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AN UPPER BOUND ON THE DIMENSION OF THE REFLEXIVITY CLOSURE

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Abstract. Let $\mathcal{V}, \mathcal{W}$ be linear spaces over an algebraically closed field, and let $\mathcal{S}$ be an n–dimensional subspace of linear operators that maps $\mathcal{V}$ into $\mathcal{W}$. We give a sharp upper bound for the dimension of the intersection of all images of nonzero operators from $\mathcal{S}$, namely $\dim\left(\bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im} A\right) \leq \dim \mathcal{V} - n + 1$. As an application, we also give a sharp upper bound for the dimension of the reflexivity closure $\operatorname{Ref}\mathcal{S}$ of $\mathcal{S}$, namely $\dim(\operatorname{Ref}\mathcal{S}) \leq n(n+1)/2$.

1. Introduction and statements of the main results

Let $\mathbb{F}$ be a commutative field. If $\mathbb{F}$ is algebraically closed, the spectrum of any n–by–n matrix $A \in \mathcal{M}_n(\mathbb{F})$ is nonempty. That is, at least one of the matrices $A + \lambda\,\mathrm{Id}$ is singular, as λ runs over all the scalars. As one can check by some simple matrix manipulations, there is an equivalent way of formulating this fact in terms of images of any given n–by–n matrices A and B:

$$\dim \bigcap_{(\lambda_0,\lambda)\in\mathbb{F}^2\setminus\{0\}} \operatorname{Im}(\lambda_0 A + \lambda B) \leq n - 1.$$

Our main result from below is the generalization of the above estimate to the case of several rectangular (not necessarily square) matrices.

Theorem 1.1. *Let $\mathbb{F}$ be an algebraically closed field, let $n, m \geq 1$, and suppose $A_1, \ldots, A_k \in \mathcal{M}_{m\times n}(\mathbb{C})$ are m–by–n matrices, with $1 \leq k \leq n+1$. Then,*

$$\dim \bigcap_{(\xi_1,\ldots,\xi_k)\neq 0} \operatorname{Im}(\xi_1 A_1 + \cdots + \xi_k A_k) \leq n - k + 1. \tag{1.1}$$

Remark 1.2. Although for m–by–n matrices $\dim(\operatorname{Im} X) \leq \min\{m, n\}$, we cannot replace, in general, in (1.1) the right side with $\min\{m, n\} - k + 1$. We refer to the last section for more details.

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Note that if $A_1, \dots, A_k$ are linearly dependent, the formula (1.1) is automatically true, because the image of some linear combination of them is zero. Otherwise, $A_1, \dots, A_k$ span a k-dimensional subspace of m–by–n matrices. So there is a more compact, but equivalent, version of Theorem 1.1:

Theorem 1.3. *Let $\mathbb{F} = \overline{\mathbb{F}}$, let $m, n \geq 1$ and let $0 \leq k \leq n$. If $\mathcal{S} \subseteq \mathcal{M}_{m\times n}(\mathbb{F})$ is a subspace of dimension at least $(k+1)$, then*

$$\dim\Big(\bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im} A\Big) \leq n-k.$$

To prove Theorem 1.3, we will require a deep result from determinantal varieties. We state it in a form which resembles [4, Lemma 2.5]:

Lemma 1.4. *Let $\mathbb{F}$ be an algebraically closed field, let r, s be integers with $1 \leq r \leq s$, and let $\mathcal{T} \subseteq \mathcal{M}_{r\times s}(\mathbb{F})$ be a linear subspace. If all matrices from $\mathcal{T}$ have maximal rank $= r$, then $\dim \mathcal{T} \leq s - r + 1$.*

Proof. The statement follows by combining Proposition 11.4 and Proposition 12.2 of [8]. For alternate proofs we refer either to [13, Corollary I] or to [10, Theorem 13.10], which estimates the codimension of the algebraic variety R_t of all r–by–s matrices of rank $< t$. Our hypothesis means $\mathcal{T} \cap R_r = \{0\}$, and we can use the estimate for $t = r$, namely $\operatorname{codim} R_r \leq s - r + 1$. □

The requirement that $\mathbb{F}$ be algebraically closed is necessary for the estimate in Lemma 1.4. For an arbitrary field, things can be more complicated; see for instance [9] for certain results in the real case.

Proof of Theorem 1.3. By replacing $\mathcal{S}$ by a $(k+1)$–dimensional subspace, we can assume that $\dim \mathcal{S} = k+1$. We thus have to show that if U, V are linear spaces over an algebraically closed field and $\mathcal{S} \subset Hom(U,V)$ is a linear subspace of the space $Hom(U,V)$ of all linear maps from U to V, with

$$\dim \mathcal{S} \leq \dim U + 1,$$

then

$$\dim \bigcap_{0\neq S\in\mathcal{S}} \operatorname{Im}(S) \leq \dim U - \dim \mathcal{S} + 1.$$

Let

$$V_0 = \bigcap_{0\neq S\in\mathcal{S}} \operatorname{Im}(S).$$

The statement is clear if $V_0 = \{0\}$. Assume then that $V_0 \neq \{0\}$ and let V_1 be a direct complement of V_0 in V. Let $\mathcal{T} \subset Hom(U, V/V_1)$ be the image of $\mathcal{S}$ under the map $Hom(U,V) \to Hom(U, V/V_1)$ obtained by composition with the canonical factorization $V \to V/V_1$. Note that $\dim \mathcal{S} = \dim \mathcal{T}$. Set $r := \dim V/V_1$ $(= \dim V_0)$ and $s := \dim U$. One easily checks that there exist $0 \neq T \in \mathcal{T}$, and moreover all such $T \neq 0$ from $\mathcal{T}$ are surjective. Hence $s = \dim U \geq \dim V/V_1 = r$ and $\mathcal{T} \cap R_r = 0$, where R_r is the algebraic variety of all operators in $Hom(U, V/V_1)$ of rank $< \dim V/V_1$ $(= r)$. By Lemma 1.4, we obtain that

$$\dim \mathcal{T} \leq s - r + 1.$$

Therefore

$$\dim \mathcal{S} = \dim \mathcal{T} \leq s - r + 1 = \dim U - \dim V_0 + 1,$$

and Theorem 1.3 is proved. □

We conclude this section with a dual version of Theorem 1.3. As usual, given a set of subspaces $\mathcal{X}_\lambda \subseteq \mathcal{W}$, we let $\bigvee \mathcal{X}_\lambda$ stand for the linear span of $\bigcup \mathcal{X}_\lambda$.

Corollary 1.5. *Under all the assumptions of Theorem* 1.3*, except for* $0 \le k \le n$ *replaced by* $0 \le k \le m$*, we have*

$$\dim \bigvee_{A \in \mathcal{S}\setminus\{0\}} \operatorname{Ker} A \ge n - m + k.$$

Proof. Regard m–by–n matrices as linear operators from $\mathcal{V} := \mathbb{F}^n$ into $\mathcal{W} := \mathbb{F}^m$. Given $A \in \mathcal{S}$, its adjoint, A^*, maps the dual space $\mathcal{W}^*$ of $\mathcal{W}$ into the dual space $\mathcal{V}^*$. Now, the space of adjoint operators $\mathcal{S}^* := \{A^*;\ \ A \in \mathcal{S}\} \subseteq \operatorname{Hom}(\mathcal{W}^*, \mathcal{V}^*)$ is of the same dimension as $\mathcal{S}$ and can be identified with transposed matrices from $\mathcal{S}$. In particular, $\mathcal{S}^* \subseteq \mathcal{M}_{n\times m}(\mathbb{F})$. Also, $\operatorname{Ker} A = (\operatorname{Im} A^*)_\perp$, where $\mathcal{X}_\perp := \{\mathbf{x} \in \mathcal{V};\ \ f(\mathbf{x}) = 0\ \forall f \in \mathcal{X}\}$ is the pre-annihilator of a subspace $\mathcal{X} \subseteq \mathcal{V}^*$. Consequently, since spaces are finite-dimensional,

$$\begin{aligned}\dim\Big(\bigvee_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Ker} A\Big) &= \dim \bigvee_{A\in\mathcal{S}\setminus\{0\}} (\operatorname{Im} A^*)_\perp = \dim\Big(\bigcap_{A^*\in\mathcal{S}^*\setminus\{0\}} \operatorname{Im} A^*\Big)_\perp \\ &= n - \dim\Big(\bigcap_{A^*\in\mathcal{S}^*\setminus\{0\}} \operatorname{Im} A^*\Big).\end{aligned}$$

The right-hand side is the intersection of images of n–by–m matrices, so by Theorem 1.3 the right-hand side is greater than or equal to $n - (m - k)$. □

2. Application

Let $\mathcal{V}, \mathcal{W}$ be vector spaces over a field $\mathbb{F}$ and let $\mathcal{S} \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ be a subspace of $\mathbb{F}$-linear operators from $\mathcal{V}$ into $\mathcal{W}$. The reflexivity closure $\operatorname{Ref}\mathcal{S}$ is the set of all linear operators $T \in \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ such that for every $\mathbf{x} \in \mathcal{V}$ there exists some $S = S_{\mathbf{x}} \in \mathcal{S}$ with $T\mathbf{x} = S\mathbf{x}$. Equivalently, $T\mathbf{x} \in \mathcal{S}\mathbf{x} := \{S\mathbf{x};\ \ S \in \mathcal{S}\}$ for each $\mathbf{x}$. It is immediate that $\operatorname{Ref}\mathcal{S}$ is also a subspace and that it contains $\mathcal{S}$. When $\dim \mathcal{S} < \infty$ we introduce the quantity $\operatorname{rd}\mathcal{S} := \dim(\operatorname{Ref}\mathcal{S}) - \dim\mathcal{S} = \dim((\operatorname{Ref}\mathcal{S})/\mathcal{S})$, which measures how much larger $\operatorname{Ref}\mathcal{S}$ is compared to $\mathcal{S}$. Following Delai [6], we call this integer the *reflexivity defect* of $\mathcal{S}$.

On the one extreme, it may happen that $\operatorname{Ref}\mathcal{S} = \mathcal{S}$. Such spaces are called reflexive and have been extensively studied [3, 7, 11, 12].

Example 2.1. If $\mathcal{S} = \operatorname{Lin}\{T\}$ is a one-dimensional subspace, then the elementary exercise validates $\operatorname{Ref}\mathcal{S} = \mathcal{S}$, so $\operatorname{rd}\mathcal{S} = 0$. More generally, $\operatorname{rd}\mathcal{S} = 0$ whenever $\operatorname{Ref}\mathcal{S} = \mathcal{S}$, that is, whenever $\mathcal{S}$ is reflexive.

But on the other extreme, it may happen that the space is far from being reflexive.

Example 2.2. The ideal $\mathcal{F} \subseteq \mathcal{B}(\mathcal{X})$ of finite-rank operators on some Banach space $\mathcal{X}$ is transitive; that is, $\mathcal{F}\mathbf{x} = \mathcal{X}$ for every nonzero vector $\mathbf{x} \in \mathcal{X}$. It follows that $\operatorname{Ref}\mathcal{F} = \mathcal{B}(\mathcal{X})$. Inversely, if $\operatorname{Ref}\mathcal{S} = \mathcal{B}(\mathcal{X})$ for some subspace of operators, then $\mathcal{S}$ must be transitive. Therefore, transitive spaces are as far from being reflexive as one can hope for.

However, there are also subspaces which are neither reflexive nor transitive.

Example 2.3. Let $\mathsf{e}_1, \dots, \mathsf{e}_n$ be a standard basis of column vectors in $\mathcal{V} := \mathbb{F}^n$. With respect to this basis, $\mathrm{Hom}(\mathbb{F}^n, \mathbb{F}^n)$ can be identified with $\mathcal{M}_n(\mathbb{F})$, the algebra of n–by–n matrices. Let $N \in \mathcal{M}_n(\mathbb{F})$ be an elementary upper-triangular Jordan nilpotent, and let $\mathcal{S} := \mathrm{Lin}\{\mathrm{Id}, N, \dots, N^{n-1}\} \subseteq \mathcal{M}_n(\mathbb{F})$. Then it follows from [5, Theorem 4.3], with $q(z) := z$ and $(n_1, n_2) = (n, 0)$, that $\mathrm{Ref}\,\mathcal{S}$ consists of all upper-triangular matrices. Thus, $\mathrm{rd}\,\mathcal{S} = n(n+1)/2 - n$.

It is our aim to show that, for algebraically closed fields, the reflexivity defect is always bounded above by $n(n+1)/2 - n$, where $n := \dim \mathcal{S}$; see Theorem 2.7 below. Example 2.3 shows that this estimate is sharp. Before giving a proof, however, we introduce the following notation. Given any subspace $\mathcal{U} \subseteq \mathcal{V}$ and $\mathcal{S} \subseteq \mathrm{Hom}(\mathcal{V}, \mathcal{W})$, we let $\mathcal{S}|_{\mathcal{U}} := \{T|_{\mathcal{U}};\ \ T \in \mathcal{S}\}$ be the set of all restrictions of operators from $\mathcal{S}$. Recall that $T|_{\mathcal{U}} : \mathcal{U} \to \mathcal{W}$.

We start with a trivial observation.

Lemma 2.4. *If $\mathcal{S} \subseteq \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ and $\mathcal{U} \subseteq \mathcal{V}$ is a subspace, then $(\mathrm{Ref}\,\mathcal{S})|_{\mathcal{U}} \subseteq \mathrm{Ref}(\mathcal{S}|_{\mathcal{U}})$.*

Proof. Immediate. □

Lemma 2.5. *Suppose $\mathcal{O} \subseteq \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is a finite-dimensional subspace of linear operators from $\mathcal{V}$ into $\mathcal{W}$. If $\mathcal{U} \subseteq \mathcal{V}$ is a subspace, then*

$$\dim \mathcal{O} = \dim(\mathcal{O}|_{\mathcal{U}}) + \dim\{T \in \mathcal{O};\ \ T|_{\mathcal{U}} = 0\}.$$

In addition, if $\mathcal{U} = \hat{\mathcal{U}} + \mathrm{Lin}\{\mathbf{x}\}$ with $\mathcal{O}|_{\hat{\mathcal{U}}} = 0$, then

$$\dim \mathcal{O} = \dim(\mathcal{O}\mathbf{x}) + \dim\{T \in \mathcal{O};\ \ T|_{\mathcal{U}} = 0\}.$$

Proof. Let $\phi : Hom(\mathcal{V}, \mathcal{W}) \to Hom(\mathcal{U}, \mathcal{W})$ denote the restriction map. Then

$$\dim \mathcal{O} = \dim \mathrm{Im}(\phi_{|\mathcal{O}}) + \dim \ker(\phi_{|\mathcal{O}}),$$

which is precisely the first conclusion of the lemma. The second equality follows similarly. We omit the details. □

We will also require the following general lemma:

Lemma 2.6. *Let $r \in \mathbb{N}$, let $\mathcal{X}, \mathcal{Y}$ be vector spaces over a field $\mathbb{F}$ with $|\mathbb{F}| \geq r+3$, and let $\mathcal{O} \subseteq \mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ be a finite-dimensional subspace. Suppose the vectors $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$ satisfy $r := \dim \mathcal{O}\mathbf{x} \geq \dim \mathcal{O}(\mathbf{x} + \lambda\mathbf{x}')$ for $\lambda \in \mathbb{F}$. If $\mathbf{y} \in \mathcal{O}(\mathbf{x} + \lambda\mathbf{x}')$ for each nonzero $\lambda \in \mathbb{F}$, then also $\mathbf{y} \in \mathcal{O}\mathbf{x}$.*

Proof. There is nothing to prove when $r = 0$.

Assume $r \geq 1$. Let $S_1, \dots, S_n$ be a basis for $\mathcal{O}$. If necessary we re-index this basis such that the first r vectors $S_1\mathbf{x}, \dots, S_r\mathbf{x}$ are linearly independent, while $S_{r+1}\mathbf{x}, \dots, S_n\mathbf{x}$ are their linear combinations.

Clearly, the vectors $S_1\mathbf{x}, \dots, S_n\mathbf{x},\ S_1\mathbf{x}', \dots, S_n\mathbf{x}'$ span the finite-dimensional subspace $\mathcal{O}\mathbf{x} + \mathcal{O}\mathbf{x}' \subseteq Y$, and $\mathcal{O}(\mathbf{x} + \lambda\mathbf{x}') \subseteq \mathcal{O}\mathbf{x} + \mathcal{O}\mathbf{x}'$ for every λ. Choose and fix an arbitrary basis of $\mathcal{O}\mathbf{x} + \mathcal{O}\mathbf{x}'$. With respect to this basis, we may identify $\mathcal{O}\mathbf{x} + \mathcal{O}\mathbf{x}'$ with $\mathbb{F}^d$, for some $d \geq r$. By doing so, we may assume that $\mathbf{y},\ S_1\mathbf{x}, \dots, S_n\mathbf{x},\ S_1\mathbf{x}', \dots, S_n\mathbf{x}'$ are already column vectors from $\mathbb{F}^d$.

Construct a $d \times (n+1)$ matrix

$$\Xi(\lambda) := \Big[\mathbf{y} \mid S_1(\mathbf{x} + \lambda\mathbf{x}') \mid \cdots \mid S_n(\mathbf{x} + \lambda\mathbf{x}')\Big]$$

by concatenating the column vectors one after another. Now, by assumptions, $\mathbf{y} \in \mathcal{O}(\mathbf{x} + \lambda\mathbf{x}')$, so $\mathbf{y}$ is a linear combination of $S_1(\mathbf{x} + \lambda\mathbf{x}'), \dots, S_n(\mathbf{x} + \lambda\mathbf{x}')$, for every $\lambda \neq 0$. Moreover, due to $r = \dim \mathcal{O}\mathbf{x} \geq \dim \mathcal{O}(\mathbf{x} + \lambda\mathbf{x}')$, there are at most r linearly independent vectors among $S_1(\mathbf{x} + \lambda\mathbf{x}'), \dots, S_n(\mathbf{x} + \lambda\mathbf{x}')$.

Equivalently stated, $\operatorname{rk} \Xi(\lambda) \leq r$ for every $\lambda \neq 0$. So, every $(r+1) \times (r+1)$ minor of $\Xi(\lambda)$ is identically zero, for $\lambda \neq 0$. But note that $S_i(\mathbf{x} + \lambda\mathbf{x}') = S_i\mathbf{x} + \lambda S_i\mathbf{x}'$ implies that these minors are polynomials in variable λ of degree at most $r + 1$. Since they vanish for $\lambda \neq 0$ and the field $\mathbb{F}$ has at least $r + 2$ nonzero elements, every $(r + 1) \times (r + 1)$ minor is a zero polynomial. Therefore, they also vanish at $\lambda = 0$, which gives $\operatorname{rk} \Xi(\lambda)|_{\lambda=0} \leq r$.

By assumptions, $S_1\mathbf{x}, \dots, S_r\mathbf{x}$ are linearly independent, which means that the second, third, $\dots$, $(r + 1)$-th columns of $\Xi(0)$ are also linearly independent. But then, $\operatorname{rk} \Xi(0) \leq r$ implies that the first column of $\Xi(0)$, that is, the vector $\mathbf{y}$, must be a linear combination of the vectors $S_1\mathbf{x}, \dots, S_r\mathbf{x}$. Equivalently, $\mathbf{y} \in \operatorname{Lin}\{S_1\mathbf{x}, \dots, S_r\mathbf{x}\} = \mathcal{O}\mathbf{x}$. □

We can now prove our main result of this section.

Theorem 2.7. *Suppose that $\mathcal{V}, \mathcal{W}$ are vector spaces over an algebraically closed field, and let $\mathcal{S} \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ be a finite-dimensional subspace of operators from $\mathcal{V}$ to $\mathcal{W}$. Then,*

$$\dim(\operatorname{Ref} \mathcal{S}) \leq \frac{(\dim \mathcal{S})(1 + \dim \mathcal{S})}{2}.$$

Proof. To shorten the arguments we write $n := \dim \mathcal{S}$. We first verify the claim for the restriction of $\mathcal{S}$ to finite-dimensional vector subspaces of $\mathcal{V}$.

So suppose $\mathcal{V}_k \subseteq \mathcal{V}$ is a subspace of dimension k, and consider $\mathcal{B}_0 := \operatorname{Ref}(\mathcal{S}|_{\mathcal{V}_k})$. Fix a vector $\mathbf{x}_1 \in \mathcal{V}_k$ such that $\dim \mathcal{B}_0\mathbf{x}_1 = \max_{\mathbf{x} \in \mathcal{V}_k} \dim \mathcal{B}_0\mathbf{x}$ is maximal. By the definition of reflexive closure, $\mathcal{B}_0\mathbf{x} \subseteq \mathcal{S}\mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}_k$, giving $\dim \mathcal{B}_0\mathbf{x} \leq n$. Now, let $\mathcal{B}_1 := \{A \in \mathcal{B}_0;\quad A\mathbf{x}_1 = 0\}$. We next construct inductively vectors $\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_k \in \mathcal{V}_k$ and subspaces $\mathcal{B}_2, \dots, \mathcal{B}_k \subseteq \mathcal{B}_0$ such that $\dim \mathcal{B}_{i-1}\mathbf{x}_i = \max_{\mathbf{x} \in \mathcal{V}_k} \dim \mathcal{B}_{i-1}\mathbf{x}$ and $\mathcal{B}_i := \{A \in \mathcal{B}_0;\quad A\mathbf{x}_1 = \cdots = A\mathbf{x}_i = 0\}$. Clearly we may assume that the vectors $\mathbf{x}_1, \dots, \mathbf{x}_k$ are linearly independent, so they form a basis of $\mathcal{V}_k$. Then we have $\mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \cdots \supseteq \mathcal{B}_k = \{0\}$. Moreover, the operators from $\mathcal{B}_0$ are determined by prescribing their values on basis elements of $\mathcal{V}_k$, so that

$$\dim \mathcal{B}_i \leq \dim \mathcal{B}_0 \leq \dim \mathcal{V}_k \cdot \max_{\mathbf{x} \in \mathcal{V}_k} \dim \mathcal{S}\mathbf{x} \leq kn < \infty.$$

We proceed by showing that $\mathcal{B}_{i-1}\mathbf{x}_i \subseteq \mathcal{B}_{i-2}\mathbf{x}_{i-1}$, $(i \geq 2)$. Let $i \geq 2$ and let $A \in \mathcal{B}_{i-1}$ be arbitrary. For each $\lambda \in \mathbb{F} \setminus \{0\}$ we have

$$\begin{aligned} A\mathbf{x}_i = A(\lambda^{-1}\mathbf{x}_{i-1} + \mathbf{x}_i) &\in \mathcal{B}_{i-1}(\lambda^{-1}\mathbf{x}_{i-1} + \mathbf{x}_i) = \mathcal{B}_{i-1}(\mathbf{x}_{i-1} + \lambda\mathbf{x}_i) \\ &\subseteq \mathcal{B}_{i-2}(\mathbf{x}_{i-1} + \lambda\mathbf{x}_i) \qquad (\lambda \neq 0). \end{aligned}$$

Since $\dim \mathcal{B}_{i-2}\mathbf{x}_{i-1} = \max_{\mathbf{x} \in \mathcal{V}_k} \dim \mathcal{B}_{i-2}\mathbf{x}$, and as algebraically closed fields have infinite cardinality, Lemma 2.6 for $\mathcal{O} := \mathcal{B}_{i-2}$ and $\mathbf{y} := A\mathbf{x}_i$ indeed gives $A\mathbf{x}_i \in \mathcal{B}_{i-2}\mathbf{x}_{i-1}$, as anticipated.

We now claim that

$$\mathcal{B}_{s-1}\mathbf{x}_s \subseteq \bigcap_{(\xi_1, \dots, \xi_s) \in \mathbb{F}^s \setminus \{0\}} \mathcal{S}\Big(\sum_{i=1}^{s} \xi_i \mathbf{x}_i\Big); \qquad (s \geq 1). \tag{2.1}$$

When $s = 1$ this follows from the definition of reflexivity closure. So assume $s \geq 2$, and let $A \in \mathcal{B}_{s-1}$. Choose any s-tuple $(\xi_1, \dots, \xi_s) \in \mathbb{F}^s \setminus \{0\}$, and let j be the last index with $\xi_j \neq 0$; so $\xi_{j+1} = 0 = \cdots = \xi_s$. Now,

$$A\mathbf{x}_s \in \mathcal{B}_{s-1}\mathbf{x}_s \subseteq \mathcal{B}_{s-2}\mathbf{x}_{s-1} \subseteq \cdots \subseteq \mathcal{B}_{j-1}\mathbf{x}_j = \mathcal{B}_{j-1}\Big(\mathbf{x}_j + \sum_{i=1}^{j-1} \xi_j^{-1}\xi_i\mathbf{x}_i\Big).$$

Since $\mathcal{B}_{j-1} \subseteq \operatorname{Ref}(\mathcal{S}|_{\mathcal{V}_k})$, we further have, by the definition of the reflexivity closure,

$$\mathcal{B}_{j-1}\Big(\mathbf{x}_j + \sum_{i=1}^{j-1} \xi_j^{-1}\xi_i\mathbf{x}_i\Big) \subseteq \mathcal{S}\Big(\mathbf{x}_j + \sum_{i=1}^{j-1} \xi_j^{-1}\xi_i\mathbf{x}_i\Big) = \mathcal{S}\Big(\sum_{i=1}^{j} \xi_i\mathbf{x}_i\Big);$$

and since $A \in \mathcal{B}_{s-1}$ was arbitrary, we deduce (2.1).

Fix a basis $S_1, \dots, S_n$ of $\mathcal{S}$. Then, $\widehat{\mathcal{W}} := \mathcal{S}\mathcal{V}_k = \operatorname{Lin}\{S_i\mathbf{x}_j;\ \ 1 \leq i \leq n,\ 1 \leq j \leq k\}$ is a finite-dimensional subspace of $\mathcal{W}$. Actually, its dimension, $m := \dim \widehat{\mathcal{W}}$, satisfies $m \leq kn$. So we may identify $\widehat{\mathcal{W}}$ with $\mathbb{F}^m$ and associate to each vector $\mathbf{x}_j$ the m–by–n matrix, given by the columns

$$\mathfrak{S}_j := \big[S_1\mathbf{x}_j | S_2\mathbf{x}_j | \cdots | S_n\mathbf{x}_j\big]. \tag{2.2}$$

Given a vector $\mathbf{x} = \xi_1\mathbf{x}_1 + \cdots + \xi_j\mathbf{x}_j$, it is immediate that $\mathcal{S}\mathbf{x} = \operatorname{Im}(\xi_1\mathfrak{S}_1 + \cdots + \xi_j\mathfrak{S}_j)$. Consequently, we can restate (2.1) as

$$\begin{aligned} \dim \mathcal{B}_{s-1}\mathbf{x}_s &\leq \dim \bigcap_{(\xi_1,\dots,\xi_s)\in\mathbb{F}^s\setminus\{0\}} \mathcal{S}\Big(\sum_{i=1}^{s} \xi_i\mathbf{x}_i\Big) \\ &= \dim \bigcap_{(\xi_1,\dots,\xi_s)\in\mathbb{F}^s\setminus\{0\}} \operatorname{Im}(\xi_1\mathfrak{S}_1 + \cdots + \xi_s\mathfrak{S}_s). \end{aligned}$$

By Theorem 1.1,

$$\dim \mathcal{B}_{s-1}\mathbf{x}_s \leq \dim \bigcap_{(\xi_1,\dots,\xi_s)\in\mathbb{F}^s\setminus\{0\}} \operatorname{Im}(\xi_1\mathfrak{S}_1 + \cdots + \xi_s\mathfrak{S}_s) \leq n - s + 1, \tag{2.3}$$

wherefrom, with a repeated use of Lemma 2.5 on a nest of subspaces $\mathcal{U}_j := \operatorname{Lin}\{\mathbf{x}_1, \dots, \mathbf{x}_j\} \subseteq \mathcal{V}_k$:

$$\begin{aligned} \dim \operatorname{Ref}(\mathcal{S}|_{\mathcal{V}_k}) &= \dim \mathcal{B}_0 \leq \dim(\mathcal{B}_0|_{\operatorname{Lin}\{\mathbf{x}_1\}}) + \dim \mathcal{B}_1 \\ &\leq \dim(\mathcal{B}_0|_{\operatorname{Lin}\{\mathbf{x}_1\}}) + \dim(\mathcal{B}_1|_{\operatorname{Lin}\{\mathbf{x}_1,\mathbf{x}_2\}}) + \dim \mathcal{B}_2 \\ &\leq \cdots \\ &\leq \dim(\mathcal{B}_0|_{\operatorname{Lin}\{\mathbf{x}_1\}}) + \dim(\mathcal{B}_1|_{\operatorname{Lin}\{\mathbf{x}_1,\mathbf{x}_2\}}) + \cdots + \dim(\mathcal{B}_{k-1}|_{\{\mathbf{x}_1,\dots,\mathbf{x}_k\}}) + \dim \mathcal{B}_k \\ &\leq \dim \mathcal{B}_0\mathbf{x}_1 + \dim \mathcal{B}_1\mathbf{x}_2 + \cdots + \dim \mathcal{B}_{k-1}\mathbf{x}_k + \dim \mathcal{B}_k \\ &\leq \begin{cases} n + (n-1) + \cdots + (n-k+1) + 0; & k \leq n \\ n + (n-1) + \cdots + 1 + 0; & k \geq n \end{cases} \leq \frac{n(n+1)}{2}. \end{aligned} \tag{2.4}$$

By Lemma 2.4, and in view of (2.4), $\dim\big((\operatorname{Ref}\mathcal{S})|_{\mathcal{V}_k}\big) \leq \dim \operatorname{Ref}(\mathcal{S}|_{\mathcal{V}_k}) \leq n(n+1)/2$ holds for every finite-dimensional subspace $\mathcal{V}_k \subseteq \mathcal{V}$. Therefore, $\dim(\operatorname{Ref}\mathcal{S}) \leq n(n+1)/2$. $\square$

3. Non-algebraically closed fields

The estimate in Theorem 2.7 is not true for non-algebraically closed fields.

Example 3.1. Consider the two-dimensional subspace $\mathcal{S}$ of $\mathcal{M}_2(\mathbb{R})$ generated by the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check that for each nonzero vector $\mathbf{x}$ the vectors $A_1\mathbf{x}$ and $A_2\mathbf{x}$ are linearly independent and therefore span all $\mathbb{R}^2$. Hence $\operatorname{Ref}\mathcal{S} = \mathcal{M}_2(\mathbb{R})$ and $\dim\operatorname{Ref}\mathcal{S} = 4$ (in the complex case $\dim\mathcal{S} = 2$ implies $\dim\operatorname{Ref}\mathcal{S} \le 3$, by Theorem 2.7).

We show that for any infinite field we have $\dim\operatorname{Ref}\mathcal{S} \le (\dim\mathcal{S})^2$. First we need the following reduction:

Lemma 3.2. *Let $n \in \mathbb{N}$, and let $\mathbb{F}$ be a field satisfying $|\mathbb{F}| \ge n+3$. Let $\mathcal{V}, \mathcal{W}$ be vector spaces over $\mathbb{F}$, and let $\mathcal{S}$ be an n-dimensional subspace of* $\operatorname{Hom}(\mathcal{V}, \mathcal{W})$. *Then there exist vector spaces $\mathcal{W}' \subseteq \mathcal{W}$ and $\mathcal{S}' \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W}')$ such that* $\dim\mathcal{W}' \le n$, $\dim\mathcal{S}' \le n$ *and* $\dim\operatorname{Ref}\mathcal{S}' \ge \dim\operatorname{Ref}\mathcal{S}$.

Proof. Fix a vector $\mathbf{x} \in \mathcal{V}$ such that the dimension $\dim\mathcal{S}\mathbf{x}$ is maximal. Set $\mathcal{W}' = \mathcal{S}\mathbf{x}$. Clearly $\dim\mathcal{W}' \le n$.

Fix a projection $P : \mathcal{W} \to \mathcal{W}$ with $\operatorname{Im} P = \mathcal{W}'$. Let $\mathcal{S}' = P\mathcal{S} = \{PA;\ \ A \in \mathcal{S}\}$. Then $\mathcal{S}' \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W}')$ and $\dim\mathcal{S}' \le \dim\mathcal{S} = n$.

Let $A \in \operatorname{Ref}\mathcal{S}$ and $A\mathbf{x} = 0$. We show that $\operatorname{Im} A \subseteq \mathcal{W}'$. Indeed, let $\mathbf{x}' \in \mathcal{V}$ be arbitrary. For each nonzero $\lambda \in \mathbb{F}$ we have

$$A\mathbf{x}' = A(\mathbf{x}' + \lambda^{-1}\mathbf{x}) \in \mathcal{S}(\mathbf{x}' + \lambda^{-1}\mathbf{x}) = \mathcal{S}(\mathbf{x} + \lambda\mathbf{x}').$$

By Lemma 2.6, we have $A\mathbf{x}' \in \mathcal{S}\mathbf{x} = \mathcal{W}'$.

We have just proved that $A \in \operatorname{Ref}\mathcal{S}$ and $A\mathbf{x} = 0$ imply $A = PA \in \mathcal{S}'$. Consequently,

$$\begin{aligned} \dim\operatorname{Ref}\mathcal{S} &= \dim\mathcal{S}\mathbf{x} + \dim\{A \in \operatorname{Ref}\mathcal{S};\ \ A\mathbf{x} = 0\} \\ &\le \dim\mathcal{S}'\mathbf{x} + \dim\{B \in \operatorname{Ref}\mathcal{S}';\ \ B\mathbf{x} = 0\} = \dim\operatorname{Ref}\mathcal{S}'. \end{aligned}$$ □

Theorem 3.3. *Let $n \in \mathbb{N}$, and let $\mathbb{F}$ be a field satisfying $|\mathbb{F}| \ge n+3$. Let $\mathcal{V}, \mathcal{W}$ be vector spaces over $\mathbb{F}$, and let $\mathcal{S}$ be an n-dimensional subspace of* $\operatorname{Hom}(\mathcal{V}, W)$. *Then* $\dim\operatorname{Ref}\mathcal{S} \le n^2$.

Proof. Suppose on the contrary that $\dim\operatorname{Ref}\mathcal{S} > n^2$. By Lemma 3.2, we may assume that $\dim\mathcal{W} \le n$. Consider the space $\mathcal{S}^* = \{A^*;\ \ A \in \mathcal{S}\} \subseteq \operatorname{Hom}(\mathcal{W}^*, \mathcal{V}^*)$. Then $\dim\mathcal{S}^* = \dim\mathcal{S} = n$ and $\dim\operatorname{Ref}\mathcal{S}^* = \dim(\operatorname{Ref}\mathcal{S})^* > n^2$ (see [5, Proposition 2.1]). Also by the previous lemma, there exist subspaces $\mathcal{V}' \subseteq \mathcal{V}^*$ and $\mathcal{S}' \subseteq \operatorname{Hom}(\mathcal{W}^*, \mathcal{V}')$ such that $\dim\mathcal{S} \le n$ and $\dim\operatorname{Ref}\mathcal{S}' > n^2$. This is a contradiction, since $\dim\operatorname{Hom}(\mathcal{W}^*, \mathcal{V}') = \dim\mathcal{W}^*\dim\mathcal{V}' \le n^2$. □

It is perhaps worth noting that the only place in the proof of Theorem 2.7 where we needed that the field is algebraically closed was in the estimates (2.3) and (2.4). In all other places the arguments demand only $|\mathbb{F}| \ge 3 + \dim\mathcal{B}_{i-2}\mathbf{x}_{i-1}$ when invoking Lemma 2.6 to show that $\mathcal{B}_{i-1}\mathbf{x}_i \subseteq \mathcal{B}_{i-2}\mathbf{x}_{i-1}$. However, $\dim\mathcal{B}_{i-2}\mathbf{x}_{i-1} = \max_{\mathbf{x}\in\mathcal{V}_k}\dim\mathcal{B}_{i-2}\mathbf{x} \le \max_{\mathbf{x}\in\mathcal{V}_k}\dim\mathcal{B}_0\mathbf{x} = \dim\mathcal{S}\mathbf{x}_1 \le n$, and hence we only need

$|\mathbb{F}| \geq n+3$. To appreciate the extra information, we use the notation from the above proof, denote by $\vec{\mathbf{x}} := (\mathbf{x}_1, \dots, \mathbf{x}_k)$ a basis for $\mathcal{V}_k$, and introduce subspaces

$$\overset{\vec{\mathbf{x}}}{\mathfrak{M}}_{s-1} := \bigcap_{(\alpha_1,\dots,\alpha_s)\in\mathbb{F}^s\setminus\{0\}} \operatorname{Im}(\alpha_1\mathfrak{S}_1 + \cdots + \alpha_s\mathfrak{S}_s), \qquad (s = 1, \dots, k).$$

Recall that the m–by–n matrices $\mathfrak{S}_j$ were introduced in (2.2). We can now record the following corollary.

Corollary 3.4. *Let $n \in \mathbb{N}$ and let $\mathcal{V}, \mathcal{W}$ be finite-dimensional vector spaces over a field with $|\mathbb{F}| \geq n+3$. Suppose $\mathcal{S} \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ is an n-dimensional subspace. Then, there exists a basis $\vec{\mathbf{x}} := (\mathbf{x}_1, \dots, \mathbf{x}_k)$ of $\mathcal{V}$ such that*

$$\dim \operatorname{Ref} \mathcal{S} \leq \dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_0 + \dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_1 + \cdots + \dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_k \leq \dim \mathcal{S} + \dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_1 + \cdots + \dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_k.$$

Corollary 3.5. *Under the notations from the previous corollary, suppose $\dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_1 = 0$ for any basis vectors $\vec{\mathbf{x}}$ of $\mathcal{V}$. Then, the space $\mathcal{S}$ is reflexive.*

Proof. This is evident from the previous corollary plus the fact that $0 = \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_1 \supseteq \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_2 \supseteq \cdots \supseteq \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_k$. □

Remark 3.6. Let $\mathcal{V}, \mathcal{W}$ be real vector spaces and let $\mathcal{S} \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ be a finite-dimensional subspace, $\dim \mathcal{S} = n$. By Theorem 3.3, $\dim \operatorname{Ref} \mathcal{S} \leq n^2$. For $n = 2$ the estimate is the best possible, see Example 3.1. The same is true for $n = 4$ and $n = 8$ (the main reason is that in these cases there are n square matrices of order n such that each nontrivial linear combination of them is invertible, see [1] and [2]). However, $n = 2, 4, 8$ are the only cases when the estimate $\dim \operatorname{Ref} \mathcal{S} \leq n^2$ is the optimal, since for other values of n such a system of n matrices does not exist. For example, for $n = 3$ each 3×3 matrix has an eigenvalue, and it is easy to show that $\dim \operatorname{Ref} \mathcal{S} \leq 7$.

Problem: What is the optimal estimate for $\dim \operatorname{Ref} \mathcal{S}$ in the real case?

4. Examples

Here we provide several examples to illuminate our results. First, it would be tempting to conjecture the more 'natural' formula $\dim \bigcap_{(\xi_1,\dots,\xi_k)\neq 0} \operatorname{Im}(\xi_1 A_1 + \cdots + \xi_k A_k) \leq \min\{m, n\} - k + 1$ in place of (1.1). But this is wrong, in general.

Example 4.1. Consider the 2–by–4 matrices

$$A_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_1 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that $\operatorname{Im}(\xi_0 A_0 + \xi_1 A_1) = \operatorname{Lin}\{(1,0)^t, (0,1)^t\}$ for every nonzero linear combination. Hence, the intersection of images has dimension 2. However, $\min\{m, n\} - k + 1 = 2 - 2 + 1 = 1$.

The next example shows that if $\mathcal{S} \subseteq \mathcal{M}_{m\times n}(\mathbb{C})$ and $\dim \mathcal{S} = k$, but $m \neq n-k+1$, then in general there exists no nonzero matrix $A \in \mathcal{S}$ with rank $\leq n - k + 1$ (cf. Lemma 1.4).

Example 4.2. Let

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ b & 0 & a \end{pmatrix} ; \ a, b, c \in \mathbb{C} \right\}.$$

Then every nonzero matrix in $\mathcal{A}$ has rank ≥ 2.

The estimates, provided in Theorem 1.1, respectively, in Theorem 1.3, are sharp. We show this in our next example.

Example 4.3. Let N be an n–by–n upper-triangular elementary Jordan nilpotent. Consider a subspace $\mathcal{S} := \operatorname{Lin}\{\operatorname{Id}, N, N^2, \dots, N^{k-1}\}$. It is easy to see that

$$\bigcap_{A \in \mathcal{S} \setminus \{0\}} \operatorname{Im} A = \bigcap_{(\xi_0, \dots, \xi_{k-1}) \neq 0} \operatorname{Im}\big(\xi_0 N^0 + \dots + \xi_{n-1} N^{k-1}\big) = \operatorname{Im} N^{k-1} = \mathbb{F}^{n-k+1} \oplus 0,$$

so for this subspace, the upper bound in Theorem 1.1 is achieved.

It would be tempting to conjecture that the inverse of the above statement is also true, up to multiplication by a fixed invertible matrix (that is, up to choosing basis vectors). In other words, if the upper bound in (1.1) is achieved, is it always $\mathcal{S} = P \operatorname{Lin}\{\operatorname{Id}, N, N^2, \dots, N^{k-1}\} Q$ for some invertible matrices P, Q? The answer is negative:

Example 4.4. Consider a subspace $\mathcal{S} \subseteq \mathcal{M}_3(\mathbb{C})$ spanned by the identity matrix A_1 and the nilpotents

$$A_2 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_3 := \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, $n = 3 = k$. One easily sees that $\xi_2 A_2 + \xi_3 A_3$ is singular for every nonzero linear combination. So, $\operatorname{Im}(\xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3) = \mathbb{C}^3$ unless $\xi_1 = 0$, in which case the image always contains the vector $(0, 1, 0)^t$. Hence,

$$\dim \bigcap_{\xi_1 \neq 0} \operatorname{Im}(\xi_1 A_1) = 3, \quad \dim \bigcap_{(\xi_1, \xi_2) \neq 0} \operatorname{Im}(\xi_1 A_1 + \xi_2 A_2) = 2,$$
$$\dim \bigcap_{(\xi_1, \xi_2, \xi_3) \neq 0} \operatorname{Im}(\xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3) = 1,$$

and the upper bound in (1.1) is always achieved.

On the other hand, no invertible matrices P, Q would force $P\mathcal{S}Q$ to be upper–triangular. Otherwise, PA_1Q and PA_2Q would be singular; hence they would have to be strictly upper-triangular. But the two-dimensional linear subspace $\operatorname{Lin}\{PA_1Q, PA_2Q\}$ of 3–by–3 strictly upper-triangular matrices necessarily contains a matrix of rank-one, a contradiction, because every nonzero linear combination of A_1, A_2 is of rank-two. In particular, this shows that $P\mathcal{S}Q$ cannot be spanned by powers of a fixed nilpotent.

The converse of Corollary 3.5 is not true, in general. It may happen that $\mathcal{S}$ is reflexive yet $\dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_1 \neq 0$.

Example 4.5. Consider a subspace $\mathcal{S} := \operatorname{Lin}\{A_1, A_2\}$ of 3–by–4 matrices spanned by

$$A_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\vec{\mathbf{x}} := (\mathsf{e}_1, \dots, \mathsf{e}_4)$ be the standard basis in $\mathbb{F}^4$. It follows from the definition (2.2) that $\mathfrak{S}_1 = \begin{bmatrix} A_1\mathbf{e}_1 \mid A_2\mathsf{e}_1 \end{bmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\mathfrak{S}_2 = \begin{bmatrix} A_1\mathbf{e}_2 \mid A_2\mathsf{e}_2 \end{bmatrix} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and so $\dim \bigcap_{(\alpha_1,\alpha_2)\neq 0} \operatorname{Im}(\alpha_1\mathfrak{S}_1 + \alpha_2\mathfrak{S}_2) = 1$. Despite this, $\mathcal{S}$ is reflexive. This can be computed directly, or else one uses the fact that every nonzero member from two-dimensional space $\mathcal{S}$ has rank 3 and then applies [12, Theorem 1.1].

This example also shows that the inequality $(\operatorname{Ref}\mathcal{S})|_{\mathcal{U}} \subseteq \operatorname{Ref}(\mathcal{S}|_{\mathcal{U}})$ in Lemma 2.4 can be strict: use $\mathcal{U} := \operatorname{Lin}\{\mathsf{e}_1, \mathsf{e}_2\} \subseteq \mathbb{F}^4$.

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