

A PROPERTY OF PEANO DERIVATIVES IN SEVERAL VARIABLES

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ABSTRACT. Let f be a function of several variables that is n times Peano differentiable. Andreas Fischer proved that if there is a number M such that $f_{\alpha} \geq M$ or $f_{\alpha} \leq M$ for each α , with $|\alpha| = n$, then f is n times differentiable in the usual sense. Here that result is improved to permit the type of one-sided boundedness to depend on α .

1. INTRODUCTION

Often authors will assume that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is n times differentiable at a point x in order to use that the value of the function can be approximated close to x by a polynomial of degree n . It seems reasonable that if such an approximation is desired, then we can simply assume that one exists and observe that it does exist if the differentiability condition is assumed. In 1891 G. Peano (see [6]) first introduced this idea of approximating a function by a polynomial, which later became known as Peano differentiation. The notion was used by de la Vallée-Poussin (see [2]) in 1908 in his study of Fourier series. Formally the concept is defined as follows.

Definition 1. Let $n \in \mathbb{N}$, let $f : \mathbb{R} \rightarrow \mathbb{R}$ and let $x \in \mathbb{R}$. Then f is n times Peano differentiable at x means that there are numbers $f_{(1)}(x), \dots, f_{(n)}(x)$ such that

$$f(x+h) = f(x) + \sum_{i=1}^n \frac{f_{(i)}(x)}{i!} h^i + o(h^n).$$

It is easy to see that if f is n times Peano differentiable at x , then the coefficients $f_{(i)}(x)$, called Peano derivatives, are unique.

By the classical version of Taylor's Theorem, if f is n times differentiable in the usual sense at x , then it is also n times Peano differentiable at x and $f_{(i)}(x) = f^{(i)}(x)$ for $i = 1, \dots, n$. Except for the case $n = 1$, the converse is false. In fact it's possible for a function to be n times Peano differentiable at a point but for f' to fail to be even continuous there let alone differentiable. In fact the existence of the n -th Peano derivative at a point does not presuppose the existence of the $(n-1)$ -st Peano derivative in a neighborhood of the point.

The distinction between the usual notion of differentiation and the Peano notion is most apparent when considering functions defined on a subset of \mathbb{R} that is assumed only to be dense in itself (so that h can tend to 0 while $x+h$ remains

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in the domain of the function). In this situation it's possible for the function to be n ($n \geq 2$) times differentiable in the usual sense and not to be n times Peano differentiable. In addition, there are functions that are n times differentiable in both senses, but that the n^{th} derivatives are not equal. By definition being n times Peano differentiable says that the function can be approximated by a polynomial of degree n , but in this setting the usual notion of n^{th} order differentiation says nothing useful about the function. However there are situations when functions defined on such domains can be extended to functions defined on a connected set. (See [3].)

An extensive study of the Peano derivative was done by H. W. Oliver in 1954. (See [5].) He established that if a function f is n times Peano differentiable on an interval, then the function $f_{(n)}$ has many of the properties known for an ordinary derivative. In addition he proved that if $f_{(n)}$ is bounded above or below on the interval, then it is an ordinary derivative; i.e., $f_{(n)} = f^{(n)}$. Additional information about Peano derivatives can be found in [7]. The purpose of this paper is to establish Oliver's result for Peano derivatives in several variables.

To state the definition of Peano differentiation in several variables, we first recall the standard notation used when working with partial derivatives of functions of several variables. Let $d \in \mathbb{N}$ and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ be a d -tuple of nonnegative integers. Then $\alpha! = \alpha_1! \cdot \alpha_2! \cdot \dots \cdot \alpha_d!$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$ and if $\mathbf{x} \in \mathbb{R}^d$, then $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \dots \cdot x_d^{\alpha_d}$ with the usual conventions that $0! = 1$ and $0^0 = 1$. Also for $p = 1, 2, \dots, d$ we use $\partial_p f$ to denote the partial derivative of f with respect to the p^{th} variable. Moreover for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\partial_\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_d^{\alpha_d} f$ (where $\partial_p^k f$ means the partial with respect to the p^{th} variable taken k times). This notation presupposes the equality of the mixed partial derivatives. The last assertion of this paper gives conditions much weaker than the continuity of the mixed partial derivatives that imply their equality. (See Corollary 4.)

Definition 2. Let $n \in \mathbb{N}$, let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and let $\mathbf{x} \in \mathbb{R}^d$. Then f is n times Peano differentiable at \mathbf{x} means for each index α with $|\alpha| \leq n$ there is a number $f_{(\alpha)}(\mathbf{x})$ such that

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \sum_{1 \leq |\alpha| \leq n} \frac{f_{(\alpha)}(\mathbf{x})}{\alpha!} \mathbf{h}^\alpha + o(\|\mathbf{h}\|^n).$$

As in the one-variable case, if f is n times Peano differentiable, then the coefficients $f_{(\alpha)}$, called Peano partials, are unique. Note that Peano differentiability implies that f is differentiable at \mathbf{x} and that the first-order partials satisfy $\partial_p f(\mathbf{x}) = f_{(\mathbf{p})}(\mathbf{x})$ for each $p = 1, 2, \dots, d$, where \mathbf{p} denotes the d -tuple having 1 in the p^{th} position and 0 elsewhere.

The goal of this paper is to prove an analogue of Oliver's boundedness result for Peano derivatives in several variables. In 2008 (see [4]) Andreas Fischer proved a result of this type for $n = 2$, while for $n \geq 3$ he assumed that all of the n^{th} order Peano partials are bounded above or all bounded below. Here we show that also for $n \geq 3$ one can allow the type of boundedness to depend on the direction of the partial derivative. In other words we allow the possibility that for $|\alpha| = n$, some $f_{(\alpha)}$ are bounded from above while the others are bounded from below. (See Corollary 3 below.)

Most papers that deal with higher order differentiation in several variables assume that the functions under study are C^n in order to conclude that the mixed

partial derivatives are equal. However, as was shown by W. H. Young in 1908 (see [8]), the equality of the mixed partial derivatives can be deduced by assuming only that the partial derivatives of order $n - 1$ are differentiable functions. Additional conditions that imply equality of the mixed partial derivatives can be found in the main result, Corollary 3 and Corollary 4.

In this context it's worth mentioning that usually C^{n+1} is assumed for a function in order to conclude that its Taylor approximation has a specific formula for the remainder. The same conclusion can be drawn with a lesser assumption. For the details, see [1].

2. A PRELIMINARY LEMMA

The following lemma is known. Here we present a proof based on repeated applications of the Mean Value Theorem (MVT).

Lemma 1. *Let g be k times differentiable on an interval $I \subset \mathbb{R}$ and let $\omega, \omega + kh \in I$ with $h > 0$. Then there is $c \in (\omega, \omega + kh)$ such that*

$$(1) \quad \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} g(\omega + ih) = h^k g^{(k)}(c).$$

In particular, if $g^{(k)}$ is bounded below by K , then $\sum_{i=0}^k (-1)^{n-i} \binom{k}{i} g(\omega + ih) \geq Kh^k$ whenever $\omega, \omega + kh \in I$ with $h > 0$.

(Note that the case $k = 1$ is just the usual MVT.)

Proof. Let $F_{m,g}(u) = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} g(u + ih)$. Since g is differentiable, $F_{m,g}(u)$ is also differentiable and $F'_{m,g}(u) = F_{m,g'}(u)$. Also from the identity $\binom{m+1}{i} = \binom{m}{i-1} + \binom{m}{i}$, for $i = 1, 2, \dots, m$, it is easily seen that the sequence of functions $\{F_{m,g}(u)\}_{m=0}^\infty$ satisfies the relation

$$F_{m+1,g}(u) = F_{m,g}(u + h) - F_{m,g}(u).$$

By the MVT there is $c_1 \in (\omega, \omega + h)$ such that

$$F_{k,g}(\omega) = F_{k-1,g}(\omega + h) - F_{k-1,g}(\omega) = hF_{k-1,g'}(c_1).$$

Applying the MVT again but this time to the function $F_{k-1,g'}(u)$ we obtain that there is $c_2 \in (c_1, c_1 + h)$ such that

$$F_{k,g}(\omega) = hF_{k-1,g'}(c_1) = h^2F_{k-2,g''}(c_2).$$

Since g is k times differentiable, we can apply the MVT $k - 2$ additional times to conclude that $F_{k,g}(x) = h^k F_{0,g^{(k)}}(c_k) = h^k g^{(k)}(c_k)$ for some $c_k \in (c_{k-1}, c_{k-1} + h)$. This is precisely the first part of the lemma with $c = c_k$. The remaining part is obvious. \square

3. THE RESULTS

The following theorem is the fundamental tool used to reach the goal. The notation $B_r(\mathbf{w})$ denotes the ball of radius r centered at \mathbf{w} . Recall that for $p = 1, 2, \dots, d$, the symbol \mathbf{p} will denote the d -tuple that has 1 in the p^{th} position and 0 elsewhere. Consequently $n\mathbf{p}$ is the d -tuple that has n in the p^{th} position and 0 elsewhere.

Theorem 2. *Let $f : B_2(\mathbf{w}) \rightarrow \mathbb{R}$ be n times Peano differentiable at \mathbf{w} and let $\partial_1^n f$ exist on $B_2(\mathbf{w})$. Suppose $\partial_1^n f$ is bounded from below by a constant $K < 0$. Then $\partial_1 f$ is $n - 1$ times Peano differentiable at \mathbf{w} and $(\partial_1 f)_{(\gamma)}(\mathbf{w}) = f_{(\gamma+1)}(\mathbf{w})$ for all $|\gamma| \leq n - 1$.*

Proof. We will prove the special case where $f(\mathbf{w}) = 0$, and $f_{(\alpha)}(\mathbf{w}) = 0$ for all $|\alpha| \leq n$. Under these assumptions, we have to prove that $\partial_1 f$ is $n - 1$ times Peano differentiable at \mathbf{w} and $(\partial_1 f)_{(\gamma)}(\mathbf{w}) = 0$ for all $|\gamma| \leq n - 1$. This is equivalent to $\partial_1 f(\mathbf{x}) = o(\|\mathbf{x} - \mathbf{w}\|^{n-1})$. The general case follows from the special case applied to

$$F(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{w}) - \sum_{1 \leq |\alpha| \leq n} \frac{f_{(\alpha)}(\mathbf{w})}{\alpha!} (\mathbf{x} - \mathbf{w})^\alpha.$$

Then F satisfies the conditions of the special case and by that case $\partial_1 F(\mathbf{x}) = o(\|\mathbf{x} - \mathbf{w}\|^{n-1})$. Differentiating $F(\mathbf{x})$ with respect to x_1 we obtain

$$\partial_1 f(\mathbf{x}) = \sum_{1 \leq |\alpha| \leq n} \frac{f_{(\alpha)}(\mathbf{w})}{\alpha!} \partial_1 (\mathbf{x} - \mathbf{w})^\alpha + o(\|\mathbf{x} - \mathbf{w}\|^{n-1}).$$

Every $\partial_1 (\mathbf{x} - \mathbf{w})^\alpha$ with $\alpha_1 = 0$ is 0 leaving only those terms in the sum with $\alpha = \gamma + 1$, where $|\gamma| \leq n - 1$. Thus

$$\begin{aligned} \partial_1 f(\mathbf{x}) &= \sum_{0 \leq |\gamma| \leq n-1} \frac{f_{(\gamma+1)}(\mathbf{w})}{(\gamma+1)!} \left((\gamma_1 + 1)(x_1 - w_1)^{\gamma_1} \prod_{j=2}^d (x_j - w_j)^{\gamma_j} \right) \\ &\quad + o(\|\mathbf{x} - \mathbf{w}\|^{n-1}) \\ &= \partial_1 f(\mathbf{w}) + \sum_{1 \leq |\gamma| \leq n-1} \frac{f_{(\gamma+1)}(\mathbf{w})}{\gamma!} (\mathbf{x} - \mathbf{w})^\gamma + o(\|\mathbf{x} - \mathbf{w}\|^{n-1}). \end{aligned}$$

Therefore $\partial_1 f$ is $n - 1$ times Peano differentiable at \mathbf{w} and $(\partial_1 f)_{(\gamma)}(\mathbf{w}) = f_{(\gamma+1)}(\mathbf{w})$ for each $|\gamma| \leq n - 1$.

Now we prove the special case. Suppose to the contrary that $\partial_1 f(\mathbf{x}) \neq o(\|\mathbf{x} - \mathbf{w}\|^{n-1})$. Then there is a number $L > 0$ such that \mathbf{w} is a limit point of

$$\{\mathbf{y} \in B_1(\mathbf{w}) : \partial_1 f(\mathbf{y}) \geq L \|\mathbf{y} - \mathbf{w}\|^{n-1}\}$$

or a limit point of

$$\{\mathbf{y} \in B_1(\mathbf{w}) : \partial_1 f(\mathbf{y}) \leq -L \|\mathbf{y} - \mathbf{w}\|^{n-1}\}.$$

We will first dispose of the case where \mathbf{w} is a limit point of $\{\mathbf{y} \in B_1(\mathbf{w}) : \partial_1 f(\mathbf{y}) \geq L \|\mathbf{y} - \mathbf{w}\|^{n-1}\}$. Fix one such \mathbf{y} . Write $\mathbf{y} = (y_1, \mathbf{y}') \in \mathbb{R} \times \mathbb{R}^{d-1}$. Notice that if $t \in [-1, 1]$, then $(y_1 + t, \mathbf{y}') \in B_2(\mathbf{w})$, and thus by the assumptions $\partial_1 f(y_1 + t, \mathbf{y}')$ is $n - 1$ times differentiable in t . By Lemma 1 with $k = n - 1$, $I = [-1, 1]$, $1/n > h > 0$, $\omega = -(n - 2)h$, and $g = \partial_1 f$,

$$\sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \partial_1 f(y_1 + (i+2-n)h, \mathbf{y}') \geq Kh^{n-1},$$

or moving the term corresponding to $i = n - 2$ to the right-hand side,

$$(2) \quad \sum_{\substack{i=0 \\ i \neq n-2}}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \partial_1 f(y_1 + (i+2-n)h, \mathbf{y}') \geq (n-1) \partial_1 f(\mathbf{y}) + Kh^{n-1}.$$

Thus by the choice of \mathbf{y} ,

$$\begin{aligned} & \sum_{\substack{i=0 \\ i \neq n-2}}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \partial_1 f(y_1 + (i+2-n)h, \mathbf{y}') \\ & \geq (n-1)L \|\mathbf{y} - \mathbf{w}\|^{n-1} + Kh^{n-1}. \end{aligned}$$

Let $\delta = \min \left\{ n^{-1} \sqrt{\frac{L}{2|K|}} \|\mathbf{y} - \mathbf{w}\|, 1/n \right\}$. The last inequality implies that for all $0 < h \leq \delta$,

$$\begin{aligned} (3) \quad & \sum_{\substack{i=0 \\ i \neq n-2}}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \partial_1 f(y_1 + (i+2-n)h, \mathbf{y}') \\ & \geq (n-1)L \|\mathbf{y} - \mathbf{w}\|^{n-1} - \frac{L}{2} \|\mathbf{y} - \mathbf{w}\|^{n-1} \geq \frac{L}{2} \|\mathbf{y} - \mathbf{w}\|^{n-1}. \end{aligned}$$

Let

$$\begin{aligned} G(h) &= \sum_{\substack{i=0 \\ i \neq n-2}}^{n-1} \frac{(-1)^{n-1-i} \binom{n-1}{i}}{(i+2-n)} f(y_1 + (i+2-n)h, \mathbf{y}') \\ &= \sum_{\substack{i=0 \\ i \neq n-2}}^{n-1} c_i f(y_1 + (i+2-n)h, \mathbf{y}'). \end{aligned}$$

Then

$$G'(h) = \sum_{\substack{i=0 \\ i \neq n-2}}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \partial_1 f(y_1 + (i+2-n)h, \mathbf{y}')$$

and the inequality (3) yields $G'(h) \geq \frac{L}{2} \|\mathbf{y} - \mathbf{w}\|^{n-1}$ for every $0 < h \leq \delta$. By the MVT, $G(h) - G(0) \geq h \frac{L}{2} \|\mathbf{y} - \mathbf{w}\|^{n-1}$. Fix an $0 < h \leq \delta$ with $h < \|\mathbf{y} - \mathbf{w}\|$. Then from the last inequality we obtain $G(h) - G(0) \geq \frac{L}{2} \|\mathbf{y} - \mathbf{w}\|^n$. Let

$$c_{n-2} = - \sum_{\substack{i=0 \\ i \neq n-2}}^{n-1} \frac{(-1)^{n-1-i} \binom{n-1}{i}}{(i+2-n)}$$

so that $-G(0) = c_{n-2}f(\mathbf{y})$. Then

$$\sum_{i=0}^{n-1} c_i f(y_1 + (i+2-n)h, \mathbf{y}') = G(h) - G(0) \geq \frac{L}{2} \|\mathbf{y} - \mathbf{w}\|^n.$$

Thus there is an i such that

$$(4) \quad |c_i f(y_1 + (i+2-n)h, \mathbf{y}')| \geq \frac{L}{2n} \|\mathbf{y} - \mathbf{w}\|^n.$$

Let $\mathbf{z} = (y_1 + (i+2-n)h, \mathbf{y}')$. Then

$$\|\mathbf{z} - \mathbf{w}\| \leq \|\mathbf{y} - \mathbf{w}\| + \|\mathbf{z} - \mathbf{y}\| \leq \|\mathbf{y} - \mathbf{w}\| + (n-1)h \leq n \|\mathbf{y} - \mathbf{w}\|.$$

Hence from (4)

$$|f(\mathbf{z})| \geq \frac{L}{2|c_i|n^{n+1}} \|\mathbf{z} - \mathbf{w}\|^n.$$

Since $\mathbf{z} \rightarrow \mathbf{w}$ as $\mathbf{y} \rightarrow \mathbf{w}$, the last inequality contradicts the assumption that f is n times Peano differentiable at \mathbf{w} .

Finally if \mathbf{w} is a limit point of $\{\mathbf{y} \in B_1(\mathbf{w}) : \partial_1 f(\mathbf{y}) \leq -L \|\mathbf{y} - \mathbf{w}\|^{n-1}\}$, then the proof above should be adjusted as follows. Lemma 1 should be applied with $\omega = -(n - 1)h$, but this time with the term corresponding to $n - 1$ being moved to the right-hand side. Consequently (2) becomes

$$\sum_{i=0}^{n-2} (-1)^{n-1-i} \binom{n-1}{i} \partial_1 f(y_1 + (i + 2 - n)h, \mathbf{y}') \geq -\partial_1 f(\mathbf{y}) + Kh^{n-1}.$$

Thus as in the proof of (3),

$$\begin{aligned} & \sum_{i=0}^{n-2} (-1)^{n-1-i} \binom{n-1}{i} \partial_1 f(y_1 + (i + 2 - n)h, \mathbf{y}') \\ & \geq L \|\mathbf{y} - \mathbf{w}\|^{n-1} + Kh^{n-1} \geq \frac{L}{2} \|\mathbf{y} - \mathbf{w}\|^n \end{aligned}$$

for all $0 < h \leq \delta$, where $\delta = \min\left\{n^{-1} \sqrt{\frac{L}{2|K|}} \|\mathbf{y} - \mathbf{w}\|, 1/n\right\}$. The rest of the proof follows line by line with the appropriate adjustments. □

Theorem 2 remains true if boundedness from below is replaced by boundedness from above. This is easily seen by applying Theorem 2 to $-f$. Also it should be clear that $\partial_1^n f$ can be replaced by $\partial_p^n f$ for any $p = 2, 3, \dots, d$. One might wonder if in the statement of Theorem 2 we could replace $\partial_1 f$ by $f_{(n\mathbf{1})}$. (Recall that $n\mathbf{1} = (n, 0, \dots, 0)$.) By Oliver’s result, the boundedness of $f_{(n\mathbf{1})}$ would imply the existence of $\partial_1 f$ and the equality $\partial_1 f = f_{(n\mathbf{1})}$ so this is not much of a generalization. But this observation will be used in the proof of the following corollary, our main result. In the statement of this result “the mixed partials are equal” refers to the following assertion. If two partials of f are obtained by taking partials of f the same number of times with respect to each coordinate but in a different order, then these two partials are equal.

Corollary 3. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be n times Peano differentiable on \mathbb{R}^d and for each d -tuple, α , with $|\alpha| = n$, the n^{th} order Peano partial $f_{(\alpha)}$ is bounded from above or from below on \mathbb{R}^d . Then f is n times differentiable in the usual sense on \mathbb{R}^d ; that is, all $(n - 1)$ -st partials of f exist and they are differentiable. Moreover the values of the partials are equal to the corresponding values of Peano partials; that is, if a partial of f is obtained by taking γ_p partials with respect to x_p for $p = 1, 2, \dots, d$ in some order, then this partial is equal to the Peano partial $f_{(\gamma)}$. In particular all the mixed partials are equal.*

Proof. The proof is by induction on n . The case $n = 1$ is covered by the assumptions. So assume that $n \geq 2$ and that the assertion is true for $n - 1$. Let $\mathbf{w} \in \mathbb{R}^d$. Since for each $p = 1, 2, \dots, d$, the partial derivative $\partial_p^n f$ is bounded from above or below, we can apply Theorem 2 to deduce that $\partial_p f$ is $n - 1$ times Peano differentiable at \mathbf{w} and $(\partial_p f)_{(\gamma)}(\mathbf{w}) = f_{(\gamma+p)}(\mathbf{w})$ for all $|\gamma| \leq n - 1$. Since \mathbf{w} was an arbitrary

point in \mathbb{R}^d , we have that $\partial_p f$ is $n - 1$ times Peano differentiable on \mathbb{R}^d and that for each d -tuple, γ , with $|\gamma| = n - 1$ the $(n - 1)$ th order Peano partial $(\partial_p f)_{(\gamma)}$ is either bounded from above or from below on \mathbb{R}^d . Thus by the induction hypothesis each $\partial_p f$ is $n - 1$ times differentiable on \mathbb{R}^d in the usual sense and partials of $\partial_p f$ are equal to the corresponding values of Peano partials of $\partial_p f$. Thus f is n times differentiable in the usual sense on \mathbb{R}^d . Finally every partial of f is also a partial of $\partial_p f$ for some p . Thus there is a d -tuple $|\alpha| \leq (n - 1)$ such that this partial is equal to $(\partial_p f)_{(\alpha)}$. Since $(\partial_p f)_{(\alpha)} = f_{(\alpha+p)}$, we have that all partials of f are equal to the corresponding Peano partials. \square

We end this paper with the following result about the equality of mixed partials.

Corollary 4. *Let $U \subset \mathbb{R}^d$ be open and let $f : U \rightarrow \mathbb{R}$ have second-order partials $\partial_{pp}^2 f$ and $\partial_{qq}^2 f$ on U . Assume that $\partial_{pp}^2 f$ is bounded from below or from above on U and the same for $\partial_{qq}^2 f$. If f is twice Peano differentiable at $\mathbf{w} \in U$, then $\partial_{pq}^2 f(\mathbf{w})$ and $\partial_{qp}^2 f(\mathbf{w})$ exist and are equal.*

Proof. By Theorem 2, $\partial_p f$ is differentiable at \mathbf{w} and $\partial_{pq}^2 f(\mathbf{w}) = f_{(\mathbf{p}+\mathbf{q})}(\mathbf{w})$. Similarly $\partial_q f$ is differentiable at \mathbf{w} and $\partial_{qp}^2 f(\mathbf{w}) = f_{(\mathbf{q}+\mathbf{p})}(\mathbf{w}) = f_{(\mathbf{p}+\mathbf{q})}(\mathbf{w})$ because f is twice Peano differentiable at \mathbf{w} by assumption. \square

In the above assertion the assumption that f is twice Peano differentiable can be replaced by f is twice differentiable in the usual sense, because the second assumption implies the first. This version of the result is substantially better than the usual assumption that the second-order partials exist and are continuous.

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