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A LOCAL SPECTRAL CONDITION FOR STRONG COMPACTNESS WITH SOME APPLICATIONS TO BILATERAL WEIGHTED SHIFTS

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Abstract. An algebra of bounded linear operators on a Banach space is said to be *strongly compact* provided that its unit ball is precompact in the strong operator topology, and a bounded linear operator on a Banach space is said to be *strongly compact* provided that the algebra with identity generated by the operator is strongly compact. Our interest in this notion stems from the work of Lomonosov on the existence of invariant subspaces. We consider a local spectral condition that is sufficient for a bounded linear operator on a Banach space to be strongly compact. This condition is then applied to describe a large class of strongly compact, injective bilateral weighted shifts on Hilbert spaces, extending earlier work of Fernández-Valles and the first author. Further applications are also derived; for instance, a strongly compact, invertible bilateral weighted shift is constructed in such a way that its inverse fails to be a strongly compact operator.

1. Introduction

Let $\mathcal{B}(X)$ denote the algebra of bounded linear operators on a complex Banach space X. A subalgebra $\mathcal{R} \subseteq \mathcal{B}(X)$ is said to be *strongly compact* if its unit ball $\{R \in \mathcal{R} : \|R\| \leq 1\}$ is precompact in the strong operator topology, and an operator $T \in \mathcal{B}(X)$ is said to be *strongly compact* if the algebra with identity generated by T is strongly compact.

Our interest in this notion stems from the work of Lomonosov [7] on the existence of invariant subspaces for essentially normal operators on Hilbert space. He showed that if T is an essentially normal operator on a Hilbert space such that both its commutant $\{T\}'$ and the commutant $\{T^*\}'$ of its adjoint fail to be strongly compact algebras, then T has a nontrivial invariant subspace. Moreover, if both T and T^* fail to be strongly compact operators, then T has a nontrivial hyperinvariant subspace.

Lomonosov, Radjavi and Troitsky [8] showed that if T is an operator such that its commutant $\{T\}'$ is a localizing algebra and the commutant $\{T^*\}'$ of its adjoint is a strongly compact algebra, then the adjoint T^* has an invariant subspace.

A characterization of strongly compact, normal operators in terms of the spectral representation was given by Lomonosov, Rodríguez-Piazza and the first author [5].

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Necessary and sufficient conditions were also provided for a unilateral weighted shift to be strongly compact in terms of the sliding products of its weights. Also, some applications were derived; for instance, the restriction of a strongly compact operator to an invariant subspace need not be a strongly compact operator.

Prunaru [11] used a special case of a more general result of Lomonosov [7] to show that for a pure hyponormal, essentially normal operator T on a Hilbert space, the commutant $\{T^*\}'$ of its adjoint is a strongly compact algebra.

Rodríguez-Piazza and the first author [6] showed that the position operator on the space of square integrable functions with respect to a finite measure of compact support is strongly compact if and only if the restriction of the measure to the exterior boundary of its support is purely atomic. (A similar result was obtained earlier by Froelich and Marsalli [4] within the framework of function algebras.) Further applications were derived; for instance, the weakly closed algebra generated by a strongly compact normal operator need not be a strongly compact algebra.

A classification of operator algebras was provided by Marsalli [9]. He gave some sufficient conditions for an algebra of operators to be strongly compact. He showed that if an operator has a total set of eigenvectors, then the operator is strongly compact, and, moreover, if all the corresponding eigenvalues have finite multiplicity, then the commutant of the operator is strongly compact. Fernández-Valles and the first author [3] applied this condition to test strong compactness for several classes of operators, namely, Cesàro operators, bilateral weighted shifts, and composition operators on Hardy spaces.

A condition of a different nature seems to be needed to prove strong compactness in the absence of eigenvalues. The aim of this paper is to give a local spectral condition that is sufficient for an operator on a Banach space to be strongly compact. This condition requires from the operator that the origin must lie in the interior of its full spectrum and that there must be a total set of vectors where the local spectral radius is less than the distance from the origin to the boundary of its full spectrum. This condition can be applied to operators with no eigenvalues, and it fits like a glove to bilateral weighted shifts.

As an application, we obtain a large class of strongly compact, bilateral weighted shifts on Hilbert spaces, extending earlier work of Fernández-Valles and the first author [3]. Then, we restrict our attention to invertible bilateral weighted shifts and we derive a condition for such operators to have a strongly compact inverse. Next, we show that if an invertible bilateral weighted shift and its inverse are strongly compact operators, then the algebra generated by both of them is strongly compact. Finally, we give an example of a strongly compact, invertible bilateral weighted shift whose inverse fails to be strongly compact.

2. A local spectral condition for strong compactness

Let T denote an operator on a complex Banach space X and let $\sigma(T)$ denote its spectrum. The *spectral radius* of T is defined as $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$. It is a standard fact that the spectral radius formula yields the alternative expression

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

The *local spectral radius* of T at a vector $x \in X$ is defined as

$$r(x, T) = \limsup_{n\to\infty} \|T^n x\|^{1/n}.$$

It is clear that $r(x,T) \leq r(T)$ for every $x \in X$. Daneš [2] showed that the set of vectors $x \in X$ that satisfy the strict inequality $r(x,T) < r(T)$ is of the first category. See also the paper of Müller [10] for more information on this remarkable fact and other related results.

Let G denote the unbounded connected component of $\mathbb{C}\backslash\sigma(T)$. The *full spectrum* of T is the set $\eta(\sigma(T)) = \mathbb{C}\backslash G$. It is plain that $\sigma(T) \subseteq \eta(\sigma(T))$ and that $\mathbb{C}\backslash\eta(\sigma(T))$ is connected. Also, it follows from the maximum modulus principle that if p is any polynomial, then

$$\max\{\,|p(\lambda)| : \lambda \in \sigma(T)\} = \max\{\,|p(\lambda)| : \lambda \in \eta(\sigma(T))\}.$$

Recall that $S \subseteq X$ is a *total set* if S generates a dense linear manifold in X. It turns out that a subalgebra $\mathcal{R} \subseteq \mathcal{B}(X)$ is strongly compact if and only if there is a total set $S \subseteq X$ such that $\{Rx : R \in \mathcal{R},\ \|R\| \leq 1\}$ is precompact for every $x \in S$. The main result of this paper relies on this simple fact. See the paper of Lomonosov, Rodríguez-Piazza and the first author [5] for a proof of it, and see also the paper of Marsalli [9] for the same result under a slightly different formulation.

Theorem 2.1. *Let T be an operator on a complex Banach space X such that $0 \in \operatorname{int}\eta(\sigma(T))$ and suppose that there is a total set $S \subseteq X$ with the property that $r(x,T) < d(0, \partial\eta(\sigma(T)))$ for every $x \in S$. Then the operator T is strongly compact.*

Proof. First, consider the distance $d = d(0, \partial\eta(\sigma(T))) = \min\{|\lambda| : \lambda \in \partial\eta(\sigma(T))\}$. Since $0 \in \operatorname{int}\eta(\sigma(T))$, we have $\{\lambda \in \mathbb{C} : |\lambda| \leq d\} \subseteq \eta(\sigma(T))$. It follows from the above remarks and the spectral mapping theorem that, for any polynomial p,

$$\begin{aligned}\max\{|p(\lambda)| : |\lambda| = d\} &\leq \max\{\,|p(\lambda)| : \lambda \in \eta(\sigma(T))\}\\ &= \max\{\,|p(\lambda)| : \lambda \in \sigma(T)\}\\ &= r(p(T)) \leq \|p(T)\|.\end{aligned}$$

This inequality allows us to control the size of the coefficients for a polynomial $p(\lambda) = a_0 + a_1\lambda + \cdots + a_m\lambda^m$. Indeed, using Cauchy's integral formula for the derivatives of order $n = 0, 1, \ldots, m$ gives

$$a_n = \frac{p^{(n)}(0)}{n!} = \frac{1}{2\pi i}\oint_{|\lambda|=d} \frac{p(\lambda)}{\lambda^{n+1}}\, d\lambda,$$

and from here we get the estimate

$$|a_n| \leq \frac{\|p(T)\|}{d^n}\,.$$

Take an arbitrary vector $x \in S$. We need to show that the set

$$\{p(T)x : p \text{ is a polynomial, } \|p(T)\| \leq 1\}$$

is precompact. Let $c > 0$ with $r(x,T) < c < d$. Since $r(x,T) = \limsup \|T^n x\|^{1/n}$ as $n \to \infty$, there is an n_0 such that $\|T^n x\| \leq c^n$ whenever $n \geq n_0$. Let $\varepsilon > 0$ and choose an $n_1 \geq n_0$ such that

$$\sum_{n=n_1+1}^{\infty} \left(\frac{c}{d}\right)^n < \varepsilon.$$

Finally, consider the finite dimensional subspace $F = \operatorname{span}\{x, Tx, \ldots, T^{n_1}x\}$. We claim that $p(T)x \in F + \varepsilon B_X$. This is obvious if $m \leq n_1$. Otherwise, $p(T)x$ can be

expressed as the sum of two terms,

$$p(T)x = \sum_{n=0}^{n_1} a_n T^n x + \sum_{n=n_1+1}^{m} a_n T^n x.$$

The first term belongs to F, while the second term satisfies the estimate

$$\left\| \sum_{n=n_1+1}^{m} a_n T^n x \right\| \leq \sum_{n=n_1+1}^{m} |a_n| \cdot \|T^n x\| \leq \sum_{n=n_1+1}^{m} \left(\frac{c}{d}\right)^n < \varepsilon,$$

which completes the proof of our claim. □

It is easy to see that strong compactness is preserved under similarities; that is, if T is a strongly compact operator, C is invertible, and $T = C^{-1}RC$, then R is strongly compact. Notice that the assumption in Theorem 2.1 is also preserved under similarities. More precisely, if T satisfies the condition in Theorem 2.1, C is invertible, and $T = C^{-1}RC$, then $\sigma(T) = \sigma(R)$, $\{Cx : x \in S\}$ is a total set, and $r(Cx, R) \leq r(x, T) < d(0, \partial\eta(\sigma(T))) = d(0, \partial\eta(\sigma(R)))$ for every $x \in S$, so that R also satisfies the condition in Theorem 2.1.

We now recall a couple of definitions from general operator theory. If T is any operator on a Banach space X, then the *lower bound* of T is defined as

$$m(T) = \inf\{\|Tx\| : x \in X, \|x\| \leq 1\}.$$

We also consider the quantity

$$r_1(T) = \sup_{n\geq 1} m(T^n)^{1/n} = \lim_{n\to\infty} m(T^n)^{1/n}.$$

It turns out that if T is invertible, then $\|T^{-1}\| = m(T)^{-1}$ and $r(T^{-1}) = r_1(T)^{-1}$.

3. Applications to bilateral weighted shifts

Let W be an injective bilateral weighted shift on an infinite dimensional, separable complex Hilbert space H; that is,

$$We_n = w_n e_{n+1},$$

where (e_n) is an orthonormal basis of H, the weight sequence (w_n) is bounded, n runs through the integers, and $w_n \neq 0$ for every n. We refer to the survey by Allen Shields [12] for information on the spectral parts of weighted shifts. See also the paper of Bourhim [1] for local spectral properties of weighted shifts. Now consider the quantities

$$r^-(W) = \lim_{n\to\infty} \sup_{k>0} |w_{-n-k} \cdots w_{-k+1}|^{1/n}, \quad r^+(W) = \lim_{n\to\infty} \sup_{k\geq 0} |w_k \cdots w_{n+k-1}|^{1/n},$$

$$r_1^-(W) = \lim_{n\to\infty} \inf_{k>0} |w_{-n-k} \cdots w_{-k+1}|^{1/n}, \quad r_1^+(W) = \lim_{n\to\infty} \inf_{k\geq 0} |w_k \cdots w_{n+k-1}|^{1/n},$$

$$r_2^-(W) = \liminf_{n\to\infty} |w_{-1} \cdots w_{-n}|^{1/n}, \quad r_2^+(W) = \liminf_{n\to\infty} |w_0 \cdots w_{n-1}|^{1/n},$$

$$r_3^-(W) = \limsup_{n\to\infty} |w_{-1} \cdots w_{-n}|^{1/n}, \quad r_3^+(W) = \limsup_{n\to\infty} |w_0 \cdots w_{n-1}|^{1/n}.$$

Then the following relationships are fulfilled:

$$r_1^-(W) \leq r_2^-(W) \leq r_3^-(W) \leq r^-(W), \quad r_1^+(W) \leq r_2^+(W) \leq r_3^+(W) \leq r^+(W),$$

$$r(W) = \max\{r^-(W), r^+(W)\}, \quad r_1(W) = \min\{r_1^-(W), r_1^+(W)\}.$$

It was shown by Fernández-Valles and the first author [3] that if a bilateral weighted shift W satisfies the inequality $r_3^+(W) < r_2^-(W)$, then its commutant $\{W\}'$ is strongly compact, and it remained an open question whether or not W is strongly compact when $r_2^-(W) \leq r_3^+(W)$. The following result provides an affirmative answer to this question for a large class of bilateral weighted shifts.

Theorem 3.1. *If a bilateral weighted shift W on an infinite dimensional, separable complex Hilbert space satisfies the inequality $r_3^+(W) < r(W)$, then W is a strongly compact operator.*

Proof. The full spectrum of W is the closed disk of radius $r(W)$ centered at the origin, and the orthornormal basis (e_k) is a spanning subset of H. Thus, it suffices to show that $r(e_k, W) = r_3^+(W)$ for each k. This is trivial when $k = 0$ because $\|W^n e_0\| = |w_0 \cdots w_{n-1}|$ for every $n \geq 1$ so that

$$r(e_0, W) = \limsup_{n\to\infty} \|W^n e_0\|^{1/n} = \limsup_{n\to\infty} |w_0 \cdots w_{n-1}|^{1/n} = r_3^+(W).$$

Then, suppose that $k \neq 0$ and notice that $\|W^n e_k\| = |w_k \cdots w_{n+k-1}|$ for every $n \geq 1$. Now, there are two possibilities. On the one hand, if $k > 0$, then

$$\|W^n e_k\| = \frac{|w_0 \cdots w_{n+k-1}|}{|w_0 \cdots w_{k-1}|} = \frac{\|W^{n+k} e_0\|}{\|W^k e_0\|}$$

so that

$$\begin{aligned} r(e_k, W) &= \limsup_{n\to\infty} \|W^n e_k\|^{1/n} = \limsup_{n\to\infty} \left(\frac{\|W^{n+k} e_0\|}{\|W^k e_0\|} \right)^{1/n} \\ &= \limsup_{n\to\infty} \|W^{n+k} e_0\|^{1/n} = \limsup_{n\to\infty} \left(\|W^{n+k} e_0\|^{1/(n+k)} \right)^{(n+k)/n} \\ &= r(e_0, W) = r_3^+(W). \end{aligned}$$

On the other hand, if $k < 0$ and $n > -k$, then

$$\|W^n e_k\| = |w_k \cdots w_{-1}| \cdot |w_0 \cdots w_{n+k-1}| = \|W^{-k} e_k\| \cdot \|W^{n+k} e_0\|$$

so that

$$\begin{aligned} r(e_k, W) &= \limsup_{n\to\infty} \|W^n e_k\|^{1/n} = \limsup_{n\to\infty} \left(\|W^{-k} e_k\| \cdot \|W^{n+k} e_0\| \right)^{1/n} \\ &= \limsup_{n\to\infty} \|W^{n+k} e_0\|^{1/n} = \limsup_{n\to\infty} \left(\|W^{n+k} e_0\|^{1/n+k} \right)^{(n+k)/n} \\ &= r(e_0, W) = r_3^+(W), \end{aligned}$$

as we wanted. □

Notice that $r_2^-(W) \leq r(W)$, so that the condition in Theorem 3.1 is satisfied whenever $r_3^+(W) < r_2^-(W)$. Hence, Theorem 3.1 is an extension of the previous result by Fernández-Valles and the first author [3].

Corollary 3.2. *Let W be an invertible bilateral weighted shift defined on an infinite dimensional, separable complex Hilbert space. If W satisfies $r_1(W) < r_2^-(W)$, then its inverse W^{-1} is a strongly compact operator.*

Proof. If W is an invertible bilateral weighted shift with weight sequence (w_n), then

$$W^{-1}e_n = \frac{1}{w_{n-1}}e_{n-1}.$$

Consider the unitary operator U defined by $Ue_n = e_{-n}$ so that $U^* = U = U^{-1}$. Now define $V = U^*W^{-1}U$ and notice that V is another bilateral weighted shift, namely, $Ve_n = v_n e_{n+1}$, where the sequence of weights (v_n) is given by

$$v_n = \frac{1}{w_{-(n+1)}}.$$

Recall the remarks at the end of the proof of Theorem 2.1 that strong compactness is preserved under similarities. Hence, it suffices to show that V is strongly compact. A quick computation yields $r_3^+(V) = r_2^-(W)^{-1} < r_1(W)^{-1} = r(V)$. □

Theorem 3.3. *Let W be an invertible bilateral weighted shift defined on an infinite dimensional, separable complex Hilbert space. If both W and W^{-1} are strongly compact, then the algebra generated by W and W^{-1} is strongly compact.*

Proof. It suffices to show that the set

$$C_k = \{p(W)e_k + q(W^{-1})e_k : p, q \text{ are polynomials, } q(0) = 0, \|p(W) + q(W^{-1})\| \leq 1\}$$

is precompact for every integer k. Since W and W^{-1} are both strongly compact, the sets

$$\begin{aligned} C_k^+ &= \{p(W)e_k : p \text{ is a polynomial, } \|p(W)\| \leq 1\}, \\ C_k^- &= \{q(W^{-1})e_k : q \text{ is a polynomial, } q(0) = 0,\ \|q(W^{-1})\| \leq 1\} \end{aligned}$$

are both precompact. Notice that for every pair of polynomials p, q with $q(0) = 0$ we have $\|p(W)e_k\| \leq \|p(W)e_k + q(W^{-1})e_k\|$ and $\|q(W^{-1})e_k\| \leq \|p(W)e_k + q(W^{-1})e_k\|$. Now it follows that $C_k \subseteq C_k^+ + C_k^-$ so that C_k is precompact, as we wanted. □

Corollary 3.4. *Let W be an invertible bilateral weighted shift defined on an infinite dimensional, separable complex Hilbert space. If both inequalities $r_3^+(W) < r(W)$ and $r_1(W) < r_2^-(W)$ are fulfilled, then the algebra generated by W and W^{-1} is strongly compact.*

Proof. Since $r_3^+(W) < r(W)$, it follows from Theorem 3.1 that W is strongly compact, and since $r_1(W) < r_2^-(W)$, it follows from Corollary 3.2 that W^{-1} is also strongly compact. Hence, the desired result is a consequence of Theorem 3.3. □

Example. Consider a bilateral weighted shift W whose sequence of weights (w_n) satisfies the conditions $1 \leq |w_n| \leq 2$ for each $n \geq 0$ and $w_n = 1$ for each $n < 0$. It is plain that such a weighted shift is bounded and invertible and that $\|W^{-1}\| = 1$. We claim that W^{-1} fails to be strongly compact. Otherwise the sequence $e_{-n} = W^{-n}e_0$ would have a norm convergent subsequence as $n \to \infty$, a contradiction. Then, the sequence of weights can be chosen in such a way that

$$\limsup_{n\to\infty} |w_0 \cdots w_{n-1}|^{1/n} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \sup_{k\geq 0} |w_k \cdots w_{n+k-1}|^{1/n} = 2.$$

There are many possible choices for such a sequence of weights; for instance, if we set $w_k = 2$ whenever $2^n \leq k \leq 2^n + n$ and $w_k = 1$ otherwise. Therefore, $r_3^+(W) = 1$ and $r(W) = r^+(W) = 2$. It follows from Theorem 3.1 that W is a strongly compact operator. Summarizing, we constructed a strongly compact bilateral weighted shift whose inverse fails to be strongly compact.

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