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ON THE GRADIENT ESTIMATE OF CHENG AND YAU

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Abstract. We improve the well-known local gradient estimate of Cheng and Yau in the case when the Ricci curvature has a negative lower bound.

1. Introduction

Let M be an m-dimensional complete noncompact Riemannian manifold. The local Cheng-Yau gradient estimate is a standard result in Riemannian geometry; see [2], also cf. [4]. It asserts that for $f : B_p(2R) \to \mathbb{R}$ harmonic and positive, if the Ricci curvature on $B_p(2R)$ has a lower bound $Ric \geq -(m-1)K$ for some $K \geq 0$, then

$$\sup_{B_p(R)} |\nabla \log f| \leq (m-1)\sqrt{K} + \frac{C}{R}. \tag{1}$$

Notice that if $K = 0$, it follows that a harmonic function with sublinear growth on a manifold with nonnegative Ricci curvature is constant. This result is clearly sharp since on $\mathbb{R}^n$ there exist harmonic functions which are linear.

When $K > 0$, we can rescale the metric so that we take $K = 1$. In particular, if f is positive harmonic on M and $Ric \geq -(m-1)$ on M, then it follows that $|\nabla \log f| \leq m-1$ on M. This result is also sharp; in fact, the equality case was recently characterized in [4].

This means that for $K = 1$ in (1) the factor $m-1$ on the right-hand side is sharp. However, the correction term that depends on the radius is not sharp anymore. The purpose of this paper is to prove a sharp version of the local Cheng-Yau gradient estimate in the following form.

Theorem 1. *Let M be a complete noncompact Riemannian manifold of dimension m with Ricci curvature bounded from below on a geodesic ball $B_p(2R)$ by*

$$Ric \geq -(m-1).$$

There exist constants C_1 depending only on m and C_2 a universal constant such that if $f : B_p(2R) \to \mathbb{R}$ is a positive harmonic function, then

$$\sup_{x \in B_p(R)} |\nabla \log f|(x) \leq m - 1 + \frac{C_1}{R} \exp(-C_2 R).$$

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Certainly, our estimate becomes meaningful for large R. Let us also remark that by rescaling the metric we obtain that if $Ric \geq -(m-1)K$ on $B_p(2R)$ and f is positive harmonic on $B_p(2R)$, then

$$\sup_{B_p(R)} |\nabla \log f| \leq (m-1)\sqrt{K} + \frac{C_1}{R}\exp\left(-C_2\sqrt{K}R\right).$$

The Cheng-Yau gradient estimate is a fundamental tool in geometric analysis. The technique has been applied in various situations; for an overview of the subject see e.g. [5, 3]. In many applications the value of the constants in the estimate is quite important. For example, the sharp version of the global estimate in the case when $Ric \geq -(m-1)$ on M has been very instrumental in rigidity theorems; see [4] and [6]. The importance of our theorem, besides being sharp, is that the estimate does not diverge when integrated on minimizing geodesics. For example, in Corollary 1 in this paper we have established a sharp lower bound for the Green's function on a nonparabolic manifold with a negative lower bound for Ricci curvature. Similar sharp upper or lower bound estimates can be established for harmonic functions defined on manifolds with boundary.

2. Proof of the result

Proof. The proof of the theorem follows the standard argument of Cheng and Yau. We apply the Bochner technique to $\phi^2 |\nabla \log f|^2$, where ϕ is a cutoff function with support in $B_p(R)$. The difference here is that we use a judicious choice of cutoff ϕ which improves the argument.

We will prove the following statement: Let $f : B_p(2R) \to \mathbb{R}$ be positive and harmonic, where $R \geq R_0(m)$ with R_0 depending only on m.

Assume that $Ric \geq -(m-1)$ on $B_p(2R)$. Then

$$|\nabla \log f|(p) \leq m - 1 + C_1 \exp(-C_2 R). \tag{2}$$

It is evident that (2) proves Theorem 1, because we can apply (2) for each $x \in B_p(R)$. Of course, since (2) is true for any radius, we can apply it for the radius $\frac{R}{2}$, with R the same as in Theorem 1.

Let $h = \log f$. Consider also the function $\phi : [0, \infty) \to \mathbb{R}$ defined by

$$\begin{aligned} \phi(0) &= 1, \\ \phi(r) &= 1 - \exp\alpha\left(R - \frac{R^2}{r}\right) \quad \text{for } \ 0 < r \leq R, \\ \phi(r) &= 0 \quad \text{for } \ r > R, \end{aligned}$$

where $\alpha > 0$ is a (small) number that will be set later.

We compute directly that for $r < R$,

$$\begin{aligned} \phi'(r) &= -\alpha\frac{R^2}{r^2}\exp\alpha\left(R - \frac{R^2}{r}\right), \\ \phi''(r) &= \left(2\alpha\frac{R^2}{r^3} - \alpha^2\frac{R^4}{r^4}\right)\exp\alpha\left(R - \frac{R^2}{r}\right), \\ \phi'(0) &= \phi''(0) = 0. \end{aligned} \tag{3}$$

By applying ϕ to the distance function from p we obtain a cutoff function on M, with support in $B_p(R)$, which we continue to denote with ϕ, i.e. $\phi(x) = \phi(r(x))$.

Let us now consider $G : B_p(R) \to \mathbb{R}$,

$$G = \phi^2 |\nabla h|^2 .$$

Since G is nonnegative on $B_p(R)$ and $G = 0$ on $\partial B_p(R)$, it follows that G attains a maximum point in the interior of $B_p(R)$. Let x_0 be this maximum point. We will assume from now on that $r(x) = d(p, x)$ is smooth at x_0, so ϕ and G are also smooth at x_0. If this is not the case; then one can use a support function at x_0, which is smooth. The computations will be similar and will imply the same result. In this standard argument, it is essential that $\phi(r)$ be nonincreasing, which is true in our case; see [1].

Hence at x_0 we have, by the maximum principle:

$$\begin{aligned} |\nabla G(x_0)| &= 0, \\ \Delta G(x_0) &\leq 0. \end{aligned} \tag{4}$$

The following part of the argument is well known, but we include it for completeness. By the Bochner formula, since $h = \log f$ and $\Delta h = -|\nabla h|^2$, we get

$$\begin{aligned} \frac{1}{2}\Delta |\nabla h|^2 &= |h_{ij}|^2 + \langle \nabla h, \nabla(\Delta h)\rangle + Ric(\nabla h, \nabla h) \\ &\geq |h_{ij}|^2 - \left\langle \nabla h, \nabla |\nabla h|^2 \right\rangle - (m-1)|\nabla h|^2 . \end{aligned} \tag{5}$$

Moreover, choosing an orthonormal frame $\{e_i\}_i$ such that $e_1 = \frac{\nabla h}{|\nabla h|}$, we have

$$\begin{aligned} |h_{ij}|^2 &\geq |h_{11}|^2 + \sum_{\alpha>1} |h_{\alpha\alpha}|^2 + 2\sum_{\alpha>1} |h_{1\alpha}|^2 \\ &\geq |h_{11}|^2 + 2\sum_{\alpha>1} |h_{1\alpha}|^2 + \frac{1}{m-1}\left|\sum_{\alpha>1} h_{\alpha\alpha}\right|^2 \\ &= |h_{11}|^2 + 2\sum_{\alpha>1} |h_{1\alpha}|^2 + \frac{1}{m-1}\left||\nabla h|^2 + h_{11}\right|^2 \\ &\geq \frac{m}{m-1}\left(|h_{11}|^2 + \sum_{\alpha>1} |h_{1\alpha}|^2\right) + \frac{1}{m-1}|\nabla h|^4 + \frac{2}{m-1} h_{11}|\nabla h|^2 . \end{aligned}$$

On the other hand, notice that

$$\begin{aligned} \left\langle \nabla |\nabla h|^2, \nabla h \right\rangle &= 2h_{ij}h_i h_j = 2h_{11}|\nabla h|^2 \quad \text{and} \\ \left|\nabla |\nabla h|^2\right|^2 &= 4|h_{ij}h_j|^2 = 4h_{1i}^2 |\nabla h|^2 , \end{aligned}$$

which imply

$$|h_{ij}|^2 \geq \frac{m}{4(m-1)}|\nabla h|^{-2}\left|\nabla |\nabla h|^2\right|^2 + \frac{1}{m-1}|\nabla h|^4 + \frac{1}{m-1}\left\langle \nabla |\nabla h|^2, \nabla h\right\rangle .$$

Then using this in (5) we conclude that

$$\begin{aligned} \frac{1}{2}\Delta |\nabla h|^2 \geq{}& \frac{m}{4(m-1)}|\nabla h|^{-2}\left|\nabla |\nabla h|^2\right|^2 + \frac{1}{m-1}|\nabla h|^4 \\ &- \frac{m-2}{m-1}\left\langle \nabla |\nabla h|^2, \nabla h\right\rangle - (m-1)|\nabla h|^2 . \end{aligned} \tag{6}$$

By (6) we have

$$\frac{1}{2}\Delta G \geq \frac{m}{4(m-1)}\phi^4 G^{-1}\left|\nabla\left(\phi^{-2}G\right)\right|^2 + \frac{1}{m-1}\phi^{-2}G^2$$
$$-\frac{m-2}{m-1}\phi^2\left\langle\nabla\left(\phi^{-2}G\right),\nabla h\right\rangle - (m-1)G + \frac{1}{2}\phi^{-2}\left(\Delta\phi^2\right)G + \left\langle\nabla\phi^2,\nabla\left(\phi^{-2}G\right)\right\rangle.$$

At the maximum point x_0 it follows from (4) that

$$\begin{aligned} 0 \quad \geq \quad & \frac{1}{m-1}G^2 - (m-1)\phi^2 G + \frac{m}{m-1}\left|\nabla\phi\right|^2 G + \frac{2(m-2)}{m-1}\phi\left\langle\nabla\phi,\nabla h\right\rangle G \\ & + \frac{1}{2}\left(\Delta\phi^2\right)G - 4\left|\nabla\phi\right|^2 G. \end{aligned}$$

Since $\phi\left\langle\nabla\phi,\nabla h\right\rangle \geq -\left|\nabla\phi\right| G^{\frac{1}{2}}$ we infer from the above inequality that

$$\begin{aligned} 0 \quad \geq \quad & \frac{1}{2}(m-1)\left(\Delta\phi^2\right)G - (3m-4)\left|\nabla\phi\right|^2 G - (m-1)^2\phi^2 G \\ & -2(m-2)\left|\nabla\phi\right| G^{\frac{3}{2}} + G^2. \end{aligned}$$

This can be written as

$$-(m-1)\phi\Delta\phi + (2m-3)\left|\nabla\phi\right|^2 + (m-1)^2\phi^2 + 2(m-2)\left|\nabla\phi\right| G^{\frac{1}{2}} \geq G. \tag{7}$$

We point out that (7) is true for any cutoff ϕ, but now we want to estimate the left-hand side of (7) from above using our choice of cutoff ϕ. In existing arguments in the literature one would bound $(m-1)^2\phi^2 \leq (m-1)^2$ and then deal with the terms involving $|\nabla\phi|$ and $\Delta\phi$. Our strategy is to make use of the fact that ϕ is small near the boundary. So, if x_0 is close to $\partial B_p(R)$, then it will be easy to see that $G(x_0) \leq (m-1)^2$. On the other hand, if that is not true, then $|\nabla\phi|$ and $\Delta\phi$ will be very small for our choice of ϕ, because ϕ decays very slowly until the boundary.

First, observe that using a standard local gradient estimate (1), we can estimate

$$\sup_{B_p(R)}\left|\nabla h\right| \leq (m-1) + \frac{c_1}{R},$$

where c_1 is a constant depending only on m. Therefore, choosing $R_0(m)$ sufficiently large, we can guarantee that $(m-1) + \frac{c_1}{R} \leq m$. Consequently,

$$G^{\frac{1}{2}}(x_0) = \left|\nabla h\right|(x_0)\,\phi(x_0) \leq \sup_{B_p(R)}\left|\nabla h\right| \leq m.$$

Next, let us consider the function $\theta : [0,R] \to \mathbb{R}$,

$$\theta(r) = \frac{R^2}{r^2}\exp\alpha\left(R - \frac{R^2}{r}\right).$$

Then we see that

$$\theta'(r) = \left(\alpha\frac{R^4}{r^4} - 2\frac{R^2}{r^3}\right)\exp\alpha\left(R - \frac{R^2}{r}\right),$$

so by assuming $R_0 > \frac{2}{\alpha}$ it follows that θ is increasing on $[0,R]$. Consequently,

$$\theta(r) \leq \theta(R) = 1.$$

By (3), we have thus proved that

$$\left|\nabla\phi\right| \leq \alpha. \tag{8}$$

Furthermore, the Laplacian comparison theorem states that

$$\Delta r\left(x_0\right) \leq (m-1)\left(1+\frac{1}{r\left(x_0\right)}\right),$$

so that we find

$$\begin{aligned} -\Delta\phi &= -\phi'\Delta r-\phi'' \\ &\leq \left(\alpha(m-1)\frac{R^2}{r^2}\left(1+\frac{1}{r}\right)+\left(-2\alpha\frac{R^2}{r^3}+\alpha^2\frac{R^4}{r^4}\right)\right)\exp\alpha\left(R-\frac{R^2}{r}\right) \\ &= \alpha\left((m-1)\frac{R^2}{r^2}+(m-3)\frac{R^2}{r^3}+\alpha\frac{R^4}{r^4}\right)\exp\alpha\left(R-\frac{R^2}{r}\right). \end{aligned}$$

Consequently, since $0 \leq \phi \leq 1$ we get

$$-\phi\Delta\phi \leq \alpha\left((m-1)\frac{R^2}{r^2}+(m-3)\frac{R^2}{r^3}+\alpha\frac{R^4}{r^4}\right)\exp\alpha\left(R-\frac{R^2}{r}\right).$$

Plugging all these into (7) and using (3) and (8) it follows that at x_0,

$$\begin{aligned} G &\leq (m-1)^2\phi-(m-1)\phi\Delta\phi+(2m-3)\alpha\left|\nabla\phi\right|+2m(m-2)\left|\nabla\phi\right| \\ &\leq (m-1)^2\left(1-\exp\alpha\left(R-\frac{R^2}{r}\right)\right) \\ &\quad +\{\alpha\left(2m(m-2)+(2m-3)\alpha+(m-1)^2\right)\frac{R^2}{r^2} \\ &\qquad +\alpha(m-1)(m-3)\frac{R^2}{r^3}+(m-1)\alpha^2\frac{R^4}{r^4}\}\exp\alpha\left(R-\frac{R^2}{r}\right). \end{aligned}$$

We can simplify this using that

$$\frac{R^2}{r^2} \leq \frac{R^4}{r^4}, \quad \frac{R^2}{r^3} \leq \frac{R^4}{r^4},$$

and furthermore that for $\alpha \leq 2/3$,

$$2m(m-2)+(3m-4)\alpha+(m-1)^2+(m-1)(m-3) \leq 4(m-1)^2.$$

It follows that at x_0 we have

$$G \leq (m-1)^2+(m-1)^2\left(4\alpha\frac{R^4}{r^4}-1\right)\exp\alpha\left(R-\frac{R^2}{r}\right). \tag{9}$$

Let us now denote $u:[0,R]\to\mathbb{R}$,

$$u(r)=\left(4\alpha\frac{R^4}{r^4}-1\right)\exp\alpha\left(R-\frac{R^2}{r}\right).$$

We want to find the maximum value of u on $[0,R]$, which by (9) will imply the desired estimate for $\left|\nabla h\right|(p)$. We have the following two cases:

Case 1: $r \geq (4\alpha)^{\frac{1}{4}}R$

Then it is easy to see that $u(r) \leq 0$.

Case 2: $r < (4\alpha)^{\frac{1}{4}}R$

In this case it follows trivially that

$$u(r) \leq 4\alpha\frac{R^4}{r^4}\exp\alpha\left(R-\frac{R^2}{r}\right).$$

It is then easy to see that the function $w : [0, R] \to \mathbb{R}$,

$$w(r) = \frac{R^4}{r^4} \exp \alpha \left(R - \frac{R^2}{r} \right),$$

is increasing on $[0, R]$ if we assume $R_0 \geq \frac{4}{\alpha}$.

Indeed,

$$w'(r) = \left(\alpha \frac{R^6}{r^6} - 4 \frac{R^4}{r^5} \right) \exp \alpha \left(R - \frac{R^2}{r} \right) \geq 0$$

if

$$r \leq \frac{\alpha}{4} R^2.$$

But this is clearly true using that $R \geq R_0 \geq \frac{4}{\alpha}$ and that $r \leq R$.

We have proved that w is increasing on $[0, R]$; therefore for any $r < (4\alpha)^{1/4} R$ we get

$$\begin{aligned} u(r) &\leq 4\alpha w(r) \leq 4\alpha w\left((4\alpha)^{1/4} R \right) \\ &= \exp\left(-\alpha \left((4\alpha)^{-1/4} - 1 \right) R \right) \\ &= \exp(-C_2 R). \end{aligned}$$

Note that α can be chosen to be a number not depending on m, and then C_2 is independent of m, too. For example, we can take $\alpha = 2^{-6}$ and then $C_2 = 2^{-6}$.

Summing up, based on Case 1 and Case 2, it follows that for $R \geq R_0$,

$$\begin{aligned} |\nabla \log f|^2 (p) &\leq G(x_0) \\ &\leq (m-1)^2 + (m-1)^2 \exp(-C_2 R), \end{aligned}$$

which is what we claimed.

This proves the local gradient estimate, when the distance function from p is smooth at x_0, the maximum point of G. As mentioned in the beginning, if this is not the case, we consider $\gamma(t)$, the minimizing geodesic from p to x_0. Certainly, we may assume that $x_0 \neq p$, since otherwise the estimates are trivial based on the definition of ϕ.

Then take $q = \gamma(\varepsilon)$ for ε small and define

$$\psi(x) := \phi(d(q, x) + \varepsilon).$$

Since x is in the cut locus of p, then it is not in the cut locus of q, so ψ is smooth at x_0. Moreover, we have $d(q, x) + \varepsilon \geq r(x)$ for any $x \in M$ and $d(q, x_0) + \varepsilon = r(x_0)$. Therefore, using that ϕ is decreasing on $\mathbb{R}$ it follows that $\psi(x) \leq \phi(x)$ for any $x \in M$ and $\psi(x_0) = \phi(x_0)$.

This means that x_0 is the maximum point of $\widetilde{G} := \psi^2 |\nabla h|^2$, which now is smooth at x_0. Performing all the above computations and letting $\varepsilon \to 0$ it is not difficult to see that we still obtain (9) at x_0. The rest of the argument is the same.

This proves (2), which as explained in the beginning of the proof also proves the theorem. □

As a consequence, we get the following sharp lower bound for the Green's function, by integrating the estimate in Theorem 1 along minimizing geodesics. We call M nonparabolic if it admits a positive symmetric Green's function.

Corollary 1. *Let M be a complete noncompact Riemannian manifold of dimension m with $Ric \geq -(m-1)$ on M. If M is nonparabolic, then there exists a constant C depending only on m such that*

$$\sup_{x \in \partial B_p(R)} G(p,x) \geq C \inf_{y \in \partial B_p(1)} G(p,y) \cdot e^{-(m-1)R},$$

where $G(p,x)$ is the positive symmetric Green's function with a pole at $p \in M$.

On $\mathbb{H}^n$ let us now take Green's function $f(x) := G(q,x)$ with a pole at q. It is known that we have the following formula:

$$f(x) = \int_{d(q,x)}^{\infty} \frac{dt}{A(t)}, \tag{10}$$

where $A(t)$ is the area of $\partial B_q(t)$. Then it can be checked that for p such that $d(p,q) = R \geq 1$,

$$(n-1) + C^{-1}e^{-2R} \leq |\nabla \log f|(p) \leq (n-1) + Ce^{-2R}.$$

This shows that the exponential-type decay in our theorem is sharp. Also, (10) shows that the decay estimate in the corollary is sharp.

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