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SHARP LOCAL LOWER L^p-BOUNDS FOR DYADIC-LIKE MAXIMAL OPERATORS

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Abstract. We provide sharp lower L^p-bounds for the localized dyadic maximal operator on $\mathbb{R}^n$ when the local L^1 and the local L^p norm of the function are given. We actually do that in the more general context of homogeneous trees in probability spaces. For this we use an effective linearization for such maximal operators on an adequate set of functions.

1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is a useful tool in analysis and is defined by

$$M_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|\,du : x \in Q,\, Q \subseteq \mathbb{R}^n \text{ is a dyadic cube}\right\} \tag{1.1}$$

for every $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ where the dyadic cubes are the cubes formed by the grids $2^{-N}\mathbb{Z}^n$ for $N = 0, 1, 2, \ldots$.

It follows easily from the Lebesgue differentiation theorem that $M_d\phi \geq |\phi|$ almost everywhere. Also for any dyadic cube Q it is trivial that $M_d\phi \geq \sup_{R:Q\subseteq R} \mathrm{Av}_R(|\phi|)$ everywhere on Q where the supremum is taken over all dyadic cubes R containing Q and $\mathrm{Av}_R(|\phi|) = \frac{1}{|R|}\int_R |\phi|$. Therefore for any $p > 1$, for any dyadic cube Q, and for any $\phi \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ we have

$$\frac{1}{|Q|}\int_Q (M_d\phi)^p \geq \max\left(\frac{1}{|Q|}\int_Q |\phi|^p, (\sup_{R:Q\subseteq R} \mathrm{Av}_R(|\phi|))^p\right). \tag{1.2}$$

The purpose of this paper is to examine whether the above more or less trivial lower bound for the localized behavior of the maximal function can be improved, aiming at sharpness. To give a precise estimate of the left-hand side of (1.2) we define for any $p > 1$ the following Bellman function (see [8]):

$$\mathcal{B}_{\mathbf{d},p}(F, f, L) = \inf\left\{\frac{1}{|Q|}\int_Q (M_d\phi)^p : \mathrm{Av}_Q(\phi^p) = F, \mathrm{Av}_Q(\phi) = f, \sup_{R:Q\subseteq R} \mathrm{Av}_R(\phi) = L\right\}, \tag{1.3}$$

where the infimum taken over all nonnegative measurable functions ϕ (the definition of this function uses a fixed cube Q, but in fact due to scaling the function is independent of the fixed cube). Our aim is to find what exactly this is.

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Actually, as in [7] we will take the more general approach of defining Bellman functions with respect to the maximal operator on a nonatomic probability space (X, μ) equipped with an N-homogeneous tree-like family $\mathcal{T}$ (as discussed in section 2 the dyadic subcubes of say $[0,1]^n$ form a 2^n-homogeneous tree), thus defining, whenever F, f, L are positive real numbers with $f \leq L$ and $f^p \leq F$,

$$\mathcal{B}^{\mathcal{T}}(F, f, L) = \inf\{\int_X \max(M_{\mathcal{T}}\phi, L)^p d\mu : \phi \geq 0 \text{ measurable with} \\ \int_X \phi d\mu = f, \quad \int_X \phi^p d\mu = F\}. \tag{1.4}$$

Then our main theorem is the following.

Theorem 1. *For any nonatomic probability space (X, μ), any N-homogeneous tree-like family $\mathcal{T}$ and any F, f, L with $f \leq L$ and $f^p \leq F$ the corresponding Bellman function is given by*

$$\mathcal{B}^{\mathcal{T}}(F, f, L) = L^p + \frac{N^p - 1}{N^p - N}(F - L^{p-1}f)^+, \tag{1.5}$$

where $x^+ = \max(x, 0)$.

Thus in particular for the dyadic maximal operator in $\mathbb{R}^n$ we get for any $\phi \geq 0$ measurable and supported in the cube $Q_0 = [0,1]^n$ that the following sharp estimate holds with $L = \sup_{R: Q_0 \subseteq R} \mathrm{Av}_R(|\phi|)$:

$$\int_{Q_0} (M_d\phi)^p \geq L^p + \frac{2^{np} - 1}{2^{np} - 2^n}(\int_{Q_0} |\phi|^p - L^{p-1}\int_{Q_0} |\phi|)^+. \tag{1.6}$$

Also by taking $N \to \infty$ we conclude that there is no uniform lower estimate, other than the trivial one (1.2), holding for all homogeneous tree-like families $\mathcal{T}$, which shows the dependence on the dimension in the case of the dyadic maximal operators. Note that the situation for the upper bound (the corresponding sup Bellman function) is quite different since the expression does not depend on $\mathcal{T}$ at all (see [4]). However, see the last section in [7], where this phenomenon has been encountered.

Next, taking $L = f$ in the above theorem we get the following.

Proposition 1. *For any N-homogeneous tree-like family $\mathcal{T}$ and any F, f with $f^p \leq F$, we have*

$$\mathcal{B}^{\mathcal{T}}(F, f, f) = f^p + \frac{N^p - 1}{N^p - N}(F - f^p). \tag{1.7}$$

We have stated this as a separate proposition because it will be our main step in proving Theorem 1. Equation (1.7) with $p = 2$ shows the exact effect of the variance of ϕ.

As for a corollary of a more global nature we have the following L^p- improvement on the a.e. bound $M_d\phi \geq |\phi|$ in $\mathbb{R}^n$.

Corollary 1. *If $\phi \in L^p(\mathbb{R}^n)$ and $\int_{B(0,\rho)} |\phi| = o(\rho^{(p-1)n})$ as $\rho \to \infty$ (in particular, if ϕ is in $L^q(\mathbb{R}^n)$ where $1 \leq q < p$), then*

$$\int_{\mathbb{R}^n} (M_d\phi)^p \geq \frac{2^{np} - 1}{2^{np} - 2^n}\int_{\mathbb{R}^n} |\phi|^p. \tag{1.8}$$

This can be easily deduced by applying Proposition 1 to the systems of the dyadic subcubes of the 2^n types of cubes $\prod_{i=1}^n [0, \pm 2^m]$ (each equipped with normalized Lebesgue measure), adding the corresponding (1.7) inequalities and then letting $m \to \infty$.

In section 2 we give the definitions and basic properties of N-homogeneous trees $\mathcal{T}$ and the corresponding maximal operators and a general procedure (introduced in [4]) that can be used to approach Bellman functions related to $M_{\mathcal{T}}\phi$. In section 3 we will prove Proposition 1, and then in section 4 we will use it to prove Theorem 1.

For more on Bellman functions and their relation to harmonic analysis we refer to [8], [9], [10] and [18]. For the exact evaluation of Bellman functions in certain cases we refer to [1], [2], [4], [6], [7], [11], [12], [13], [15], [16] and [17]. We also note the approach initiated in [11], and also used in [17], to certain Bellman functions via PDE methods which has given alternative proofs of the results in [4] plus certain more general ones.

2. Trees and maximal operators

As in [4] we let (X, μ) be a nonatomic probability space (i.e. $\mu(X) = 1$). We give the following.

Definition 1. We call a set $\mathcal{T}$ of measurable subsets of X an N-homogeneous tree (where $N > 1$ is an integer) if the following conditions are satisfied:

(i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ there corresponds a finite subset $\mathcal{C}(I) \subseteq \mathcal{T}$ containing N elements each having measure equal to $N^{-1}\mu(I)$ such that the elements of $\mathcal{C}(I)$ are pairwise disjoint subsets of I and $I = \bigcup \mathcal{C}(I)$.

(ii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_{(m)}$, where $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_{(m)}} \mathcal{C}(I)$.

(iii) The family $\mathcal{T}$ differentiates $L^1(X, \mu)$.

We could replace the disjointness condition in (ii) above by asking that the pairwise intersections have measure 0 instead, but then one could replace X by $X \backslash \bigcup_{I \in \mathcal{T}} \bigcup_{J_1, J_2 \in \mathcal{C}(I),\ J_1 \neq J_2} (J_1 \cap J_2)$, which has full measure.

Examples. 1) If Q_0 is the unit cube $\mathbb{R}^n$ we let E be the union of all the boundaries of all dyadic cubes in Q_0 and then let $X = Q_0 \backslash E$ and $\mathcal{T}$ be the set of all open dyadic cubes $Q \subseteq Q_0$. Here $N = 2^n$ and each $\mathcal{C}(Q)$ is the set of the 2^n subcubes of Q obtained by bisecting its sides. More generally, for any integer $m > 1$ we may consider all m-adic cubes $Q \subseteq Q_0$ with $\mathcal{C}(Q)$ being the set of the m^n open subcubes of Q obtained by dividing each side of it into m equal parts.

2) Given the integers $d_1, ..., d_n \geq 1$ and $m > 1$ we can define $\mathcal{T}$ on X equal to Q_0 minus a certain set of measure 0 by setting for each open parallelepiped R the family $\mathcal{C}(R)$ consisting of the open parallelepipeds formed by dividing the dimensions of R into $m^{d_1}, ..., m^{d_n}$ equal parts respectively. For example, if $n = 2, m = 2, d_1 = 1$ and $d_2 = 2$, we get the set of dyadic parabolic rectangles contained in $[0, 1]^2$.

An easy induction shows that each family $\mathcal{T}_{(m)}$ consists of pairwise disjoint sets each having measure N^{-m} and whose union is X. Moreover, if $x \in X$, the set $\mathcal{A}(x) = \{I \in \mathcal{T} : x \in I\}$ forms a chain $I_0(x) = X \supsetneq I_1(x) \supsetneq ...$ with $I_m(x) \in \mathcal{C}(I_{m-1}(x))$ for every $m > 0$. From this remark it easily follows that if $I, J \in \mathcal{T}$ and $I \cap J$ is nonempty, then $I \subseteq J$ or $J \subseteq I$. In particular, for any $I, J \in \mathcal{T}$ we have either $I \cap J = \varnothing$ or one of them is contained in the other. The following gives another property of $\mathcal{T}$ that will be useful later. For a proof in a more general context see [4].

Lemma 1. *For every $I \in \mathcal{T}$ and every α such that $0 < \alpha < 1$ there exists a subfamily $\mathcal{F}(I) \subseteq \mathcal{T}$ consisting of pairwise disjoint subsets of I such that $\mu(\bigcup_{J\in\mathcal{F}(I)} J) = \sum_{J\in\mathcal{F}(I)} \mu(J) = (1-\alpha)\mu(I)$.*

Proof. Write $\alpha = \sum_{j=1}^{\infty} d_j N^{-j}$ in the N-ary system and then use d_j elements of each scale $\mathcal{T}_{(j)}$, noting that all these can be made pairwise disjoint since $d_j < N$, and property (i) in Definition 1. □

Now given any such $\mathcal{T}$ we define the maximal operator associated to it as follows:

$$M_{\mathcal{T}}\psi(x) = \sup\{\mathrm{Av}_I(|\psi|) : x \in I \in \mathcal{T}\} \tag{2.1}$$

for every $\psi \in L^1(X,\mu)$, where for any nonnegative $\phi \in L^1(X,\mu)$ and for any $I \in \mathcal{T}$ we have written $\mathrm{Av}_I(\phi) = \frac{1}{\mu(I)}\int_I \phi d\mu$.

Let ϕ be a nonnegative nonconstant $\mathcal{T}$-step function; that is, there exist integers $m > 0$ and $\lambda_P \geq 0$ for each $P \in \mathcal{T}_{(m)}$ such that

$$\phi = \sum_{P\in\mathcal{T}_{(m)}} \lambda_P \chi_P \tag{2.2}$$

(where χ_P denotes the characteristic function of P). For every $x \in X$ we let $I_\phi(x)$ denote the unique *largest* element of the set $\{I \in \mathcal{T} : x \in I \text{ and } M_{\mathcal{T}}\,\phi(x) = \mathrm{Av}_I(\phi)\}$ (which is nonempty since $\mathrm{Av}_J(\phi) = \mathrm{Av}_P(\phi)$ whenever $P \in \mathcal{T}_{(m)}$ and $J \subseteq P$). Next, for any $I \in \mathcal{T}$ we define the set

$$A_I = A(\phi, I) = \{x \in X : I_\phi(x) = I\} \tag{2.3}$$

and we let $\mathcal{S} = \mathcal{S}_\phi$ denote the set of all $I \in \mathcal{T}$ such that A_I is nonempty. It is clear that each such A_I is a union of certain P's from $\mathcal{T}_{(m)}$, and moreover

$$M_{\mathcal{T}}\,\phi = \sum_{I\in\mathcal{S}} \mathrm{Av}_I(\phi)\chi_{A_I}. \tag{2.4}$$

We define the correspondence $I \to I^*$ with respect to $\mathcal{S}$ as follows: for any $I \in \mathcal{S}$, I^* is the minimal element in the set of all $J \in \mathcal{S}$ that properly contain I. This is defined for every I in $\mathcal{S}$ that is not maximal with respect to $\subseteq$. We also write $y_I = \mathrm{Av}_I(\phi)$ for every $I \in \mathcal{S}$.

The main properties of the above are given in the following (see also [4] and [5]).

Lemma 2. *(i) For every $I \in \mathcal{S}$ we have $I = \bigcup_{\mathcal{S}\ni J\subseteq I} A_J$.*

(ii) For every $I \in \mathcal{S}$ we have $A_I = I \setminus \bigcup_{J\in\mathcal{S}:J^*=I} J$, and so $\mu(A_I) = \mu(I) - \sum_{J\in\mathcal{S}:J^*=I}\mu(J)$ and $\mathrm{Av}_I(\phi) = \frac{1}{\mu(I)}\sum_{J\in\mathcal{S}:J\subseteq I}\int_{A_J}\phi d\mu$.

(iii) For a $I \in \mathcal{T}$ we have $I \in \mathcal{S}$ if and only if $\mathrm{Av}_Q(\phi) < \mathrm{Av}_I(\phi)$ whenever $I \subseteq Q \in \mathcal{T}$, $I \neq Q$. In particular, $X \in \mathcal{S}$, and so $I \to I^*$ is defined for all $I \in \mathcal{S}$ such that $I \neq X$.

(iv) If $I, J \in \mathcal{S}$ are such that $J^* = I$, then

$$y_I < y_J \leq N y_I. \tag{2.5}$$

Proof. (i) Clearly $X = \bigcup_{J\in\mathcal{S}} A_J$. Fix $I \in \mathcal{S}$. Supposing that $x \in A(\phi, J) \cap I$ for some J we have $x \in I \cap J \neq \varnothing$, and so either $I \subseteq J$ or $J \subseteq I$. Suppose now that $I \subsetneq J$. Then also $\mathrm{Av}_J(\phi) = M_{\mathcal{T}}\phi(x) \geq \mathrm{Av}_I(\phi)$, and so I cannot be an $I_\phi(z)$ for

any $z \in I$. Therefore $A(\phi, I) = \varnothing$, contradicting the assumption $I \in \mathcal{S}$. Hence we must have $J \subseteq I$, and this easily implies that I is the union of all A_J's for $J \subseteq I$.

(ii) Follows easily from (i).

(iii) One direction follows from the definition of the I_ϕ's. For the other assume that $I \in \mathcal{T}_{(s)}$ satisfies the assumption. Since

$$\mathrm{Av}_J(\phi) = \frac{\sum_{F \in \mathcal{C}(J)} \mu(F)\, \mathrm{Av}_F(\phi)}{\sum_{F \in \mathcal{C}(J)} \mu(F)}, \tag{2.6}$$

we conclude that for each $J \in \mathcal{T}$ there exists $J' \in \mathcal{C}(J)$ such that $\mathrm{Av}_{J'}(\phi) \leq \mathrm{Av}_J(\phi)$. Starting from I and applying the above $m - s$ times we get a chain $I = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_{m-s}$ such that $I_{(r)} \in \mathcal{T}_{(s+r)}$ for each s, and moreover $\mathrm{Av}_{I_{m-s}}(\phi) \leq \mathrm{Av}_{I_{m-s+1}}(\phi) \leq \ldots \leq \mathrm{Av}_{I_1}(\phi) \leq \mathrm{Av}_{I_0}(\phi) = \mathrm{Av}_I(\phi)$. Now from this and the assumption on I it is clear that $I_\phi(x) = I$ for every $x \in I_{m-s}$ and therefore $I \in \mathcal{S}$.

(iv) The inequality $y_I < y_J$ follows from (iii). For the other inequaity let F be the unique element of the whole family $\mathcal{T}$ such that $J \in \mathcal{C}(F)$. Note that $F \subseteq I$. We claim that $\mathrm{Av}_F(\phi) \leq y_I$. Indeed $I \in \mathcal{S}$ implies that $\mathrm{Av}_Q(\phi) < y_I$ whenever $I \subseteq Q$, $I \neq Q$, and so if $\mathrm{Av}_F(\phi) > y_I$ there would exist $F' \in \mathcal{T}$ such that $F \subseteq F' \subseteq I$, $F' \neq I$ and $\mathrm{Av}_{F'}(\phi) > \mathrm{Av}_Q(\phi)$ whenever $F' \subseteq Q$, $F' \neq Q$. But this combined with (iii) implies that F' must be in $\mathcal{S}$, contradicting our assumption $J^* = I$. Thus we get, since $J \subseteq F$,

$$y_J = \frac{1}{\mu(J)} \int_J \phi d\mu \leq \frac{1}{\mu(J)} \int_F \phi d\mu = \frac{\mu(F)}{\mu(J)} \mathrm{Av}_F(\phi) \leq \frac{\mu(F)}{\mu(J)} y_I = N y_I, \tag{2.7}$$

which completes the proof. □

The above lemma shows that this linearization $M_{\mathcal{T}}\, \phi$ may be viewed as a multi-scale version of the classical Calderon-Zygmund decomposition.

3. Proof of Proposition 1

Here we will prove Proposition 1. Assuming that $\mathcal{T}$ is an N-homogeneous tree we let ϕ be a nonnegative $\mathcal{T}$-step function such that

$$\int_X \phi d\mu = f \text{ and } \int_X \phi^p d\mu = F \tag{3.1}$$

and let $\mathcal{S} = \mathcal{S}_\phi$ be the corresponding subtree of $\mathcal{T}$. Using the notation from section 2 we make the following two simple observations. First, by Lemma 2 (iv) we have $y_{I^*} < y_I \leq N y_{I^*}$ for all $I \in \mathcal{S} \backslash \{X\}$, and second, $\phi(t) \leq y_I$ whenever $I \in \mathcal{S}$ and $t \in A_I$.

The second remark gives

$$\int_{A_I} \phi(t)^p d\mu(t) \leq \int_{A_I} \phi(t) y_I^{p-1} d\mu(t) = y_I^{p-1} \int_{A_I} \phi d\mu \tag{3.2}$$

for all $I \in \mathcal{S}$, and Lemma 2 (ii) implies that

$$\int_{A_I} \phi d\mu = \mu(I) y_I - \sum_{J \in \mathcal{S}: J^* = I} \mu(J) y_J.$$

Hence

$$F = \int_X \phi^p d\mu = \sum_{I\in\mathcal{S}} \int_{A_I} \phi^p d\mu \le \sum_{I\in\mathcal{S}} y_I^{p-1}(\mu(I)y_I - \sum_{J\in\mathcal{S}:J^*=I} \mu(J)y_J)$$
$$= y_X^p\mu(X) + \sum_{I\in\mathcal{S}:I\neq X} y_I^p\mu(I) - \sum_{J\in\mathcal{S}:J\neq X} y_{J^*}^{p-1}\mu(J)y_J,$$

and so

$$F \le f^p + \sum_{I\in\mathcal{S}:I\neq X} \mu(I)y_I(y_I^{p-1} - y_{I^*}^{p-1}). \tag{3.3}$$

Now (2.4) and Lemma 2 imply that

$$\int_X (M_{\mathcal{T}}\phi)^p d\mu = \sum_{I\in\mathcal{S}} a_I y_I^p = \sum_{I\in\mathcal{S}} (\mu(I) - \sum_{J\in\mathcal{S}:J^*=I} \mu(J))y_I^p$$
$$= f^p + \sum_{I\in\mathcal{S},I\neq X} \mu(I)(y_I^p - y_{I^*}^p). \tag{3.4}$$

Next, for any $I \in \mathcal{S}, I \neq X$ we have $1 < \dfrac{y_I}{y_{I^*}} \le N$. On the other hand, the function $h(t) = \dfrac{t^p - 1}{t^p - t}$ is easily seen to be strictly decreasing on $(1, +\infty)$. Therefore since $\dfrac{y_I}{y_{I^*}} \in (1, N]$ we obtain

$$\frac{y_I^p - y_{I^*}^p}{y_I(y_I^{p-1} - y_{I^*}^{p-1})} = h(\frac{y_I}{y_{I^*}}) \ge h(N) = \frac{N^p - 1}{N^p - N}. \tag{3.5}$$

Using (3.5) in (3.4) and by (3.3) we get

$$\int_X (M_{\mathcal{T}}\phi)^p d\mu \ge f^p + \frac{N^p - 1}{N^p - N}(\sum_{I\in\mathcal{S}:I\neq X} \mu(I)y_I(y_I^{p-1} - y_{I^*}^{p-1}))$$
$$\ge f^p + \frac{N^p - 1}{N^p - N}(F - f^p) \tag{3.6}$$

for all nonnegative step functions ϕ.

Now for the general case, given $\phi \ge 0$ measurable satisfying (3.1) we define ϕ_m as follows:

$$\phi_m = \sum_{I\in\mathcal{T}_{(m)}} \mathrm{Av}_I(\phi)\chi_I$$

and note that

$$M_{\mathcal{T}}\phi_m = \sum_{I\in\mathcal{T}_{(m)}} \max\{\mathrm{Av}_J(\phi) : I \subseteq J \in \mathcal{T}\}\chi_I \tag{3.7}$$

since $\mathrm{Av}_J(\phi) = \mathrm{Av}_J(\phi_m)$ whenever $I \subseteq J \in \mathcal{T}$ when $I \in \mathcal{T}_{(m)}$. Also,

$$\int_X \phi_m d\mu = \int_X \phi d\mu = f, \;\; F_m = \int_X \phi_m^p d\mu \le \int_X \phi^p d\mu \le F \tag{3.8}$$

for all m and $M_{\mathcal{T}}\phi_m$ converges monotonically to $M_{\mathcal{T}}\phi$. Also, since each ϕ_m is a $\mathcal{T}$-step function we can apply (3.6) to get

$$\int_X (M_{\mathcal{T}}\phi_m)^p d\mu \ge f^p + \frac{N^p - 1}{N^p - N}(F_m - f^p) \tag{3.9}$$

for every m. On the other hand, we have $\phi_m^p \le (M_{\mathcal{T}}\phi)^p$ everywhere and $\phi_m^p \to \phi^p$ almost everywhere by property (iii) in Definition 1. Hence by dominated convergence we conclude that $F_m = \int_X \phi_m^p d\mu \to \int_X \phi^p d\mu = F$, and so using monotone convergence for $M_{\mathcal{T}}\phi_m$ and (3.9) we get

$$\int_X (M_{\mathcal{T}}\phi)^p d\mu = \lim_{m\to\infty} \int_X (M_{\mathcal{T}}\phi_m)^p d\mu \ge f^p + \frac{N^p-1}{N^p-N}(F-f^p). \tag{3.10}$$

These prove that $\mathcal{B}_p^{\mathcal{T}}(F,f,f) \ge f^p + \frac{N^p-1}{N^p-N}(F-f^p)$.

To prove the reverse inequality we fix positive f and F with $f^p < F$ (the case $F = f^p$ being trivial) and let $X = I_0 \supseteq I_1 \supseteq ...I_s \supseteq I_{s+1} \supseteq ...$ be a chain such that $I_s \in \mathcal{T}_{(s)}$ for all $s \ge 0$ (and so $\mu(I_s) = N^{-s}$).

For a strictly increasing sequence of nonnegative integers $m_0 < m_1 < ... < m_k < ...$ to be chosen later, we define

$$\phi = f\sum_{k=0}^{\infty} N^{m_k-k}\chi_{I_{m_k}\setminus I_{m_k+1}}. \tag{3.11}$$

We have

$$\int_X \phi d\mu = f\sum_{k=0}^{\infty} N^{m_k-k}(N^{-m_k} - N^{-m_k-1}) = f\sum_{k=0}^{\infty} N^{-k}(1-\frac{1}{N}) = f, \tag{3.12}$$

$$\int_X \phi^p d\mu = f^p\sum_{k=0}^{\infty} N^{m_kp-kp}(N^{-m_k}-N^{-m_k-1}) = f^p(1-\frac{1}{N})\sum_{k=0}^{\infty} N^{m_k(p-1)-kp} \stackrel{\text{def}}{=} F_0, \tag{3.13}$$

say, and if $m_{k-1} < s \le m_k$ where $k \ge 0$ (setting $m_{-1} = -1$), then

$$\mathrm{Av}_{I_s}(\phi) = N^s f\sum_{j=k}^{\infty} N^{m_j-j}(N^{-m_j} - N^{-m_j-1}) = fN^{s-k}, \tag{3.14}$$

and this increases as s increases (if $s = m_k$, then $\mathrm{Av}_{I_s}(\phi) = \mathrm{Av}_{I_{s+1}}(\phi)$). We next claim that $M_{\mathcal{T}}\phi(x) = \mathrm{Av}_{I_s}(\phi)$ whenever $x \in I_s\setminus I_{s+1}$ and $s \ge 0$. Indeed suppose that $x \in I_s\setminus I_{s+1}$ and let J be the unique element of $\mathcal{T}_{(s+1)}$ such that $x \in J$ (clearly $J \in \mathcal{C}(I_s)$ and $J \ne I_s$). Then the set of all I's in $\mathcal{T}$ containing x consists of $I_0, ..., I_s$ and J and certain subintervals of J. But $\mathrm{Av}_{I_s}(\phi) \ge \mathrm{Av}_{I_r}(\phi)$ for all $0 \le r < s$, and since ϕ is either 0 on J (if s is not an m_k) or if $s = m_k$ (so $\phi = fN^{m_k-k}$ on J) is equal to $\mathrm{Av}_{I_s}(\phi)$ on J, we get that $M_{\mathcal{T}}\phi(x) = \mathrm{Av}_{I_s}(\phi)$. Hence using (3.14) we get

$$M_{\mathcal{T}}\phi = f\sum_{s=0}^{\infty} N^{s-k(s)}\chi_{I_s\setminus I_{s+1}}, \tag{3.15}$$

where $k(s)$ is the smallest integer k with $m_k \ge s$. This implies that

$$\int_X (M_{\mathcal{T}}\phi)^p d\mu = f^p\sum_{s=0}^{\infty} N^{ps-pk(s)}(N^{-s} - N^{-s-1}) = f^p(1-\frac{1}{N})\sum_{s=0}^{\infty} N^{(p-1)s-pk(s)}. \tag{3.16}$$

Next we compute

$$\sum_{s=0}^{\infty} N^{(p-1)s-pk(s)} = \sum_{j=0}^{\infty} N^{-pj} \sum_{s:k(s)=j} N^{(p-1)s} = \sum_{j=0}^{m_0} N^{(p-1)s} + \sum_{j=1}^{\infty} \sum_{s=m_{k-1}+1}^{m_k} N^{(p-1)s-pj}$$

$$= (N^{p-1}-1)^{-1}\left((1-N^{-p})N^{p-1}(\sum_{k=0}^{\infty} N^{m_k(p-1)-kp}) - 1\right) \tag{3.17}$$

$$= (N^{p-1}-1)^{-1}\left((1-N^{-p})N^{p-1}(\frac{F_0}{(1-\frac{1}{N})f^p}) - 1\right).$$

Therefore

$$\int_X (M_{\mathcal{T}}\phi)^p d\mu = f^p + \frac{N^p-1}{N^p-N}(F_0 - f^p). \tag{3.18}$$

Hence to complete the proof of Proposition 1 it suffices to show that a sequence (m_k) as above can be found such that F_0 as defined in (3.13) equals our given F. But this will follow by applying the next lemma to the real number $a = \dfrac{F}{f^p} > 1$.

Lemma 3. *Suppose $N > 1$ is an integer and $p > 1,\ a > 1$ are real numbers. Then there exist integers $0 \le m_0 < m_1 < ... < m_k < ...$ such that*

$$a = (1-\frac{1}{N})\sum_{k=0}^{\infty} N^{m_k(p-1)-kp}. \tag{3.19}$$

Proof. Since $a > 1$ there exists a maximal $j_0 \ge 0$ such that $N^{j_0(p-1)} \le a$. Set $a_0 = a,\ m_0 = j_0$ and inductively define $a_r \ge 1,\ j_r \ge 0,\ m_r > m_{r-1}$ by choosing j_r to be the maximal integer such that $N^{j_r(p-1)} \le a_r$, setting $m_r = m_{r-1} + j_r + 1 > m_{r-1}$ and

$$a_{r+1} = \frac{N(a_r - (1-\frac{1}{N})N^{j_r(p-1)})}{N^{j_r(p-1)}} \ge 1. \tag{3.20}$$

An easy induction shows that for any $r > 0$,

$$a = N^{m_r(p-1)-rp-1}a_{r+1} + (1-\frac{1}{N})\sum_{k=0}^{r} N^{m_k(p-1)-kp}. \tag{3.21}$$

Next, for any $r > 0$, by the way j_r is chosen we have $a_r < N^{(j_r+1)(p-1)}$. Hence

$$a_r - (1-\frac{1}{N})N^{j_r(p-1)} < (1-\frac{N-1}{N^p})a_r, \tag{3.22}$$

and so using (3.20) and $m_r = m_{r-1} + j_r + 1$ we conclude that

$$N^{m_r(p-1)-rp}a_{r+1} < N^{m_r(p-1)-rp}\frac{N}{N^{j_r(p-1)}}(1-\frac{N-1}{N^p})a_r$$

$$= (1-\frac{N-1}{N^p})N^{m_{r-1}(p-1)-(r-1)p}a_r, \tag{3.23}$$

and so

$$N^{m_r(p-1)-rp-1}a_{r+1} < (1-\frac{N-1}{N^p})^r\frac{a_1}{N}. \tag{3.24}$$

Taking $r \to \infty$ in (3.24) and using (3.21) completes the proof of the lemma. □

This completes the proof of Proposition 1.

4. Proof of Theorem 1

Assume f, F, L are positive such that $L > f$, $f^p \le F$. We consider two cases:

Case 1 ($F \ge L^{p-1} f$). Let ϕ be nonnegative and measurable, satisfying (3.1). Consider the set $K = \bigcup\{J \in \mathcal{T} : \mathrm{Av}_J(\phi) \ge L\}$, which clearly is equal to the union of pairwise disjoint (maximal) elements I_j of $\mathcal{T}$. Setting

$$\alpha_j = \int_{I_j} \phi^p d\mu,\ \beta_j = \frac{1}{\mu(I_j)} \int_{I_j} \phi d\mu \text{ and } \lambda_j = \mu(I_j) \tag{4.1}$$

and using Proposition 1 for ϕ restricted to I_j and for the tree $\mathcal{T}(I_j)$ on the probability space $(I_j, \frac{1}{\mu(I_j)}\mu)$ consisting of all elements of $\mathcal{T}$ contained in I_j, we get

$$\frac{1}{\mu(I_j)} \int_{I_j} (M_{\mathcal{T}}(\phi))^p d\mu \ge \frac{1}{\mu(I_j)} \int_{I_j} (M_{\mathcal{T}(I_j)}(\phi\chi_{Ij}))^p d\mu \ge \beta_j^p + \frac{N^p - 1}{N^p - N}\left(\frac{a_j}{\mu(I_j)} - \beta_j^p\right), \tag{4.2}$$

and so multiplying by $\mu(I_j)$ and adding over all j's give

$$\int_K (M_{\mathcal{T}}(\phi))^p d\mu \ge \frac{N^p - 1}{N^p - N} \sum \alpha_j - \frac{N - 1}{N^p - N} \sum \lambda_j \beta_j^p. \tag{4.3}$$

Noting that $M_{\mathcal{T}}(\phi) < L$ off K we have

$$\int_X \max(M_{\mathcal{T}}(\phi), L)^p d\mu \ge L^p(1 - \sum \lambda_j) - \frac{N^p - 1}{N^p - N} \sum \alpha_j - \frac{N - 1}{N^p - N} \sum \lambda_j \beta_j^p. \tag{4.4}$$

But since also $\phi(t) \le M_{\mathcal{T}}(\phi)(t) < L$ on $X \backslash K$ we have $F - \sum \alpha_j = \int_{X\backslash K} \phi^p \le L^{p-1} \int_{X\backslash K} \phi = L^{p-1}(f - \sum \lambda_j \beta_j)$, so (4.4) gives

$$\begin{aligned} \int_X \max(M_{\mathcal{T}}(\phi), L)^p d\mu &\ge L^p + (F - L^{p-1} f) \frac{N^p - 1}{N^p - N} \\ &\quad - \sum \lambda_j \left(L^p - \frac{N^p - 1}{N^p - N} \beta_j L^{p-1} + \frac{N - 1}{N^p - N} \beta_j^p\right). \end{aligned} \tag{4.5}$$

Now we use the fact that each β_j belongs to the interval $[L, NL]$ (since the I_j's are maximal), and, combined with the observation that the convex function $g(x) = 1 - \frac{N^p-1}{N^p-N} x + \frac{N-1}{N^p-N} x^p$ satisfies $g(1) = g(N) = 0$, we infer that $g(\frac{\beta_j}{N}) \le 0$ for all j; thus the sum in (4.5) is nonpositive. Therefore (4.5) implies

$$\int_X \max(M_{\mathcal{T}}(\phi), L)^p d\mu \ge L^p + \frac{N^p - 1}{N^p - N}(F - L^{p-1} f). \tag{4.6}$$

Conversely, by applying Lemma 1 we take I_j to be pairwise disjoint members of $\mathcal{T}$ such that $\sum \mu(I_j) = \frac{f}{L} \in (0, 1)$ and for each j, we use the proof of Proposition 1 to take ϕ_j on I_j such that

$$\frac{1}{\mu(I_j)} \int_{I_j} \phi_j d\mu = L,\ \sum_j \int_{I_j} \phi_j^p d\mu = F \tag{4.7}$$

(which is possible since $F \ge L^{p-1} f$ implies that we can find $\alpha_j \ge \mu(I_j) L^p$ such that $\sum a_j = F$) and

$$\frac{1}{\mu(I_j)} \int_{I_j} (M_{\mathcal{T}(I_j)}(\phi\chi_{Ij}))^p d\mu = L^p + \frac{N^p - 1}{N^p - N}\left(\frac{1}{\mu(I_j)} \int_{I_j} \phi_j^p d\mu - L^p\right). \tag{4.8}$$

Then we define $\phi = \sum_j \phi_j \chi_{I_j}$ and note that since $\phi = 0$ off $\bigcup_j I_j$ and $\mathrm{Av}_{I_j}(\phi) = L$ for all j's, we have $M_{\mathcal{T}}(\phi) < L$ on $X \setminus \bigcup_j I_j$, and so $M_{\mathcal{T}}(\phi) = M_{\mathcal{T}(I_j)}(\phi_j)$ on each I_j. Hence

$$\begin{aligned}
&\int_X \max(M_{\mathcal{T}}(\phi), L)^p d\mu \\
&= L^p(1 - \sum \mu(I_j)) + L^p \sum \mu(I_j) + \frac{N^p - 1}{N^p - N} \sum \Big(\int_{I_j} \phi_j^p d\mu - L^p \mu(I_j)\Big) \\
&= L^p + \frac{N^p - 1}{N^p - N}(F - L^p \sum \mu(I_j)) = L^p + \frac{N^p - 1}{N^p - N}(F - L^{p-1} f). \qquad (4.9)
\end{aligned}$$

Case 2 ($F < L^{p-1} f$). Here we have the trivial bound $\int_X \max(M_{\mathcal{T}}(\phi), L)^p d\mu \geq L^p$. But also there exists $\kappa \geq f$ such that $\kappa^{p-1} f = F$, and by our assumption we also have $\kappa < L$. We choose a measurable $K \subseteq X$ with $\mu(K) = \dfrac{f}{\kappa} \in (0, 1]$ and take $\phi = \kappa \chi_K$. Then $\int_X \phi d\mu = \kappa \mu(K) = f$, $\int_X \phi^p d\mu = \kappa^p \mu(K) = F$, and since $L > \kappa = \|\phi\|_\infty$, we have $M_{\mathcal{T}}(\phi) < L$ on X. Thus $\int_X \max(M_{\mathcal{T}}(\phi), L)^p d\mu = L^p$, and this completes the proof of Theorem 1.

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References

[1] D. L. Burkholder, Martingales and Fourier analysis in Banach spaces, C.I.M.E. Lectures (Varenna (Como), Italy, 1985), *Lecture Notes in Mathematics* 1206 (1986), 61-108. MR864712 (88c:42017)

[2] D. L. Burkholder, Boundary value problems and sharp inequalities for martingale transforms, *Ann. of Prob.* **12** (1984), 647-702. MR744226 (86b:60080)

[3] L. Grafakos, S. Montgomery-Smith. Best constants for uncentered maximal functions, *Bull. London Math. Soc.* **29** (1997), no. 1, 60-64. MR1416408 (98b:42031)

[4] A. D. Melas, The Bellman functions of dyadic-like maximal operators and related inequalities, *Adv. in Math.* **192** (2005), 310-340. MR2128702 (2005k:42052)

[5] A. D. Melas, A sharp L^p inequality for dyadic A_1 weights in $\mathbb{R}^n$. *Bull. Lond. Math. Soc.* **37** (2005), 919-926. MR2186725 (2006h:42035)

[6] A. D. Melas, Sharp general local estimates for dyadic-like maximal operators and related Bellman functions, *Adv. in Math.* **220**, no. 2 (2009), 367-426. MR2466420 (2010g:42044)

[7] A. D. Melas, Dyadic-like maximal operators on $L \log L$ functions, *Jour. Funct. Anal.* **257**, no. 6 (2009), 1631-1654. MR2540986 (2010g:42045)

[8] F. Nazarov, S. Treil, The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis, *Algebra i Analiz* **8**, no. 5 (1996), 32-162. MR1428988 (99d:42026)

[9] F. Nazarov, S. Treil, A. Volberg, The Bellman functions and two-weight inequalities for Haar multipliers. *J. Amer. Math. Soc.* **12**, no. 4 (1999), 909-928. MR1685781 (2000k:42009)

[10] F. Nazarov, S. Treil, A. Volberg, Bellman function in stochastic optimal control and harmonic analysis (how our Bellman function got its name), *Oper. Theory: Advances and Appl.* **129** (2001), 393-424, Birkhäuser Verlag. MR1882704 (2003b:49024)

[11] L. Slavin, A. Stokolos, V. Vasyunin. Monge-Ampère equations and Bellman functions: The dyadic maximal operator. *C. R. Acad. Paris Sér. I Math.* **346**, no. 9-10 (2008), 585-588. MR2412802 (2009c:42036)

[12] L. Slavin, V. Vasyunin, Sharp results in the integral-form John-Nirenberg inequality, *Trans. Amer. Math. Soc.* **363**, no 8 (2011), 4135-4169. MR2792983 (2012c:42005)

[13] L. Slavin, A. Volberg. The explicit BF for a dyadic Chang-Wilson-Wolff theorem. The s-function and the exponential integral. *Contemp. Math.* **444** (2007), 215-228. MR2423630 (2009c:26048)

[14] E. M. Stein. Note on the class $L \log L$, *Studia Math.* **32** (1969), 305-310. MR0247534 (40:799)
[15] V. Vasyunin. The sharp constant in the reverse Holder inequality for Muckenhoupt weights. *Algebra i Analiz* **15** (2003), no. 1, 73-117. MR1979718 (2004h:42017)
[16] V. Vasyunin, A. Volberg. The Bellman functions for a certain two weight inequality: the case study. *Algebra i Analiz* **18** (2006), No. 2. MR2244935 (2007k:47053)
[17] V. Vasyunin, A. Volberg. Monge-Ampère equation and Bellman optimization of Carleson embedding theorem, *Amer. Math. Soc. Translations Series* 2, **226** (2009). MR2500520 (2011f:42027)
[18] A. Volberg, Bellman approach to some problems in harmonic analysis, *Seminaire des Equations aux derivées partielles, Ecole Polytéchnique,* 2002, exposé. XX, 1-14.
[19] G. Wang, Sharp maximal inequalities for conditionally symmetric martingales and Brownian motion, *Proc. Amer. Math. Soc.* **112** (1991), 579-586. MR1059638 (91i:60121)

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