

## SIERPIŃSKI-ZYGMUND FUNCTIONS AND OTHER PROBLEMS ON LINEABILITY

JOSÉ L. GÁMEZ-MERINO, GUSTAVO A. MUÑOZ-FERNÁNDEZ, VÍCTOR M. SÁNCHEZ,  
AND JUAN B. SEOANE-SEPÚLVEDA

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*Dedicated to Professor Richard M. Aron on his 65<sup>th</sup> anniversary*

**ABSTRACT.** We find *large* algebraic structures inside the following sets of *pathological* functions: (i) perfectly everywhere surjective functions, (ii) differentiable functions with almost nowhere continuous derivatives, (iii) differentiable nowhere monotone functions, and (iv) Sierpiński-Zygmund functions. The conclusions obtained on (i) and (iii) are improvements of some already known results.

### 1. INTRODUCTION AND NOTATION

Lately it has become a sort of trend in Mathematical Analysis to search for what are often large algebraic structures (vector spaces or algebras, among others) of functions on  $\mathbb{K}$  ( $\mathbb{R}$  or  $\mathbb{C}$ ) that enjoy certain *pathological* properties. This paper continues this ongoing search. As it is becoming a standard concept, given a property we say that the subset  $M$  of functions on  $\mathbb{K}$  satisfying it is  $\alpha$ -*lineable* if  $M \cup \{0\}$  contains a vector space of dimension  $\alpha$  (finite or infinite). If  $M$  contains an infinite-dimensional vector space, it will be called *lineable* for short. We shall also say that  $M$  is  $(\alpha, \beta)$ -algebrable if there exists an algebra  $\mathcal{B}$  such that  $\mathcal{B} \subset M \cup \{0\}$ ,  $\dim \mathcal{B} = \alpha$ , and  $\text{card } S = \beta$ , where  $\alpha$  and  $\beta$  are two cardinal numbers and  $S$  is a minimal system of generators of  $\mathcal{B}$ . ( $S = \{z_\gamma\}_{\gamma \in \Gamma}$  is said to be a *minimal set of generators of  $\mathcal{B}$*  if  $\mathcal{B} = \mathcal{A}(S)$  is the algebra generated by  $S$  and for every  $\gamma_0 \in \Gamma$ ,  $z_{\gamma_0} \notin \mathcal{A}(S \setminus \{z_{\gamma_0}\})$ .) At times, we shall say that  $M$  is *algebrable* if  $M \cup \{0\}$  contains an infinitely generated algebra.

This notion of lineability was coined by Gurariy and first introduced in [4], while the word algebrability appeared just recently (see, for instance, [5, 6]). In recent years a wide variety of examples of vector spaces of functions on  $\mathbb{R}$  or  $\mathbb{C}$  enjoying certain pathological properties has been constructed. One of the earliest results in this direction was proved by Gurariy [16, 17], who showed that the set of nowhere

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differentiable functions on  $[0, 1]$  is lineable. (This famous result was improved later and even Banach spaces of such functions were constructed.) More recently, in [14] the authors constructed, among other examples, a Banach space of differentiable functions on  $\mathbb{R}^n$  failing the Denjoy-Clarkson property, a Banach space of non-Riemann-integrable bounded functions that nevertheless have an antiderivative at each point of an interval, and a Banach space of infinitely differentiable functions that vanish at infinity and are not the Fourier transform of any Lebesgue-integrable function. Also, in [5], the authors constructed, given any set  $E \subset \mathbb{T}$  of Lebesgue measure zero, an infinite-dimensional, infinitely generated dense subalgebra of  $\mathcal{C}(\mathbb{T})$  every non-zero element of which has a Fourier series expansion divergent in  $E$ . More authors have been working on these types of problems related to lineability and algebraicity in the framework of vector measures, operator theory, polynomials, and holomorphy. (We refer to, e.g., [1, 7, 8, 10, 11, 21, 22] for some recent advances in this theory.)

This paper is arranged as follows. In Section 2 we introduce the terms *strongly everywhere surjective* function and *perfectly everywhere surjective* function and show that these are particular cases of everywhere surjectivity. (See [4]: a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is said to be everywhere surjective if  $f(I) = \mathbb{R}$  for every non-trivial interval  $I$ .) In [4, Theorem 4.3] the authors proved that the set of everywhere surjective functions on  $\mathbb{R}$  is  $2^{\mathfrak{c}}$ -lineable, where  $\mathfrak{c}$  stands for the continuum. Here, we improve [4, Theorem 4.3] by showing that the set of strongly everywhere surjective functions and the set of perfectly everywhere surjective functions are both  $2^{\mathfrak{c}}$ -lineable as well.

In Section 3 we study the lineability of the set of differentiable functions whose derivatives have a large set of discontinuities. In particular, in this section we prove that the set of differentiable functions on  $\mathbb{R}$  whose derivatives are discontinuous almost everywhere is  $\mathfrak{c}$ -lineable.

Section 4 deals with differentiable nowhere monotone functions. In [4] the authors proved that the set  $\mathcal{DNM}(\mathbb{R})$  of differentiable functions on  $\mathbb{R}$  that are nowhere monotone is  $\aleph_0$ -lineable in  $\mathcal{C}(\mathbb{R})$ . Here we show that this set is actually  $\mathfrak{c}$ -lineable, and that this result is optimal in terms of dimension.

Section 5 focuses on the class of Sierpiński-Zygmund functions. These are highly pathological functions that are not Borel functions on any set with “large” cardinality. We prove that this class is  $\mathfrak{c}^+$ -lineable and therefore, under the Generalized Continuum Hypothesis (GCH), even  $2^{\mathfrak{c}}$ -lineable. We also show that, under any assumptions, it is  $(\mathfrak{c}, \mathfrak{c})$ -algebraic.

Throughout the paper  $\alpha^+$  will denote the successor cardinal number of  $\alpha$  and  $\omega_{\mathfrak{c}}$  the first ordinal number of cardinality  $\mathfrak{c}$ .

## 2. STRONG AND PERFECT EVERYWHERE SURJECTIVITY ON $\mathbb{R}$

Lebesgue [15, 19] was probably the first to exhibit a somewhat surprising example of a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with the property that on every non-trivial interval  $I$ ,  $f(I) = \mathbb{R}$ . In [4] the authors proved that the set of such *everywhere surjective* functions is  $2^{\mathfrak{c}}$ -lineable, which is the *best* possible result in terms of dimension. One might think that, in terms of surjectivity, these everywhere surjective functions are in some sense the *most pathological*. In this section we shall show that even *more pathological* surjective functions exist. In order to do this we shall show that there exists a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that for every perfect set  $P$  and every  $r \in \mathbb{R}$ , the set  $\{x \in P : f(x) = r\}$  has cardinality  $\mathfrak{c}$ . First we show that there exists a function

$f: \mathbb{R} \rightarrow \mathbb{R}$  such that for every perfect set  $P$ ,  $f(P) = \mathbb{R}$ . This construction can be found in, for example, [13, pp. 124, 125], but for the sake of completeness of the paper, we include it here as well.

**Example 2.1.** *A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that for every perfect set  $P$ ,  $f(P) = \mathbb{R}$ .*

First of all let us recall that if  $P$  is a perfect set, then  $P$  has cardinality  $\mathfrak{c}$ . Now, the cardinality of the collection of pairs  $(P, y)$  where  $P$  is a perfect set and  $y \in \mathbb{R}$  is  $\mathfrak{c}$ . Let  $\{A_\alpha\}_{\alpha < \omega_{\mathfrak{c}}}$  be a well-ordering of this set and notice that each perfect set  $P$  occurs  $\mathfrak{c}$  times at a first element of a pair  $(P, y)$ . Assume that  $A_0 = (P_0, y_0)$ , choose  $x_0 \in P_0$ , and define  $f(x_0) = y_0$ . Let us assume now that for each  $\beta < \alpha$ ,  $A_\beta = (P_\beta, y_\beta)$  and a point  $x_\beta \in P_\beta$  has been chosen in such a way that  $x_\beta \neq x_\gamma$  for every  $\gamma < \beta$  and  $f(x_\beta) = y_\beta$ . Next, let  $A_\alpha = (P_\alpha, y_\alpha)$ . Since the cardinality of the set  $\{x_\beta : \beta < \alpha\}$  is less than  $\mathfrak{c}$ , there exists  $x_\alpha \in P_\alpha \setminus \{x_\beta : \beta < \alpha\}$ . Now, define  $f(x_\alpha) = y_\alpha$ . Thus, for each  $\alpha < \omega_{\mathfrak{c}}$ , a point has been chosen so that  $x_\alpha \neq x_\beta$  for  $\beta < \alpha$  and  $f(x_\alpha) = y_\alpha$ . If  $x \notin \{x_\alpha : \alpha < \omega_{\mathfrak{c}}\}$ , we define  $f(x) = 0$ . By doing this,  $f$  is now defined on the whole real line. Now, if  $P$  is any perfect set and  $y \in \mathbb{R}$ , then there exists  $\alpha < \omega_{\mathfrak{c}}$  such that  $(P, y) = A_\alpha$ . Therefore there exists  $x = x_\alpha \in P$  with  $f(x) = y$ , and thus  $f(P) = \mathbb{R}$ .

One could think of constructing an even *more extreme* function that is surjective on any set of cardinality  $\mathfrak{c}$ , but this is just not possible. Indeed, if  $f$  were such a function and  $S$  had cardinality  $\mathfrak{c}$ , the set  $S^* = S \setminus f^{-1}(0)$  would have cardinality  $\mathfrak{c}$  too, but  $f$  would not be surjective on  $S^*$ . An important point in the construction in Example 2.1 is that the family of perfect sets in  $\mathbb{R}$  has cardinality  $\mathfrak{c}$ . This cannot be accomplished if we take the family of all sets of cardinality  $\mathfrak{c}$  instead.

It must also be noted that a function such as the one presented in Example 2.1 actually attains every real value  $\mathfrak{c}$  times on any perfect set. Indeed, it is well-known that if  $P \subset \mathbb{R}$  is perfect, it must contain a Cantor-like set  $C_P$ , and this can be decomposed into  $\mathfrak{c}$  disjoint Cantor-like sets  $C_P^i$ ,  $i \in \mathbb{R}$ . Since every value is attained at least once in any  $C_P^i$ , our function must consequently attain this value  $\mathfrak{c}$  times in  $C_P$  and therefore in  $P$ .

From now on we shall be referring to a function such as the one described in Example 2.1 as *perfectly everywhere surjective*. We shall use the notation

$$\mathcal{ES}(\mathbb{R}) = \{ f: \mathbb{R} \rightarrow \mathbb{R} : f \text{ is everywhere surjective} \}$$

and

$$\mathcal{PES}(\mathbb{R}) = \{ f: \mathbb{R} \rightarrow \mathbb{R} : f \text{ is perfectly everywhere surjective} \}.$$

Clearly, if  $f$  is perfectly everywhere surjective, then  $f$  is everywhere surjective; i.e.,  $\mathcal{PES}(\mathbb{R}) \subset \mathcal{ES}(\mathbb{R})$ . The converse is false, as we can see in the following construction:

**Example 2.2.** *An everywhere surjective function that is not perfectly everywhere surjective.*

Let  $(I_n)_{n \in \mathbb{N}}$  be the collection of all open intervals with rational endpoints. Of course  $I_1$  contains a Cantor-type set; call it  $C_1$ . Now,  $I_2 \setminus C_1$  also contains a Cantor-type set; call it  $C_2$ . Next,  $I_3 \setminus (C_1 \cup C_2)$  contains a Cantor type set as well, which we call  $C_3$ . Inductively, we construct a family of pairwise disjoint Cantor-type sets,  $(C_n)_{n \in \mathbb{N}}$ , such that for every  $n \in \mathbb{N}$ ,  $I_n \setminus (\bigcup_{k=1}^{n-1} C_k) \supset C_n$ . Now, take (for every

$n \in \mathbb{N}$ )  $\phi_n$  to be any bijection between  $C_n$  and  $\mathbb{R}$ , and define  $f: \mathbb{R} \rightarrow \mathbb{R}$  as follows:

$$f(x) = \begin{cases} \phi_n(x) & \text{if } x \in C_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $f$  is clearly everywhere surjective (and also zero almost everywhere!). Indeed, let  $I$  be any interval in  $\mathbb{R}$ . There exists  $k \in \mathbb{N}$  such that  $I_k \subset I$ , and thus

$$f(I) \supset f(I_k) \supset f(C_k) = \phi_k(C_k) = \mathbb{R}.$$

Of course, by construction,  $f$  cannot be perfectly everywhere surjective, since it takes each real value only once on the  $C_k$ 's.

It is easy to prove that an everywhere surjective function must take all real values at least  $\aleph_0$  times on any interval (but not more: the function from the previous example attains *exactly*  $\aleph_0$  times every real value except 0, which is attained  $\mathfrak{c}$  times). One might wonder whether a function that takes every real value  $\mathfrak{c}$  times on every interval must be *a fortiori* perfectly everywhere surjective. Our next example shows that this is false.

**Example 2.3.** *A function that takes every real value  $\mathfrak{c}$  times on any interval but is not perfectly everywhere surjective.*

Let  $I_n$  and  $C_n$ ,  $n \in \mathbb{N}$ , be as in Example 2.2. For every  $n \in \mathbb{N}$ , decompose  $C_n$  into  $\mathfrak{c}$  disjoint Cantor-like sets,  $C_n^i$  for  $i \in \mathbb{R}$ . Take, for any  $n \in \mathbb{N}$  and  $i \in \mathbb{R}$ ,  $\phi_n^i$  to be any bijection between  $C_n^i$  and  $\mathbb{R}$ . Similarly to Example 2.2, define

$$f(x) = \begin{cases} \phi_n^i(x) & \text{if } x \in C_n^i, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that  $f$  takes all values  $\mathfrak{c}$  times on any interval. Also,  $f$  is not perfectly everywhere surjective, because  $f$  attains each value just once on each  $C_n^i$ .

We shall say that a function is *strongly everywhere surjective* if it takes every real value  $\mathfrak{c}$  times on any interval, as in the previous example, and we write

$$\mathcal{SES}(\mathbb{R}) = \{ f: \mathbb{R} \rightarrow \mathbb{R} : f \text{ is strongly everywhere surjective} \}.$$

Our previous considerations show that

$$\mathcal{PES}(\mathbb{R}) \subset \mathcal{SES}(\mathbb{R}) \subset \mathcal{ES}(\mathbb{R}).$$

All these inclusions are proper. Indeed, observe that the function in Example 2.2 is actually in  $\mathcal{ES}(\mathbb{R}) \setminus \mathcal{SES}(\mathbb{R})$ , whereas the function in Example 2.3 lies in  $\mathcal{SES}(\mathbb{R}) \setminus \mathcal{PES}(\mathbb{R})$ .

*Remark 2.4.* It is easy to prove that the functions in  $\mathcal{PES}(\mathbb{R})$  are non-measurable whereas the functions from Examples 2.2 and 2.3 are zero almost everywhere and therefore measurable.

Now that we have sorted out all the previous concepts, our aim is to show that the set of perfectly everywhere surjective functions is  $2^{\mathfrak{c}}$ -lineable. In order to do so, let us recall a result ([4, Prop. 4.2]) that will be needed in this section.

**Lemma 2.5** (Aron, Gurariy, Seoane-Sepúlveda). *There exists a vector space  $\Lambda$  of functions  $\mathbb{R} \rightarrow \mathbb{R}$  with the following two properties:*

- (i) *Every non-zero element of  $\Lambda$  is an onto function, and*
- (ii)  $\dim \Lambda = 2^{\mathfrak{c}}$ .

This result can be proved by considering the vector space

$$\text{span}\{H_C \circ \Phi : \emptyset \neq C \subset \mathbb{R}\},$$

where the linearly independent functions  $H_C : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$  are given by

$$H_C(x, x_1, x_2, x_3, \dots) = x \prod_{i=1}^{\infty} \chi_C(x_i)$$

and  $\Phi$  is any bijection between  $\mathbb{R}^{\mathbb{N}}$  and  $\mathbb{R}$ .

**Theorem 2.6.** *The set  $\mathcal{PES}(\mathbb{R})$  is  $2^{\mathfrak{c}}$ -lineable.*

*Proof.* Let  $\Lambda$  be as in Lemma 2.5 and fix any perfectly everywhere surjective function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Now, consider the vector space  $V = \{g \circ f : g \in \Lambda\}$ . Clearly,  $\dim V = 2^{\mathfrak{c}}$ . Now, take  $g \circ f \in V$  and any  $r \in \mathbb{R}$ . Since  $g$  is onto, there exists  $s \in \mathbb{R}$  such that  $g(s) = r$ . Now, since  $f$  is perfectly everywhere surjective, for every perfect set  $P$  there exists  $x \in P$  such that  $f(x) = s$ . Thus,  $(g \circ f)(x) = r$ , and we are done.  $\square$

Notice that [4, Theorem 4.3] follows directly: as straightforward consequences of Theorem 2.6, we obtain that  $\mathcal{SES}(\mathbb{R})$  and  $\mathcal{ES}(\mathbb{R})$  are both  $2^{\mathfrak{c}}$ -lineable.

As we saw in Example 2.2, the set of everywhere surjective functions that are not strongly everywhere surjective is not empty. Moreover, this set is (surprisingly) lineable. The same can be said about the set of strongly but not perfectly everywhere surjective functions.

**Theorem 2.7.**

- (i)  $\mathcal{SES}(\mathbb{R}) \setminus \mathcal{PES}(\mathbb{R})$  is  $2^{\mathfrak{c}}$ -lineable.
- (ii)  $\mathcal{ES}(\mathbb{R}) \setminus \mathcal{SES}(\mathbb{R})$  is  $\mathfrak{c}$ -lineable.
- (iii)  $\mathcal{S}(\mathbb{R}) \setminus \mathcal{ES}(\mathbb{R})$  is  $2^{\mathfrak{c}}$ -lineable, where  $\mathcal{S}(\mathbb{R})$  denotes the family of all surjective functions on  $\mathbb{R}$ .

*Proof.* (i) To prove that  $\mathcal{SES}(\mathbb{R}) \setminus \mathcal{PES}(\mathbb{R})$  is  $2^{\mathfrak{c}}$ -lineable, we can follow a similar argument to the one from Theorem 2.6, keeping in mind that the composition of the function from Example 2.3 with any surjective function is constant a.e. and thus cannot belong to  $\mathcal{PES}(\mathbb{R})$ .

(ii) In this case, the proof is also similar to that of Theorem 2.6, taking  $f$  to be the function from Example 2.2. Let us consider the following vector space of continuous functions on  $\mathbb{R}$ :

$$V = \text{span}\{e^{rx} - e^{-rx} : r > 0\}.$$

It is easy to see that  $\dim V = \mathfrak{c}$  and that every  $g \in V \setminus \{0\}$  is surjective. Also, every non-zero function in the vector space  $W = \{g \circ f : g \in V\}$  is everywhere surjective. We just need to show that if  $h \in W$  with  $h \neq 0$ , then  $h$  is not strongly everywhere surjective. Of course  $h$  can be written as  $g \circ f$ , with  $0 \neq g \in V$ . By the definition of  $g$ , if  $\alpha \in \mathbb{R}$ ,  $g$  takes the value  $\alpha$  just a finite number of times. Now, if  $C_n$  is one of the Cantor-type sets in the construction of Example 2.2, the function  $h = g \circ f$  also takes the value  $\alpha$  just a finite number of times on  $C_n$ , since  $f : C_n \leftrightarrow \mathbb{R}$  is a bijection. Consequently, it takes every value (except 0) no more than  $\aleph_0$  times, and we are done.

We leave the easy details of the proof of (iii) to the interested reader.  $\square$

Let us remark that the dimensions obtained in Theorem 2.6 and in (i) and (iii) of Theorem 2.7 are clearly the highest possible; that of Theorem 2.7 (ii), however, may be less than optimal. Our impression is that this is indeed the case, and this set is actually  $2^{\mathfrak{c}}$ -lineable.

The structure of the set of everywhere surjective functions has been intensively studied in recent years by several authors in [2, 6]. Actually, more can be said about the set of perfectly everywhere surjective functions. A similar argument to that in Example 2.1 can be applied to construct perfectly everywhere surjective functions from  $\mathbb{C}$  to  $\mathbb{C}$ . Using the arguments from [6] and any perfectly everywhere surjective function  $f: \mathbb{C} \rightarrow \mathbb{C}$ , one can easily show that there exists an infinitely generated algebra of these functions:

**Theorem 2.8.** *The set of perfectly everywhere surjective functions on  $\mathbb{C}$  is algebraable.*

Also, notice that while the set of perfectly everywhere surjective functions (and thus the set of surjective functions) is lineable, this is not the case for the set of injective functions from  $\mathbb{R}$  to  $\mathbb{R}$ , as seen in the following result:

**Theorem 2.9.** *The set of injective functions is not lineable. Moreover, if  $V$  is a vector space every non-zero element of which is an injective function on  $\mathbb{R}$ , then  $\dim V = 1$ .*

*Proof.* Let us suppose that there exists a 2-dimensional vector space of injective functions,  $V$ , generated by  $f$  and  $g$ , two linearly independent injective functions. Then, take  $x \neq y$  and  $\alpha = \frac{f(x)-f(y)}{g(y)-g(x)} \in \mathbb{R}$ . Next, consider the function  $h = f + \alpha g \in V \setminus \{0\}$ . By construction we have  $h(x) = h(y)$ , and we are done.  $\square$

### 3. DIFFERENTIABLE FUNCTIONS WITH ALMOST EVERYWHERE DISCONTINUOUS DERIVATIVES

The space of functions that are the derivatives of differentiable functions has dimension  $\mathfrak{c}$ . In this section we find that the space of derivatives contains a subspace of dimension  $\mathfrak{c}$  of mappings “enjoying” the special property of being discontinuous almost everywhere.

Let us recall some basic definitions needed to understand this section properly.

**Definition 3.1.** Let  $E \subset \mathbb{R}$ . We say that  $x \in \mathbb{R}$  is a point of density of  $E$  if

$$\liminf_{\varepsilon \rightarrow 0^+} \frac{m(E \cap (x - \varepsilon, x + \varepsilon))}{2\varepsilon} = 1,$$

where  $m$  stands for the Lebesgue measure on  $\mathbb{R}$ . We write

$$\text{dens } E = \{x \in \mathbb{R} : x \text{ is a density point of } E\}.$$

**Definition 3.2.** We say that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is approximately continuous at  $x_0 \in \mathbb{R}$  if there exists  $E \subset \mathbb{R}$  such that  $x_0 \in \text{dens } E$  and

$$\lim_{\substack{x \rightarrow x_0 \\ x \in E}} f(x) = f(x_0).$$

If  $f$  is approximately continuous at every  $x_0 \in \mathbb{R}$ , we simply say that  $f$  is approximately continuous.

The density topology can be defined as the initial topology for the approximately continuous functions. Derivatives are closely related to approximately continuous functions. Actually, every bounded approximately continuous function is the derivative of a differentiable function ([12, Theorem 5.5(a), p. 21]). In order to prove the main result of this section we shall use some of the results in the literature about approximate continuity. The next result ([20, Theorem 8.1]) will be crucial in our discussion.

**Theorem 3.3** (Zahorski). *For every  $G_\delta$ , density-closed set  $E$ , there exists an approximately continuous mapping  $f_0: \mathbb{R} \rightarrow [0, 1]$  such that its zero set  $Z_{f_0}$  is equal to  $E$ .*

A direct consequence of this theorem is the following result. The proof can be found in [20, pp. 1, 2], but we give it here for completeness.

**Lemma 3.4.** *There exists an approximately continuous mapping  $f_0: \mathbb{R} \rightarrow [0, 1]$  satisfying the following properties:*

- (i)  $Z_{f_0}$  is a  $G_\delta$ , dense set with Lebesgue measure zero.
- (ii)  $f_0$  is discontinuous at every  $x \in \mathbb{R} \setminus Z_{f_0}$ .

*Proof.* Let  $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$  and define

$$E = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left( r_n - \frac{1}{2^{m+n}}, r_n + \frac{1}{2^{m+n}} \right).$$

Then  $E$  is a  $G_\delta$ , dense set and its measure is zero. According to a standard property, since  $E$  has measure zero, it is density-closed. Hence, by Theorem 3.3 there exists an approximately continuous mapping  $f_0 : \mathbb{R} \rightarrow [0, 1]$  such that  $Z_{f_0} = E$ . This proves (i). Now if  $x \notin E$  and  $(r_{n_k})$  is a subsequence of  $(r_n)$  converging to  $x$ , then  $f_0(r_{n_k}) = 0$  for all  $k \in \mathbb{N}$  since  $(r_{n_k}) \subset E = Z_{f_0}$ , and therefore  $f_0$  is not continuous at  $x$  since  $f_0(x) > 0$ . This concludes the proof.  $\square$

*Remark 3.5.* In the proof of the next theorem, as well as in other results below, the following will be used: If  $\alpha_1, \dots, \alpha_n > 0$  and  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ , then the mapping defined by  $p(x) := \sum_{k=1}^n \lambda_k x^{\alpha_k}$  for every  $x \geq 0$  has infinitely many zeros only if  $\lambda_1 = \dots = \lambda_n = 0$ . Notice that the latter is equivalent to the fact that  $\{x^\alpha : \alpha > 0\}$  is a linearly independent set of continuous functions on  $[0, \infty)$ .

**Theorem 3.6.** *The set of bounded approximately continuous mappings defined on  $\mathbb{R}$  that are discontinuous almost everywhere is  $\mathfrak{c}$ -lineable.*

*Proof.* Let  $f_0 : \mathbb{R} \rightarrow [0, 1]$  be the mapping whose existence has been proved in Lemma 3.4. Now consider the set  $\{f_\alpha : \alpha \in (0, \infty)\}$ , where  $f_\alpha(x) = e^{-\alpha|x|}f_0(x)$  for all  $x \in \mathbb{R}$ . Then  $\{f_\alpha : \alpha \in (0, \infty)\}$  is clearly a linearly independent set of bounded mappings since  $\mathbb{R} \ni x \mapsto e^{-\alpha|x|}$ , with  $\alpha \in (0, \infty)$ , are linearly independent and bounded and  $f_0$  is bounded. Moreover, if

$$f(x) := \sum_{k=1}^n \lambda_k f_{\alpha_k}(x) = \left( \sum_{k=1}^n \lambda_k e^{-\alpha_k|x|} \right) f_0(x)$$

for all  $x \in \mathbb{R}$ , with  $\lambda_k \in \mathbb{R}$  for  $k = 1, \dots, m$ , then  $f$  is bounded. Since the mapping defined by  $g(x) := \sum_{k=1}^n \lambda_k e^{-\alpha_k|x|}$  for all  $x \in \mathbb{R}$  is continuous, the set of continuity points of  $f$  is contained in the union of the sets of continuity points of  $f_0$  and  $Z_g$ .

This concludes the proof, since  $Z_g$  is a finite set (see Remark 3.5) and therefore its measure is zero.  $\square$

Recall that the bounded Pompeiu derivatives (i.e., derivatives that vanish on a dense set) form a Banach space ([12, p. 33]). Notice that the space constructed in the above theorem is a subspace of it.

Since all approximately continuous and bounded mappings are derivatives, we obtain as an easy consequence of Theorem 3.6 the following result:

**Theorem 3.7.** *The set of differentiable functions on  $\mathbb{R}$  whose derivatives are discontinuous almost everywhere is  $\mathfrak{c}$ -lineable.*

Both the space of approximately continuous functions on  $\mathbb{R}$  and the space of differentiable functions on  $\mathbb{R}$  are  $\mathfrak{c}$ -dimensional. Therefore the results proved in Theorem 3.6 and Theorem 3.7 are optimal. Notice also that with little effort one can prove that if  $I$  is any non-void compact interval in  $\mathbb{R}$ , then the set of differentiable functions whose derivatives are discontinuous almost everywhere on  $I$  is  $\mathfrak{c}$ -lineable as well.

Theorem 3.7 can be extended to dense-lineability for differentiable functions on a non-void compact interval in  $\mathbb{R}$ . Indeed, let us recall ([3]) that a subset  $M$  of a topological vector space  $X$  is said to be *dense-lineable* in  $X$  if there exists a linear manifold  $Y \subset M \cup \{0\}$  that is dense in  $X$ . Also, let  $A$  and  $B$  be subsets of a vector space  $X$ . We say that  $A$  is *stronger* than  $B$  if  $A + B \subset A$  (see [3] as well).

In [3] the authors obtained the following sufficient condition for a set to be dense-lineable in a topological vector space.

**Theorem 3.8** (Aron, García-Pacheco, Pérez-García, Seoane-Sepúlveda). *Let  $X$  be a separable Banach space, and consider two subsets  $A$  and  $B$  of  $X$  such that  $A$  is lineable and  $B$  is dense-lineable. If  $A$  is stronger than  $B$ , then  $A$  is dense-lineable.*

Now, let  $I$  be any non-void compact interval in  $\mathbb{R}$ . In the previous theorem, if we take  $A$  to be the set of differentiable functions whose derivatives are discontinuous almost everywhere on  $I$  and consider  $B$  to be the set of polynomials, then the Weierstrass Approximation Theorem leads us to the following trivial consequence of Theorem 3.8:

**Corollary 3.9.** *Let  $I$  be any non-void compact interval in  $\mathbb{R}$ . The set of differentiable functions whose derivatives are discontinuous almost everywhere on  $I$  is dense-lineable in the space  $C(I)$  of continuous real functions on  $I$ , endowed with the supremum norm.*

#### 4. DIFFERENTIABLE NOWHERE MONOTONE FUNCTIONS

In this section we prove that the set  $\mathcal{DNM}(\mathbb{R})$  of differentiable functions on  $\mathbb{R}$  that are nowhere monotone is  $\mathfrak{c}$ -lineable. In [4] it was proved that this set is  $\aleph_0$ -lineable using a different technique, and in [3] it has been shown that the set  $\mathcal{DNM}([a, b])$  is dense-lineable for any non-void interval  $[a, b]$ . Here we use approximately continuous functions and the properties of the density topology to obtain the above statement (see [12, 13, 20]).

**Theorem 4.1.** *The set of bounded approximately continuous functions on  $\mathbb{R}$  that are positive on a dense subset of  $\mathbb{R}$  and negative on another is  $\mathfrak{c}$ -lineable.*

*Proof.* First, let  $A = (x_n)_{n \in \mathbb{N}}$  and  $B = (y_n)_{n \in \mathbb{N}}$  be two disjoint dense subsets of  $\mathbb{R}$ . Since each set of Lebesgue measure 0 is density-closed and the density topology is completely regular, for each  $n \in \mathbb{N}$  there exist two approximately continuous functions  $f_n, g_n : \mathbb{R} \rightarrow [0, 1]$  such that  $f_n(x_m) = 0$  for all  $m \in \mathbb{N}$ ,  $f_n(y_n) = 1$ ,  $g_n(y_m) = 0$  for all  $m \in \mathbb{N}$ , and  $g_n(x_n) = 1$ . The function

$$h = \sum_{n=1}^{\infty} \frac{1}{2^n} (g_n - f_n)$$

is bounded and approximately continuous and satisfies  $h(y_m) < 0 < h(x_m)$  for all  $m \in \mathbb{N}$ .

We consider now, for each  $\alpha > 0$ , the bounded approximately continuous function  $h_\alpha(x) = e^{-\alpha|x|}h(x)$ , which also satisfies  $h_\alpha(y_m) < 0 < h_\alpha(x_m)$  for all  $m \in \mathbb{N}$ . The set  $\{h_\alpha : \alpha > 0\}$  obviously generates a linear space of bounded approximately continuous functions and in addition is linearly independent. Indeed, let us consider the function  $\sum_{k=1}^n \lambda_k h_{\alpha_k}$  with  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$  and  $\alpha_1, \dots, \alpha_n > 0$ . If  $\sum_{k=1}^n \lambda_k h_{\alpha_k} \equiv 0$ , then the equation  $\sum_{k=1}^n \lambda_k e^{-\alpha_k|x|} = 0$  would have infinitely many solutions, and then  $\lambda_1 = \dots = \lambda_n = 0$  (see Remark 3.5).

Now, define  $C_+ = \{x \in \mathbb{R} : \sum_{k=1}^n \lambda_k e^{-\alpha_k|x|} > 0\}$  and  $C_- = \{x \in \mathbb{R} : \sum_{k=1}^n \lambda_k e^{-\alpha_k|x|} < 0\}$ . Since  $\sum_{k=1}^n \lambda_k h_{\alpha_k}(x) > 0$  for all  $x \in (C_+ \cap A) \cup (C_- \cap B)$  and  $\sum_{k=1}^n \lambda_k h_{\alpha_k}(x) < 0$  for all  $x \in (C_+ \cap B) \cup (C_- \cap A)$ , these two sets being disjoint and dense in  $\mathbb{R}$ , the proof is concluded.  $\square$

Since all approximately continuous and bounded mappings are derivatives, we obtain as an easy consequence of Theorem 4.1 the following result:

**Theorem 4.2.** *The set  $DN\mathcal{M}(\mathbb{R})$  of differentiable functions on  $\mathbb{R}$  that are nowhere monotone is  $\mathfrak{c}$ -lineable.*

Notice that the space constructed in Theorem 4.1 is a subspace of the Banach space of Pompeiu derivatives. It is known that the set of derivatives that are positive on a dense set and negative on another is a dense  $G_\delta$  set in the Banach space of Pompeiu derivatives ([12, Theorem 6.7, p. 34]). It would be interesting to see whether this set is also dense-lineable.

### 5. SIERPIŃSKI-ZYGMUND FUNCTIONS

An immediate consequence of the classic Luzin’s Theorem is that for every measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , there is a measurable set  $S \subset \mathbb{R}$  of infinite measure such that  $f|_S$  is continuous. It is natural to wonder whether similar results could be achieved for arbitrary functions that are not necessarily measurable; that is, given any arbitrary function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , can we find a “big” set  $S \subset \mathbb{R}$  such that  $f|_S$  is continuous?

One of the meanings that we can give to the word “big” is “dense”. In this direction, Blumberg gave in [9] an affirmative answer to this question.

**Theorem 5.1** (Blumberg). *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an arbitrary function. There exists a dense subset  $S \subset \mathbb{R}$  such that the function  $f|_S$  is continuous.*

A careful reading of the proof of this result (see, e.g., [18, p. 154]) shows that the set  $S$  obtained there is countable. Naturally, we could wonder whether we can choose the subset  $S$  in Blumberg’s theorem to be uncountable. A (partial) negative answer was given in [24] by Sierpiński and Zygmund.

**Theorem 5.2** (Sierpiński-Zygmund). *There exists a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that for any set  $Z \subset \mathbb{R}$  of cardinality the continuum, the restriction  $f|_Z$  is not a Borel map. (In particular, this restriction is not continuous.)*

Obviously, if the Continuum Hypothesis holds, the restriction of this function to any uncountable set cannot be continuous. For the proof of this result, see the original paper or [18, pp. 165, 166]. The Continuum Hypothesis is necessary in this setting. Shinoda proved in [23] that if Martin's Axiom and the negation of the Continuum Hypothesis hold, then for every  $f: \mathbb{R} \rightarrow \mathbb{R}$  there exists an uncountable set  $Z \subset \mathbb{R}$  such that  $f|_Z$  is continuous.

A function such as the one obtained in Theorem 5.2 is indeed very pathological: These functions are never measurable and, although it is possible to construct them to be injective, they are nowhere monotone in a very strong sense. (Their restriction to any set of cardinality  $\mathfrak{c}$  is not monotone.)

In what follows, we shall say that a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a *Sierpiński-Zygmund function* if it satisfies the condition in Sierpiński-Zygmund's Theorem, and we write

$$\mathcal{SZ}(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} : f \text{ is a Sierpiński-Zygmund function}\}.$$

Following the ideas in the proof of Theorem 5.2, we can obtain the following result, which will be of use later:

**Lemma 5.3.** *Let  $\mathcal{F}$  be a family of functions from  $\mathbb{R}$  to  $\mathbb{R}$  whose cardinality is not bigger than  $\mathfrak{c}$ . There exists a function  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  such that for every  $Z \subset \mathbb{R}$  of cardinality  $\mathfrak{c}$ , every function of  $\mathcal{F}$  is different from  $\varphi$  on  $Z$ .*

*Proof.* We can assume without loss of generality that  $\text{card } \mathcal{F} = \mathfrak{c}$ . As the cardinality of  $\mathcal{F}$  is  $\mathfrak{c}$ , we can write this set in the form  $\mathcal{F} = \{f_\alpha : \alpha < \omega_{\mathfrak{c}}\}$ . For the same reason, we can write  $\mathbb{R} = \{x_\alpha : \alpha < \omega_{\mathfrak{c}}\}$ .

We shall define the value  $\varphi(x_\alpha)$  for every  $\alpha < \omega_{\mathfrak{c}}$  through a transfinite induction process. Let  $\alpha < \omega_{\mathfrak{c}}$ , and assume that we have defined  $\varphi(x_\beta)$  for every  $\beta < \alpha$ . Then we choose  $\varphi(x_\alpha)$  so that  $\varphi(x_\alpha) \neq f_\beta(x_\alpha)$  for every  $\beta \leq \alpha$ . (This can be done since the cardinality of  $\{\beta : \beta \leq \alpha\}$  is strictly smaller than  $\mathfrak{c}$ .) In this way, we have defined the function  $\varphi$ .

Now let  $Z \subset \mathbb{R}$  satisfy  $\text{card } Z = \mathfrak{c}$ . Choose  $f_\alpha \in \mathcal{F}$ , and let us show that  $\varphi|_Z \neq f_\alpha|_Z$ . Indeed, take an ordinal  $\beta$  with  $\alpha < \beta < \omega_{\mathfrak{c}}$  and  $x_\beta \in Z$ . (Again, we can do this because the cardinality of  $\{x_\beta : \beta \leq \alpha\}$  is strictly smaller than  $\mathfrak{c}$ .) Taking into account the way in which we defined  $\varphi$ , it must be that  $\varphi(x_\beta) \neq f_\alpha(x_\beta)$ . This concludes the proof.  $\square$

From this result, we can prove Sierpiński-Zygmund's Theorem through the use of the following lemmas:

**Lemma 5.4.** *Let  $Z \subset \mathbb{R}$  and let  $f: Z \rightarrow \mathbb{R}$  be a Borel map. There exist a Borel set  $Z^* \supset Z$  and a Borel map  $f^*: Z^* \rightarrow \mathbb{R}$  such that  $f^*|_Z = f$ .*

The proof of this lemma can be found in [18, pp. 163, 164].

**Lemma 5.5.** *Let  $Z \subset \mathbb{R}$  and let  $f: Z \rightarrow \mathbb{R}$  be a Borel map. There exists a Borel map  $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$  such that  $\hat{f}|_Z = f$ .*

*Proof.* Assume that  $Z^*$  and  $f^*$  are as in Lemma 5.4, and define  $\hat{f}$  by

$$\hat{f}(x) = \begin{cases} f^*(x), & x \in Z^*, \\ 0, & x \notin Z^*. \end{cases}$$

It is easy to verify that this is a Borel map. □

Taking into account Lemma 5.5, if in Lemma 5.3 we choose  $\mathcal{F}$  to be the family of Borel maps, we will have proven the existence of the function announced in Theorem 5.2. Actually, Lemma 5.3 suffices to prove that there is a “massive” vector space of functions like this. To be precise:

**Theorem 5.6.** *The set  $\mathcal{SZ}(\mathbb{R})$  is  $\mathfrak{c}^+$ -lineable. Assuming GCH,  $\mathcal{SZ}(\mathbb{R})$  is  $2^{\mathfrak{c}}$ -lineable.*

*Proof.* Obviously,  $\mathcal{SZ}(\mathbb{R}) \cup \{0\}$  contains a vector space ( $\{0\}!$ ), and a standard application of Zorn’s Lemma gives that there is a maximal vector space  $X$  contained in  $\mathcal{SZ}(\mathbb{R}) \cup \{0\}$ . Assume that  $\dim X \leq \mathfrak{c}$ ; we shall get a contradiction. We have  $\text{card } X \leq \mathfrak{c}$ , and if we define

$$\mathcal{F} = \{f + g : f \in X, g \text{ is a Borel map}\},$$

then  $\text{card } \mathcal{F} = \mathfrak{c}$ . Using Lemma 5.3, we can find a function  $\varphi$  that differs from every function of  $\mathcal{F}$  on every set of cardinality  $\mathfrak{c}$ . In particular,  $\varphi \notin X$ . In addition to this,  $\varphi \in \mathcal{SZ}(\mathbb{R})$ . What is more:

(1)  $\text{If } f \in X, \text{ then } \varphi + f \in \mathcal{SZ}(\mathbb{R}).$

Indeed, if  $\varphi + f \notin \mathcal{SZ}(\mathbb{R})$ , there exist a Borel map  $g$  and a set  $Z$  with cardinality  $\mathfrak{c}$  such that  $\varphi + f = g$  on  $Z$ ; but we obtain then that on  $Z$  we have  $\varphi = -f + g \in \mathcal{F}$ , which is a contradiction.

Define now  $Y = [\varphi] + X$ . Using (1), it is easy to prove that  $Y \subset \mathcal{SZ}(\mathbb{R}) \cup \{0\}$ , in contradiction to the fact that  $X$  is maximal. □

Changing just a few details in the proofs of Lemma 5.3 and Theorem 5.6, we can obtain a more general (and abstract) version of the latter. We think that this can be useful in other contexts.

**Theorem 5.7.** *Let  $A$  be a set and  $V$  a vector space. Suppose that*

$$\text{card } A = \alpha \leq \beta = \text{card } V.$$

*Let  $\mathcal{F}$  be a family of functions from  $A$  to  $V$  such that  $\text{card } \mathcal{F} \leq \alpha$ . Let*

$$\mathcal{H} = \{f : A \rightarrow V : \text{no function of } \mathcal{F} \text{ is equal to } f \\ \text{on any subset of } A \text{ of cardinality } \alpha\}.$$

*Then  $\mathcal{H}$  is  $\alpha^+$ -lineable. (Assuming GCH, we obtain  $2^\alpha$ -lineability.)*

If we want a vector space inside  $\mathcal{SZ}(\mathbb{R}) \cup \{0\}$  to contain some additional algebraic structure, we can say something in that sense too. We can find an algebra such that every non-zero element is a Sierpiński-Zygmund function. To prove this, we state a preliminary result. (Actually, the lemma given here is much stronger than what we need.)

**Definition 5.8.** Let  $S \subset \mathbb{R}$ . We say that  $\varphi : S \rightarrow \mathbb{R}$  is countably strictly monotone if there exists a collection  $\{S_n\}_{n \in \mathbb{N}}$  of subsets of  $\mathbb{R}$  such that  $S = \bigcup_{n=1}^\infty S_n$  and  $\varphi|_{S_n}$  is strictly monotone for every  $n \in \mathbb{N}$ .

**Lemma 5.9.** *Let  $f \in \mathcal{SZ}(\mathbb{R})$ , let  $S$  be a Borel set such that  $f(\mathbb{R}) \subset S$ , and let  $\varphi : S \rightarrow \mathbb{R}$  be a countably strictly monotone Borel map. Then  $\varphi \circ f$  is a Sierpiński-Zygmund function.*

*Proof.* Assume that  $\varphi \circ f \notin \mathcal{SZ}(\mathbb{R})$ , and consider  $Z \subset \mathbb{R}$  with cardinality  $\mathfrak{c}$  such that the restriction of  $\varphi \circ f$  to  $Z$  is a Borel map. Let  $\{S_n\}_{n \in \mathbb{N}}$  be as in Definition 5.8. It can easily be proved that  $\varphi$  being a Borel map, the sets  $S_n$  can be chosen to be Borel as well. It is obvious that for some  $i_0 \in \mathbb{N}$  the set  $f^{-1}(S_{i_0}) \cap Z$  must have cardinality  $\mathfrak{c}$ . Let  $Z_0 = f^{-1}(S_{i_0}) \cap Z$  and  $\varphi_0 = \varphi|_{S_{i_0}}$ . The map  $\varphi_0$  (and therefore its inverse  $\varphi_0^{-1}$ ) is an injective Borel map. The restriction of  $\varphi \circ f$  to  $Z_0$  is a Borel map. As  $\varphi \circ f$  coincides with  $\varphi_0 \circ f$  on  $Z_0$ , we have that, when restricted to  $Z_0$ , the function  $f = \varphi_0^{-1} \circ (\varphi_0 \circ f)$  is a Borel map, which is a contradiction.  $\square$

**Theorem 5.10.** *The set of Sierpiński-Zygmund functions is  $(\mathfrak{c}, \mathfrak{c})$ -algebrable.*

*Proof.* Let  $f$  be the function obtained in Theorem 5.2. First of all, let us point out that  $|f|$  is also a Sierpiński-Zygmund function, since the absolute value is obviously a  $\mathfrak{c}$ -injective continuous function. Define

$$\Lambda = \text{span}\{|f|^\alpha : \alpha > 0\}.$$

It is clear that  $\Lambda$  is an algebra. Let us show that it has the desired properties.

*Step 1* ( $\Lambda \setminus \{0\} \subset \mathcal{SZ}(\mathbb{R})$ ). Consider a linear combination  $\lambda_1|f|^{\alpha_1} + \lambda_2|f|^{\alpha_2} + \dots + \lambda_n|f|^{\alpha_n}$ , where  $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n$  and  $\lambda_i \neq 0$  for  $i = 1, 2, \dots, n$ . This function can be written as  $\varphi \circ |f|$ , where  $\varphi(x) = \lambda_1x^{\alpha_1} + \lambda_2x^{\alpha_2} + \dots + \lambda_nx^{\alpha_n}$ . Using Remark 3.5, it is clear that the derivative of  $\varphi$  has only a finite number of zeros. In consequence,  $\varphi$  is a  $\mathfrak{c}$ -injective continuous function. Therefore,  $\varphi \circ |f|$  is a Sierpiński-Zygmund function.

*Step 2.* The dimension of  $\Lambda$  as a vector space is  $\mathfrak{c}$ . Just notice that as the linear combination in the previous paragraph is always a Sierpiński-Zygmund function, it cannot be identically zero. This proves that the functions  $|f|^\alpha$ ,  $\alpha > 0$ , are linearly independent and therefore  $\dim \Lambda = \mathfrak{c}$ .

*Step 3.* The algebraic dimension of  $\Lambda$  is  $\mathfrak{c}$ . Let  $B$  be a positive Hamel basis of  $\mathbb{R}$  over  $\mathbb{Q}$  and let  $S = \{|f|^\beta : \beta \in B\}$ . By an elementary argument, we can show that  $S$  is a generator system for  $\Lambda$ . We shall see in the following that  $S$  is an algebraically independent family. Notice also that  $\text{card } S = \text{card } B = \mathfrak{c}$ .

If  $P(x_1, \dots, x_m)$  is a polynomial, it can be written in the form

$$P(x_1, \dots, x_m) = \sum_{i=1}^n \lambda_i \prod_{j=1}^m x_j^{n_{ij}},$$

where  $n_{ij} \in \mathbb{N} \cup \{0\}$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m$  and if  $n_{ij} = n_{i'j}$  for every  $j = 1, \dots, m$ , then  $i = i'$ . (This last condition states plainly that the monomials appearing in the previous equation are not similar to each other.) Assume that  $P(|f|^{\beta_1}, \dots, |f|^{\beta_m}) \equiv 0$  where  $\beta_i \in B$  for  $i = 1, \dots, m$ , and  $\beta_i \neq \beta_j$  if  $i, j \in \{1, \dots, m\}$ . Then we have

$$\sum_{i=1}^n \lambda_i |f|^{\sum_{j=1}^m n_{ij} \beta_j} \equiv 0.$$

Assume that our polynomial is non-zero; then it must be that  $\lambda_i \neq 0$  for some  $i$ . As the functions  $|f|^\alpha$ ,  $\alpha > 0$ , are linearly independent, this means that there exist  $i, i' \in \{1, \dots, n\}$ , with  $i \neq i'$ , such that

$$\sum_{j=1}^m (n_{ij} - n_{i'j})\beta_j = 0.$$

As  $B$  is linearly independent over  $\mathbb{Q}$ , we have  $n_{ij} = n_{i'j}$  for every  $j = 1, \dots, m$ , and then  $i = i'$ . This is a contradiction.  $\square$

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DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE CIENCIAS 3, MADRID 28040, SPAIN

*E-mail address:* `jlgomez@mat.ucm.es`

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE CIENCIAS 3, MADRID 28040, SPAIN

*E-mail address:* `gustavo_fernandez@mat.ucm.es`

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE CIENCIAS 3, MADRID 28040, SPAIN

*E-mail address:* `victorms@mat.ucm.es`

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE CIENCIAS 3, MADRID 28040, SPAIN

*E-mail address:* `jseoane@mat.ucm.es`