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A SCHWARZ LEMMA FOR THE MODULUS OF A VECTOR-VALUED ANALYTIC FUNCTION

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Abstract. It is proved that

$$|\nabla|f|(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D},$$

where $f : \mathbb{D} \mapsto \mathbb{B}_k$ is an analytic function from the unit disk $\mathbb{D}$ into the unit ball $\mathbb{B}_k \subset \mathbb{C}^k$. Applications to the Lipschitz condition of the modulus of a $\mathbb{C}^k$-valued function are given.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$. The classical Schwarz lemma states that if $f : \mathbb{D} \mapsto \mathbb{D}$ is an analytic function, then

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{1.1}$$

This inequality does not hold for analytic functions with values in $\mathbb{B}_k := \{w \in \mathbb{C}^k : |w| < 1\}$, $k \geq 2$. For instance, the function $f(z) = (z, 1)/\sqrt{2}$ satisfies

$$|f'(0)| = \sqrt{1-|f(0)|^2} > 1-|f(0)|^2. \tag{1.2}$$

However (1.1) can also be written as

$$|\nabla|f|(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}, \quad \text{for } z \in \mathbb{D} \text{ and } f(z) \neq 0, \tag{1.3}$$

because scalar-valued analytic functions satisfy

$$|\nabla|f|(z)| = |f'(z)|, \quad \text{for } z \in \mathbb{D} \text{ and } f(z) \neq 0. \tag{1.4}$$

If we define

$$|\nabla|f|(z)| = |f'(z)|, \quad \text{for } z \in \mathbb{D} \text{ and } f(z) = 0, \tag{1.5}$$

then the inequality in (1.3) holds for all $z \in \mathbb{D}$. It turns out that this form of Schwarz's lemma extends to vector-valued functions. Precisely:

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Theorem 1.1. *Let $f : \mathbb{D} \mapsto \mathbb{B}_k$ be an analytic function. Then*

$$|\nabla |f|(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{1.6}$$

In the standard way, we deduce the following consequence:

Theorem 1.2. *Let $f : \mathbb{D} \mapsto \mathbb{B}_k$ be an analytic function. Then*

$$\left| \frac{|f(z)| - |f(a)|}{1 - |f(z)|\,|f(a)|} \right| \le \left| \frac{z-a}{1-\bar{a}z} \right|, \quad a, z \in \mathbb{D}. \tag{1.7}$$

Of course, we have to explain the meaning of $|\nabla g|$, where $g = |f|$. If $g(z) \neq 0$, then g is $\mathbb{R}$-differentiable at z and therefore ∇g is treated as the ordinary gradient. In the general case we set

$$|\nabla g(z)| = \limsup_{h \to 0} \frac{|g(z+h) - g(z)|}{|h|}. \tag{1.8}$$

If $g(z) \neq 0$, this coincides with the ordinary definition of $|\nabla g|$. If $g(z) = 0$, then

$$\begin{aligned} |\nabla g(z)| &= \limsup_{h \to 0} \frac{|f(z+h)|}{|h|} \\ &= |f'(z)| \\ &= \Big(\sum_{j=1}^{k} |{f_j}'(z)|^2 \Big)^{1/2}, \end{aligned} \tag{1.9}$$

where $f = (f_1, f_2, \ldots, f_k)$. Let

$$g = |f| = (|f_1|^2 + |f_2|^2 + \ldots + |f_k|^2)^{1/2}.$$

Since g is real-valued, we have $|\nabla g| = 2|\partial g|$, where

$$\partial g = \frac{1}{2}\Big(\frac{\partial g}{\partial x} - i \frac{\partial g}{\partial y} \Big), \quad z = x + iy.$$

It follows that

$$\begin{aligned} |\nabla |f|(z)| &= \frac{1}{|f(z)|} \Big| \sum_{j=1}^{k} \bar{f}_j(z)\, {f_j}'(z) \Big| \\ &= \frac{1}{|f(z)|} |\langle f'(z), f(z) \rangle|, \quad \text{if } f(z) \neq 0. \end{aligned} \tag{1.10}$$

Here $\langle z, w \rangle = z_1 \bar{w}_1 + \ldots + z_k \bar{w}_k$.

As a consequence of (1.1) we have

$$|f'(0)| \le 2(1 - |f(0)|). \tag{1.11}$$

Applying this to the function $w \mapsto f(z + \varepsilon w)/M$, where $0 < \varepsilon < 1 - |z|$,

$$M = \sup_{w \in D_\varepsilon(z)} |f(w)|, \quad D_\varepsilon(z) = \{w : |w - z| < \varepsilon\},$$

we obtain

$$|f'(z)| \le \frac{2}{\varepsilon} \sup_{w \in D_\varepsilon(z)} (|f(w)| - |f(z)|). \tag{1.12}$$

The importance of this inequality lies in the fact that, when combined with a classical theorem of Hardy and Littlewood, it yields a simple proof (see [6]) of the

following result of Dyakonov [2]. Let $\Lambda_\alpha(\mathbb{D})$, $0 < \alpha \le 1$, denote the class of all functions $f : \mathbb{D} \mapsto \mathbb{C}$ for which $|f(z) - f(w)| \le \text{const.} \times |z - w|^\alpha$ for all $z, w \in \mathbb{D}$.

Theorem D. *If f is analytic and $0 < \alpha \le 1$, then $f \in \Lambda_\alpha(\mathbb{D})$ if and only if $|f| \in \Lambda_\alpha(\mathbb{D})$.*

Indeed, from the hypothesis $|f(w)| - |f(z)| \le |z - w|^\alpha$ and (1.12) it follows that $|f'(z)| \le 2(1 - |z|)^{\alpha-1}$. Now the proof is completed by the theorem of Hardy and Littlewood mentioned earlier:

Theorem HL. *A function f analytic in $\mathbb{D}$ belongs to $\Lambda_\alpha(\mathbb{D})$ $(0 < \alpha \le 1)$ if and only if*

$$|f'(z)| \le \text{const.} \times (1 - |z|)^{\alpha-1}, \quad z \in \mathbb{D}. \tag{1.13}$$

Note also that Theorem D can be viewed as an improvement of the Hardy-Littlewood theorem on conjugate functions (see [3] and [4]).

However, Theorem D does not extend to $\mathbb{C}^k$-valued functions ($k \ge 2$), which implies that the same holds for the relation (1.1) (see Section 3). So we have to consider functions with additional properties (see Theorem 3.2 and 3.3).

2. Proof of Theorem 1.1

Lemma 2.1. *If $f : \mathbb{D} \mapsto \mathbb{C}^k$ is analytic and $f(\mathbb{D}) \subset \mathbb{D}$, then*

$$|f'(0)| \le \sqrt{1 - |f(0)|^2}. \tag{2.1}$$

Proof. From the hypothesis $f(\mathbb{D}) \subset \mathbb{D}$ it follows that

$$\begin{aligned} 1 &> \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt \\ &\ge |f(0)|^2 + |f'(0)|^2 r^2, \quad 0 < r < 1. \end{aligned} \tag{2.2}$$

The result follows. □

Proof of Theorem 1.1. It suffices to prove that

$$|\nabla |f|(0)| \le 1 - |f(0)|^2 \tag{2.3}$$

and then to apply this to the functions $f \circ \sigma_a$, where

$$\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad a, z \in \mathbb{D}. \tag{2.4}$$

If $f(0) = 0$, then, by Lemma 2.1, we have $|\nabla |f|(0)| = |f'(0)| \le 1$, as desired. If $f(0) \ne 0$, then we consider the scalar-valued function $h(z) = \langle f(z), f(0)/|f(0)| \rangle$. Applying the classical Schwarz's lemma to h, we get

$$|\nabla |f|(0)| = |\langle f'(0), f(0)/|f(0)| \rangle| = |h'(0)| \le 1 - |h(0)|^2 = 1 - |f(0)|^2,$$

which completes the proof. □

Remark 2.2. If $k = 2$, then equality in (2.1) is attained for $f(z) = (z, 1)/\sqrt{2}$.

Remark 2.3. Applying (2.3) to the function $f \circ \sigma_z$, we get

$$|f'(z)| \le \frac{\sqrt{1 - |f(z)|^2}}{1 - |z|^2}. \tag{2.5}$$

3. Lipschitz conditions on the modulus

Theorem D, as noted in the Introduction, does not hold in the class of $\mathbb{C}^2$-valued analytic functions. To see this, consider the function $f(z) = ((1-z)^{\alpha/2}, 1)$, $0 < \alpha \le 1$. We have

$$\begin{aligned} \big|\, |f(z)| - |f(w)| \,\big| &= |\sqrt{|1-z|^\alpha + 1} - \sqrt{|1-w|^\alpha + 1}\,| \\ &\le \big|\, |1-w|^\alpha - |1-z|^\alpha \,\big| \\ &\le |z-w|^\alpha, \end{aligned}$$

while

$$|f(1) - f(r)| = (1-r)^{\alpha/2}, \quad 0 < r < 1.$$

This shows that in the following result the index $\alpha/2$ is optimal.

Theorem 3.1. If $f : \mathbb{D} \mapsto \mathbb{C}^k$, $k \ge 2$, is analytic and if $|f| \in \Lambda_\alpha(\mathbb{D})$, $0 < \alpha \le 1$, then $f \in \Lambda_{\alpha/2}(\mathbb{D})$.

This fact is a consequence of (2.5). See [1] for the case of analytic functions with values in a Banach space.

On the other hand, since Theorem HL remains valid for $\mathbb{C}^k$-valued functions (with the obvious definition of Λ_α), we have the following consequence of Theorem 1.1 and formula (1.10):

Theorem 3.2. *Let $f : \mathbb{D} \mapsto \mathbb{C}^k$ be an analytic function such that*

$$|f'(z)|\,|f(z)| \le K|\langle f'(z), f(z)\rangle|, \quad z \in \mathbb{D}, \tag{3.1}$$

where K is a constant independent of z. Let $0 < \alpha \le 1$. Then $f \in \Lambda_\alpha(\mathbb{D})$ if and only if $|f| \in \Lambda_\alpha(\mathbb{D})$.

In [2], Dyakonov proved another important result:

Theorem D2. *Let $f : \mathbb{D} \mapsto \mathbb{C}$ be analytic on $\mathbb{D}$ and continuous on $\mathbb{D} \cup \partial\mathbb{D}$, let $0 < \alpha < 1$, and let $\mathcal{P}g$ denote the Poisson integral of $g \in L^1(\mathbb{T})$,*

$$\mathcal{P}g(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, |d\zeta|, \quad z \in \mathbb{D}.$$

If $g = |f|$, then the following conditions are equivalent:

(i) *$f \in \Lambda_\alpha(\mathbb{D})$.*

(ii) *The restriction $g_{|\partial\mathbb{D}}$ belongs to $\Lambda_\alpha(\partial\mathbb{D})$ and there exists a constant C such that*
$$\mathcal{P}g(z) - g(z) \le C(1-|z|)^\alpha, \quad \text{for all } z \in \mathbb{D}.$$

(iii) *$g_{|\partial\mathbb{D}}$ belongs to $\Lambda_\alpha(\partial\mathbb{D})$ and there exists a constant C such that $g(\zeta) - g(r\zeta) \le C(1-r)^\alpha$ for all $\zeta \in \partial\mathbb{D}$ and $0 < r < 1$.*

(iv) *There exists a constant C such that $g(\zeta) - g(z) \le C|\zeta - z|^\alpha$ for all $\zeta \in \partial\mathbb{D}$ and $z \in \mathbb{D}$.*

Using the approach of the papers [5] and [6], one can prove the following.

Theorem 3.3. *Theorem* D2 *remains true if we assume that $f : \mathbb{D} \mapsto \mathbb{C}^k$ $(k \ge 2)$ is an analytic function that satisfies* (3.1).

Further generalizations can be made by using arbitrary majorants; we omit that discussion here.

Finally, note the following fact:

Proposition 3.4. *If f satisfies* (3.1) *and $\varphi : \mathbb{D} \mapsto \mathbb{D}$ is an analytic function, then $f \circ \varphi$ satisfies* (3.1) *(with the same constant K).*

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