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REALIZATION OF THE MAPPING CLASS GROUP OF HANDLEBODY BY DIFFEOMORPHISMS

SUSUMU HIROSE

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Abstract. For the oriented 3-dimensional handlebody constructed from a 3-ball by attaching g 1-handles, it is shown that the natural surjection from the group of orientation-preserving diffeomorphisms of it to the mapping class group of it has no section when g is at least 6.

Let M be an n-dimensional compact oriented manifold and S be a subset of ∂M. We denote the group of orientation-preserving diffeomorphisms of M whose restrictions on S are the identity by $\mathrm{Diff}(M,S)$, the subgroups of them consisting of elements that are isotopic to the identity by $\mathrm{Diff}_0(M,S)$, and the quotient group $\mathrm{Diff}(M,S)/\mathrm{Diff}_0(M,S)$ by $\mathcal{M}(M,S)$. For an element f of $\mathrm{Diff}(M,S)$, let $[f]$ be the element of $\mathcal{M}(M,S)$ represented by f. The homomorphism $\pi_{M,S}$ from $\mathrm{Diff}(M,S)$ to $\mathcal{M}(M,S)$ defined by $\pi_{M,S}(f) = [f]$ is a surjection. Let Γ be a subgroup of $\mathcal{M}(M,S)$. We call a homomorphism s from Γ to $\mathrm{Diff}(M,S)$ which satisfies $\pi_{M,S} \circ s = id_\Gamma$ a *section* for $\pi_{M,S}$ over Γ. Morita [7] showed that the natural surjection from $\mathrm{Diff}^2(\Sigma_g)$ to the mapping class group $\mathcal{M}(\Sigma_g)$ of Σ_g has no section over $\mathcal{M}(\Sigma_g)$ when $g \geq 5$. Markovic [5] (when $g \geq 6$) and Markovic and Saric [6] (when $g \geq 2$) showed that the natural surjection from $\mathrm{Homeo}(\Sigma_g)$ to $\mathcal{M}(\Sigma_g)$ has no section over $\mathcal{M}(\Sigma_g)$. By using the different method from them, Franks and Handel [2] showed that the natural surjection from $\mathrm{Diff}(\Sigma_g)$ to $\mathcal{M}(\Sigma_g)$ has no section over $\mathcal{M}(\Sigma_g)$ when $g \geq 3$.

Let H_g be an oriented 3-dimensional handlebody of genus g which is an oriented 3-manifold constructed from a 3-ball by attaching g 1-handles. Let Σ_g be an oriented closed surface of genus g; then $\partial H_g = \Sigma_g$. The restriction to the boundary defines a homomorphism $\rho_\partial : \mathrm{Diff}(H_g) \to \mathrm{Diff}(\Sigma_g)$, and ρ_∂ induces an injection $\mathcal{M}(H_g) \hookrightarrow \mathcal{M}(\Sigma_g)$ since H_g is an irreducible 3-manifold. We will show:

Theorem 1. *If $g \geq 6$, there is no section for $\pi_{H_g} : \mathrm{Diff}(H_g) \to \mathcal{M}(H_g)$ over $\mathcal{M}(H_g)$.*

For contradiction, we assume that there is a section $s : \mathcal{M}(H_g) \to \mathrm{Diff}(H_g)$. Let Γ be a subgroup of $\mathcal{M}(H_g)$, and i_Γ be the inclusion from Γ to $\mathcal{M}(H_g)$. Then Γ is a

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subgroup of $\mathcal{M}(\Sigma_g)$, and the composition $\rho_\partial \circ s \circ i_\Gamma$ is a section for $\pi_{\Sigma_g} : \mathrm{Diff}(\Sigma_g) \to \mathcal{M}(\Sigma_g)$ over Γ. Therefore, if we can find a subgroup Γ of $\mathcal{M}(H_g)$, over which there is no section for π_{Σ_g}, then Theorem 1 follows.

Let D be a 2-disk in Σ_g, and $\Sigma_{g,1}$ be $\Sigma_g \setminus int\, D$. Let c be an essential simple closed curve on Σ_g such that $\Sigma_g \setminus c$ is not connected. Then the closure of one component of $\Sigma_g \setminus c$ is diffeomorphic to $\Sigma_{g_1,1}$, and the closure of the other component of $\Sigma_g \setminus c$ is diffeomorphic to $\Sigma_{g_2,1}$. We remark that $g = g_1 + g_2$ and $g_1, g_2 \geq 1$. These diffeomorphisms induce injections $\mathcal{M}(\Sigma_{g_1,1}, \partial\Sigma_{g_1,1}) \to \mathcal{M}(\Sigma_g)$ and $\mathcal{M}(\Sigma_{g_2,1}, \partial\Sigma_{g_2,1}) \to \mathcal{M}(\Sigma_g)$ (see [8]). By these injections, we consider $\mathcal{M}(\Sigma_{g_1,1}, \partial\Sigma_{g_1,1})$ and $\mathcal{M}(\Sigma_{g_2,1}, \partial\Sigma_{g_1,1})$ as subgroups of $\mathcal{M}(\Sigma_g)$. From Theorem 1.6 in [2] proved by Franks and Handel, we see:

Theorem 2 ([2]). *Let Γ_1 be a nontrivial finitely generated subgroup of $\mathcal{M}(\Sigma_{g_1,1}, \partial\Sigma_{g_1,1})$ such that $H^1(\Gamma_1, \mathbb{R}) = 0$ and μ be an element of $\mathcal{M}(\Sigma_{g_2,1}, \partial\Sigma_{g_2,1})$ which is represented by a pseudo-Anosov homeomorphism on $int\, \Sigma_{g_2,1}$. Then there is no section for $\pi_{\Sigma_g} : \mathrm{Diff}(\Sigma_g) \to \mathcal{M}(\Sigma_g)$ over $\langle \Gamma_1, \mu \rangle$, where $\langle \Gamma_1, \mu \rangle$ is a subgroup of $\mathcal{M}(\Sigma_g)$ generated by elements of Γ_1 and μ.*

We assume $g \geq 6$. The 3-manifold $\Sigma_{2,1} \times [0,1]$ is diffeomorphic to H_4. Let D_1 be a 2-disk in $int\,(\partial\Sigma_{2,1} \times [0,1]) \subset \partial(\Sigma_{2,1} \times [0,1])$, D_2 and D_3 be disjoint 2-disks on ∂H_{g-6}, and D_4 be a 2-disk on ∂H_2. Along these 2-disks, we glue $\Sigma_{2,1} \times [0,1]$, H_{g-6} and H_2 such that $D_1 = D_2$, $D_3 = D_4$. Then the 3-manifold obtained as a result is diffeomorphic to H_g. By the above construction, we get two natural inclusions $\Sigma_{2,1} \times [0,1] \hookrightarrow H_g$ and $H_2 \hookrightarrow H_g$. These inclusions induce natural homomorphisms $i_1 : \mathcal{M}(\Sigma_{2,1} \times [0,1], \partial\Sigma_{2,1} \times [0,1]) \to \mathcal{M}(H_g)$ and $i_2 : \mathcal{M}(H_2, D_4) \to \mathcal{M}(H_g)$. If $[h]$ is in $\mathcal{M}(\Sigma_{2,1} \times [0,1], \partial\Sigma_{2,1} \times [0,1])$ (resp. $\mathcal{M}(H_2, D_4)$) represented by $h \in \mathrm{Diff}(\Sigma_{2,1} \times [0,1], \partial\Sigma_{2,1} \times [0,1])$ (resp. $\mathrm{Diff}(H_2, D_4)$), then $i_1([h])$ (resp. $i_2([h])$) is represented by the diffeomorphism obtained by extending h to H_g using the identity mapping on $H_g \setminus \Sigma_{2,1} \times [0,1]$ (resp. $H_g \setminus H_2$).

We define homomorphisms $\Pi : \mathrm{Diff}(\Sigma_{2,1}, \partial\Sigma_{2,1}) \to \mathrm{Diff}(\Sigma_{2,1} \times [0,1], \partial\Sigma_{2,1} \times [0,1])$ by $\Pi(h) = h \times id_{[0,1]}$, and $I_1 : \mathrm{Diff}(\Sigma_{2,1} \times [0,1], \partial\Sigma_{2,1} \times [0,1]) \to \mathrm{Diff}(H_g)$ by the identity on $H_g \setminus \Sigma_{2,1} \times [0,1]$. Then the composition $I_1 \circ \Pi$ induces a homomorphism $P : \mathcal{M}(\Sigma_{2,1}, \partial\Sigma_{2,1}) \to \mathcal{M}(H_g)$. By applying Corollary 4.2 of [8] to the subsurface $\Sigma_{2,1} \times \{0,1\} \subset \partial H_g$, the injectivity of P is shown. Korkmaz [4] showed that $H_1(\mathcal{M}(\Sigma_{2,1}, \partial\Sigma_{2,1}), \mathbb{Z}) = \mathbb{Z}/10\,\mathbb{Z}$; hence $H^1(\mathcal{M}(\Sigma_{2,1}, \partial\Sigma_{2,1}), \mathbb{R}) = 0$. Therefore, $\Gamma_1 = P(\mathcal{M}(\Sigma_{2,1}, \partial\Sigma_{2,1}))$ satisfies the assumption of Theorem 2 when $g_1 = g - 2$, $g_2 = 2$.

Fathi and Laudenbach [3] constructed a pseudo-Anosov homeomorphism ϕ on $\partial(H_2)$ which is a restriction of a homeomorphism on H_2. The definitions of pseudo-Anosov homeomorphisms and terminologies (e.g., singular foliation) related to them can be found in [1]. Any pseudo-Anosov homeomorphism preserves the set of singular points of the singular foliation which is preserved by this homeomorphism. Since the number of singular points of the singular foliation is finite, a proper power of ϕ, say ϕ^n, fixes some points. Let p be a point fixed by ϕ^n. Then ϕ^n defines a pseudo-Anosov homeomorphism on $\partial(H_2) \setminus p = int\, \Sigma_{2,1}$. Let μ be an element of $\mathcal{M}(\Sigma_{2,1}, \partial\Sigma_{2,1}) \subset \mathcal{M}(\Sigma_g)$ represented by this homomorphism. Then μ is an element of $\mathcal{M}(H_g)$ and satisfies the assumption of Theorem 2 when $g_1 = g - 2$, $g_2 = 2$.

Then $\langle P(\mathcal{M}(\Sigma_{2,1}, \partial\Sigma_{2,1})), \mu\rangle$ is a subgroup of $\mathcal{M}(H_g)$ and, by Theorem 2, there is no section over $\langle P(\mathcal{M}(\Sigma_{2,1}, \partial\Sigma_{2,1})), \mu\rangle$. Therefore, there is no section for π_{H_g} : $\mathrm{Diff}(H_g) \to \mathcal{M}(H_g)$ over $\mathcal{M}(H_g)$.

Remark 3. Two subgroups G_1 and G_2 of $\mathcal{M}(\Sigma_g)$ are *conjugate* if there is an element $h \in \mathcal{M}(\Sigma_g)$ such that $hG_1h^{-1} = G_2$. When two subgroups G_1 and G_2 of $\mathcal{M}(\Sigma_g)$ are conjugate, there is a section for π_{Σ_g} over G_1 if and only if there is a section for π_{Σ_g} over G_2. In the above proof of Theorem 1, it is shown that there is no section for π_{Σ_g} over $\mathcal{M}(H_g)$ under any identification of ∂H_g with Σ_g, since, under the different identifications of ∂H_g with Σ_g, $\mathcal{M}(H_g)$ is regarded as a conjugate subgroup of $\mathcal{M}(\Sigma_g)$.

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Department of Mathematics, Graduate School of Science and Engineering, Saga University, Saga, 840–8502 Japan

E-mail address: hirose@ms.saga-u.ac.jp