PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 3, March 2011, Pages 881–883
S 0002-9939(2010)10652-4
Article electronically published on September 29, 2010

ON ADDITIVE COMPLEMENTS. II

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(Communicated by Matthew A. Papanikolas)

Abstract. Two infinite sequences A and B of non-negative integers are called additive complements if their sum contains all sufficiently large integers. Let $A(x)$ and $B(x)$ be the counting functions of A and B and let $\limsup_{x\to\infty} A(x)B(x)/x = \alpha(A,B)$. Recently, the authors [Proceedings of the American Mathematical Society 138 (2010), 1923-1927] proved that for additive complements A and B, if $\alpha(A,B) < 5/4$ or $\alpha(A,B) > 2$, then $A(x)B(x) - x \to +\infty$ as $x \to \infty$. In this paper, we prove that for any $\varepsilon > 0$ there exist additive complements A and B with $2-\varepsilon < \alpha(A,B) < 2$ and $A(x)B(x)-x = 1$ for infinitely many positive integers x.

1. Introduction

Two infinite sequences A and B of non-negative integers are called additive complements if their sum contains all sufficiently large integers.

Let $A(x)$ and $B(x)$ be the counting functions of A and B. Namely,

$$A(x) = \sum_{\substack{a\le x\\ a\in A}} 1 \quad \text{and} \quad B(x) = \sum_{\substack{b\le x\\ b\in B}} 1.$$

For the construction of additive complements A and B with $A(x)B(x) \sim x$, one may refer to [6].

In 1964, Danzer [1] conjectured that for additive complements A and B, if

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \le 1,$$

then

$$A(x)B(x) - x \to +\infty \qquad \text{as} \qquad x \to +\infty.$$

(See also [2], [4, p. 75] and [5].) In [7], Sárközy and Szemerédi proved this conjecture. Recently, the authors [3] proved that

Received by the editors April 14, 2010.
2010 *Mathematics Subject Classification.* Primary 11B13, 11B34.
Key words and phrases. Additive complements, sequences, counting functions.
This work was supported by the National Natural Science Foundation of China, Grant No. 10771103.

Theorem A. *For additive complements A and B, if*

$$\limsup_{x\to\infty}\frac{A(x)B(x)}{x}>2 \text{ or } \limsup_{x\to\infty}\frac{A(x)B(x)}{x}<\frac{5}{4},$$

then $A(x)B(x)-x\to+\infty$ as $x\to\infty$.

Once we conjectured that if A and B are additive complements, then $A(x)B(x)-x\to+\infty$ as $x\to+\infty$. After paper [3] went to print, we found a counterexample, so we removed the conjecture from [3] when we checked the proofs. In this paper, as a complement to [3], we prove the following result.

Theorem 1. *For any integer a with $a\geq 2$, there exist additive complements A and B such that $A(x)B(x)-x=1$ for infinitely many positive integers x and*

$$\limsup_{x\to\infty}\frac{A(x)B(x)}{x}=\frac{2a+2}{a+2}.$$

2. Proofs of Theorem 1

We will use the following lemmas. Their proofs are clear.

Lemma 1. *For any integers a_1, a_2, b_1, b_2 and u, if $a_1b_2-a_2b_1\geq 0$ and $a_2x+b_2>0$ for all $0\leq x\leq u$, then for all $0\leq x\leq u$ we have*

$$\frac{a_1x+b_1}{a_2x+b_2}\leq\frac{a_1u+b_1}{a_2u+b_2}.$$

Lemma 2. *For any integers a_1, a_2, b_1, b_2 and v, if $a_2>0$, $a_1b_2-a_2b_1\geq 0$ and $a_2v+b_2>0$, then*

$$\frac{a_1v+b_1}{a_2v+b_2}\leq\frac{a_1}{a_2}.$$

Proof of Theorem 1. For any integer a with $a\geq 2$, assume that

$$A_j=\{\sum\varepsilon_i a^{2i+j}\mid\varepsilon_i=0,1,\dots,a-1\},\quad j=0,1,$$

where the summation is a finite sum. It is clear that A_0 and A_1 are additive complements.

For $x_k=a^{2k}-1$ we have

$$A_0(x_k)=A_1(x_k)=a^k,\quad A_0(x_k)A_1(x_k)-x_k=1.$$

Now we prove that

$$\limsup_{x\to\infty}\frac{A_0(x)A_1(x)}{x}=\frac{2a+2}{a+2}.$$

For $x_{l,j}=a^{2l+j}+(a-1)\sum_{i=0}^{l-1}a^{2i+j}(j=0,1)$, we have $A_j(x_{l,j})=2a^l$, $A_{1-j}(x_{l,j})=a^{l+j}$. Then

$$\frac{A_0(x_{l,j})A_1(x_{l,j})}{x_{l,j}}=\frac{2}{(1+\frac{1}{a+1})-\frac{1}{a+1}\frac{1}{a^{2l}}}\to\frac{2a+2}{a+2},\quad l\to\infty.$$

Now it suffices to prove that for any positive integers $x\neq x_{l,j}$ we have

$$\frac{A_0(x)A_1(x)}{x}\leq\frac{2a+2}{a+2}.$$

If a positive integer $x\notin A_0\bigcup A_1$, then

$$\frac{A_0(x)A_1(x)}{x}=\frac{A_0(x-1)A_1(x-1)}{x}<\frac{A_0(x-1)A_1(x-1)}{x-1}.$$

So we may assume that $x \in A_0 \bigcup A_1$. Hence, there exist integers l, j with $j \in \{0, 1\}$ and $x = \epsilon_j a^j + \epsilon_{2+j} a^{2+j} + \cdots + \epsilon_{2l+j} a^{2l+j}$, where $\epsilon_{2l+j} > 0$ and $\epsilon_{2i+1+j} = 0$ for all i. Then $A_{1-j}(x) = a^{l+j}$. If $\epsilon_{2l+j} \geq 2$, then, by $A_j(x) \leq \epsilon_{2l+j} a^l + a^l$, we have

$$\frac{A_j(x)A_{1-j}(x)}{x} \leq \frac{a^{2l+j}\epsilon_{2l+j} + a^{2l+j}}{x} \leq \frac{a^{2l+j}\epsilon_{2l+j} + a^{2l+j}}{a^{2l+j}\epsilon_{2l+j}} \leq \frac{3}{2} \leq \frac{2a+2}{a+2}.$$

So we only need to consider $\epsilon_{2l+j} = 1$. Since $x \neq x_{l,j}$, we may assume that

$$x = \sum_{i=0}^{m-1} \varepsilon_{2i+j} a^{2i+j} + (a-1) \sum_{i=m}^{l-1} a^{2i+j} + a^{2l+j},$$

where $0 < m \leq l$ and $\epsilon_{2m+j-2} < a - 1$ (if $m = l$, then the second sum is zero). Thus

$$A_j(x) \leq a^l + (a-1) \sum_{i=m}^{l-1} a^i + \epsilon_{2m+j-2} a^{m-1} + a^{m-1} = 2a^l - a^m + (\epsilon_{2m+j-2} + 1) a^{m-1}.$$

Hence by Lemmas 1 ($x = \epsilon_{2m+j-2}$) and 2 ($v = a^{2l+j}$) we have

$$\begin{aligned}
\frac{A_{1-j}(x)A_j(x)}{x} &\leq \frac{a^{l+j}(2a^l - a^m + (\epsilon_{2m+j-2} + 1)a^{m-1})}{a^{2l+j} + (a-1)(a^{2m+j} + \cdots + a^{2l+j-2}) + \epsilon_{2m+j-2} a^{2m+j-2}} \\
&= \frac{2a^{2l+j} - a^{m+l+j} + (\epsilon_{2m+j-2} + 1)a^{m+l+j-1}}{(1 + \frac{1}{a+1})a^{2l+j} - \frac{1}{a+1} a^{2m+j} + \epsilon_{2m+j-2} a^{2m+j-2}} \\
&\leq \frac{2a^{2l+j} - a^{m+l+j} + (a-2+1)a^{m+l+j-1}}{(1 + \frac{1}{a+1})a^{2l+j} - \frac{1}{a+1} a^{2m+j} + (a-2) a^{2m+j-2}} \\
&\leq \frac{2a+2}{a+2}.
\end{aligned}$$

This completes the proof of Theorem 1. □

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