\quad +q^2(a_1+b_1q)T''_{n-1}(q).
\end{aligned}$$

It follows that

$$\begin{aligned}
&T_{n+1}(q)T_{m-1}(q)-T_n(q)T_m(q) \\
&= T_{m-1}(q)\{[a_3+(b_1+b_2+b_3)q]\,T_n(q)+[a_1+a_2+(3b_1+b_2)q]qT'_n(q) \\
&\quad +q^2(a_1+b_1q)T''_n(q)\}-T_n(q)\{[a_3+(b_1+b_2+b_3)q]\,T_{m-1}(q) \\
&\quad +[a_1+a_2+(3b_1+b_2)q]qT'_{m-1}(q)+q^2(a_1+b_1q)T''_{m-1}(q)\} \\
&= [a_1+a_2+(3b_1+b_2)q]q\left[T'_n(q)T_{m-1}(q)-T_n(q)T'_{m-1}(q)\right] \\
&\quad +q^2(a_1+b_1q)\left[T''_n(q)T_{m-1}(q)-T_n(q)T''_{m-1}(q)\right]. \qquad (2.8)
\end{aligned}$$

Noting that $(T_{n,k})_{n,k\geq 0}$ is totally positive by Theorem 2.1, we derive

$$T_{n,i}T_{m,j} \geq T_{n,j}T_{m,i}$$

for $n \geq m$ and $i \geq j$. Thus, if we view $T_{n,k}$ and $T_{m-1,k}$ as a_k and b_k in Lemma 2.6, where $b_k = 0$ for $k > m$, then we can obtain that

$$T'_n(q)T_{m-1}(q)-T_n(q)T'_{m-1}(q) \geq_q 0$$

and

$$T''_n(q)T_{m-1}(q)-T_n(q)T''_{m-1}(q) \geq_q 0.$$

So, by (2.8),

$$T_{n+1}(q)T_{m-1}(q)-T_n(q)T_m(q) \geq_q 0.$$

This completes the proof. □

3. Applications

In this section, we give some applications of the main results. In the following examples, some properties considered were proved by various techniques in the literature.

Example 3.1. The Bell polynomial, or the exponential polynomial, is the generating function $B_n(q) = \sum_{k=0}^{n} S_{n,k}q^k$ of the Stirling numbers of the second kind, where the Stirling numbers of the second kind satisfy the recurrence

$$S_{n+1,k} = kS_{n,k} + S_{n,k-1},$$

which implies the recurrence

$$B_{n+1}(q) = qB_n(q) + qB'_n(q).$$

It is well-known that the Bell polynomials $B_n(q)$ have only real zeros; see [18, 29] for instance. The log-concavity of $\{S_{n_0-ai,k_0+bi}\}_{i\geq 0}$ was proved by Su *et al.* [27]. The q-log-convexity and strong q-log-convexity of $\{B_n(q)\}_{n\geq 0}$ have been proved by Liu and Wang [19] and Chen *et al.* [11], respectively.

Example 3.2. For the Jacobi-Stirling numbers $\mathrm{JS}_n^k(z)$ of the second kind, Mongelli [20] showed that $\left(\mathrm{JS}_n^k(z)\right)_{n,k\geq 0}$ is TP and all of its row and column sequences of are PF sequences. Further, we also have the following property.

Corollary 3.3. *Let a, b, n_0, k_0 be nonnegative integers. Then:*

(i) $\{JS_{n_0-an}^{k_0+bn}(z)\}_{n\geq 0}$ *is log-concave and unimodal.*
(ii) $\{JS_n(q)\}_{n\geq 0}$ *is a strongly q-log-convex sequence for $z \geq 0$.*

Remark 3.4. If $z = 1$, then $\mathrm{JS}_n^k(1)$ are the Legendre-Stirling numbers of the second kind.

Example 3.5. The *central factorial numbers* of the second kind, $T(n, k)$, are defined in Riordan's book [22, pp. 213-217] by

$$x^n = \sum_{k=0}^{n} T(n,k)\, x \prod_{i=1}^{k-1} \left(x + \frac{k}{2} - i\right). \tag{3.1}$$

Therefore, if we denote the central factorial numbers of even indices by $U(n, k) = T(2n, 2k)$, then

$$U(n, k) = U(n - 1, k - 1) + k^2U(n - 1, k).$$

For the central factorial numbers of odd indices, set $V(n, k) = 4^{n-k}T(2n+1, 2k+1)$. By the definition, we have the following recurrence relation:

$$V(n, k) = V(n - 1, k - 1) + (2k + 1)^2V(n - 1, k).$$

Corollary 3.6. *Let a, b, n_0, k_0 be nonnegative integers. Then:*

(i) *Both $(U(n, k))_{n,k\geq 0}$ and $(V(n, k))_{n,k\geq 0}$ are TP.*
(ii) *The sequences $\{U(n, k)\}_{k=0}^n$, $\{U(n, k)\}_{n\geq 0}$ and $\{V(n, k)\}_{n\geq 0}$ are all PF.*
(iii) *The sequence $\{U(n_0 - an, k_0 + bn)\}_{n\geq 0}$ is log-concave and unimodal.*
(iv) *Both $\{U_n(q)\}_{n\geq 0}$ and $\{V_n(q)\}_{n\geq 0}$ have only real zeros and are strongly q-log-convex.*

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