

THIN POSITION AND PLANAR SURFACES FOR GRAPHS IN THE 3-SPHERE

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ABSTRACT. We show that given a trivalent graph in S^3 , either the graph complement contains an essential almost meridional planar surface or, after edge slides, thin position for the graph is also bridge position. This can be viewed as an extension of a theorem of Thompson to graphs. It follows that any graph complement always contains a useful planar surface.

Thin position for a knot is a powerful tool developed by Gabai [1]. Given a knot in thin position, there is a useful planar surface in the knot complement and this planar surface plays an important role in Gabai's proof of property R [1] and Gordon-Luecke's solution of the knot complement problem [2]. Scharlemann and Thompson later generalized thin position to graphs in S^3 and used it in a new proof of Waldhausen's theorem that any Heegaard splitting of S^3 is standard; see [5] and [4, section 5].

In [6], Thompson proved that either thin position for a knot is also bridge position or the knot complement contains an essential meridional planar surface. Wu improved this result by showing that the thinnest level surface is an essential planar surface [7]. In this paper, we generalize Thompson's theorem to graphs in S^3 and show that either, after edge slides, thin position is also bridge position or there is an essential planar surface in the graph complement and all but at most one of the boundary components of this planar surface are meridians for the graph. A consequence of this theorem is that there is always a nice planar surface in any graph complement. The existence of such a nice planar surface is a key in the proof of a theorem in [3], which says that given a graph Γ in S^3 , if one glues back a handlebody $N(\Gamma)$ to $S^3 - N(\Gamma)$ via a sufficiently complicated map, then the resulting closed 3-manifold cannot be S^3 .

Definition 1. Let N be a compact orientable 3-manifold with boundary and let P be an orientable surface properly embedded in N . We say P is *essential* if either P is a compressing disk for N or P is incompressible and ∂ -incompressible. We say P is *strongly irreducible* if P is separating, P has compressing disks on both sides, and each compressing disk on one side meets each compressing disk on the other side. P is *∂ -strongly irreducible* if

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- (1) every compressing and ∂ -compressing disk on one side meets every compressing and ∂ -compressing disk on the other side, and
- (2) there is at least one compressing or ∂ -compressing disk on each side.

Let Γ be a graph in S^3 , $N(\Gamma)$ an open regular neighborhood of Γ , and P a planar surface properly embedded in $S^3 - N(\Gamma)$. We say P is *meridional* if every component of ∂P bounds a compressing disk for the handlebody $N(\Gamma)$ and we say P is an *almost meridional planar surface* if all but at most one component of ∂P bounds a compressing disk for $N(\Gamma)$. Note that a compressing disk for $S^3 - N(\Gamma)$ is an almost meridional planar surface by definition.

Since we are mainly interested in the topology of $N(\Gamma)$ and $S^3 - N(\Gamma)$, we may assume our graph Γ is trivalent.

Theorem 2. *Let $\Gamma \subset S^3$ be a trivalent graph that, via isotopy and edge slides, has been moved into thin position. Then either*

- (1) Γ is in bridge position or
- (2) $S^3 - N(\Gamma)$ contains an essential almost meridional planar surface.

The motivation of the paper is the following theorem, which says that a graph complement always contains a nice planar surface. Such a planar surface turns out to be very useful in [3].

Theorem 3. *Let Γ be any graph in S^3 . Then either*

- (1) $S^3 - N(\Gamma)$ contains a meridional planar surface that is strongly irreducible and ∂ -strongly irreducible or
- (2) $S^3 - N(\Gamma)$ contains an essential almost meridional planar surface or
- (3) $S^3 - N(\Gamma)$ contains a nonseparating incompressible almost meridional planar surface.

Note that in part (3) of Theorem 3, after some ∂ -compressions, a nonseparating incompressible planar surface becomes an essential planar surface. Furthermore, it follows from the proof of Theorem 3 that the nonseparating planar surface in part (3) of Theorem 3 can be chosen to be a punctured disk bounded by an unknotted loop in Γ after some edge slides on Γ .

Notation. Throughout this paper we denote the interior of X by $\text{int}(X)$, the closure of X (under the path-metric) by \overline{X} , and the number of components of X by $|X|$ for any space X .

The proof of the two theorems is a quick application of the techniques in [4, section 5]. Next we briefly recall some definitions and results in [4, section 5].

Suppose that the trivalent graph Γ lies in $S^3 - \{\text{two points}\}$ and let $h : S^3 - \{\text{two points}\} \rightarrow \mathbb{R}$ be the standard height function with each $h^{-1}(p)$ a 2-sphere for any point $p \in \mathbb{R}$. As in [4, Definitions 5.1 and 5.2], we may assume that each vertex of Γ is either a Y -vertex or a λ -vertex, as shown in Figure 1(a), and that $h|_{\text{edges}}$ is a Morse function. Moreover, the maxima of Γ consists of all local maxima of $h|_{\text{edges}}$ and all λ -vertices, and the minima of Γ consists of all local minima of $h|_{\text{edges}}$ and all Y -vertices. We may also assume that the maxima and minima are at different heights and say such a graph Γ is in normal form. Γ is in *bridge position* if there is a level sphere that lies above all minima and below all maxima. The width of Γ is defined in [4, Definition 5.3], and we say Γ is in *thin position* if the height function minimizes the width. For any Γ as above, its width decreases if

one pushes a maximum below a minimum, but the width remains the same if one pushes maxima past maxima and minima past minima. A level sphere P at a generic height is called a *thin sphere* if the adjacent critical height above P is a minimum (possibly a Y -vertex) and the adjacent height below P is a maximum (possibly a λ -vertex). Clearly Γ is in bridge position if and only if there is no thin sphere.

Next we define upper and lower triples according to [4, Definition 5.8]. Given Γ in normal form and P a level sphere at a generic height, let B_u and B_l be the two 3-balls above and below P respectively. Let E be a disk in $S^3 - N(\Gamma)$ transverse to P such that $\partial E = \alpha \cup \beta$, where α is an arc properly embedded in $P - N(\Gamma)$, $\partial\alpha = \partial\beta$ and β is a nontrivial arc in $\overline{\partial N(\Gamma)}$. Suppose the endpoints of α are at different boundary components of $P - N(\Gamma)$. By shrinking $N(\Gamma)$ back to Γ , we may view α as an arc in P connecting two punctures of $\Gamma \cap P$. Let v be one of the points of $\Gamma \cap P$ at an end of α and suppose none of the arc components of $\text{int}(E) \cap P$ is incident to v . Then (v, α, E) is called an *upper triple* (resp. a *lower triple*) if a small product neighborhood of α in E lies in B_u (resp. B_l).

As in [4, 5], if a thin sphere admits an upper or a lower triple (v, α, E) , then one can perform an edge slide or a broken edge slide (see [5, Section 1] for details) along the disk E . This move however does not reduce the width. Nonetheless, another combinatorial measure $W_P(\Gamma)$ is introduced in the proof of [4, Lemma 5.10] for a thin sphere P , and it is shown in [4] that the (broken) edge slide reduces $W_P(\Gamma)$. Moreover, the proof of [4, Lemma 5.10] shows that if there is always an upper or a lower triple for a thin sphere, then when the process stops, the width is reduced and the thin position for Γ is also bridge position. Although the setting in [4, 5] is that $N(\Gamma)$ is a handlebody in a Heegaard splitting of S^3 , the only assumption one needs in the proof of [4, Lemma 5.10] is that a thin sphere always admits an upper or a lower triple; see [4, Lemma 5.9]. We now summarize Scharlemann’s result as the following lemma.

Lemma 4 (Scharlemann [4]). *Let Γ be a trivalent graph in S^3 as above. Then either*

- (1) *after isotopy and edge slides, thin position for Γ is also bridge position or*
- (2) *there is a thin sphere for Γ that does not admit any upper or lower triple.*

Now we are in position to prove Theorem 2.

Proof of Theorem 2. Since a compressing disk for $S^3 - N(\Gamma)$ is an essential almost meridional planar surface, we may suppose that $S^3 - N(\Gamma)$ has incompressible boundary. By Lemma 4, we may assume that there is a thin sphere P that does not admit any upper or lower triple.

Let γ be a boundary component of the planar surface $P - N(\Gamma)$, and let D be an embedded disk in $S^3 - N(\Gamma)$ transverse to P with $\partial D = \alpha \cup \beta$, where α is an arc properly embedded in $P - N(\Gamma)$ with both endpoints in γ , $\beta \subset \overline{\partial N(\Gamma)}$ and $\partial\alpha = \partial\beta$. Then α cuts $P - N(\Gamma)$ into two planar subsurfaces P_1 and P_2 . Suppose α is an essential arc in $P - N(\Gamma)$, i.e. neither P_1 nor P_2 is a disk, and suppose β is nontrivial; i.e., β is not isotopic (in $\overline{\partial N(\Gamma)}$) relative to $\partial\beta$ to a subarc of γ . We say the disk D is a *good disk based at γ* if for some i ($i = 1$ or 2), $\text{int}(D) \cap P_i$ consists of simple closed curves (if not empty). We call the boundary component of P_i ($i = 1$ or 2) that contains α a *good loop*.

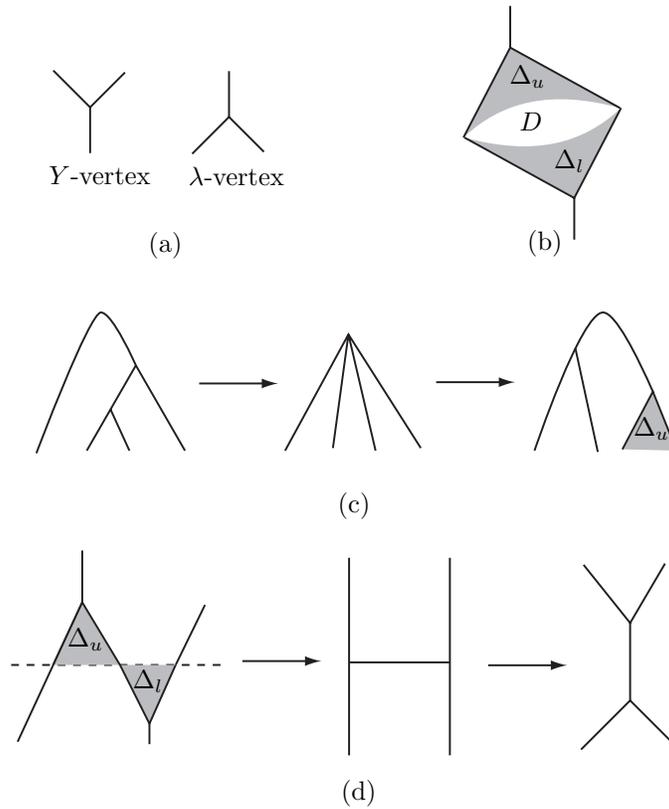


FIGURE 1

We call a simple closed curve in $P - N(\Gamma)$ a compressing simple closed curve if it bounds an embedded disk in $S^3 - N(\Gamma)$. Note that the interior of this disk may intersect P and hence may not be a compressing disk for $P - N(\Gamma)$.

Claim 1. If there are neither good loops nor compressing simple closed curves in P , then $P - N(\Gamma)$ is an essential meridional planar surface.

Proof of Claim 1. If P contains no compressing simple closed curve, then $P - N(\Gamma)$ is incompressible in the graph complement. So it remains to show that $P - N(\Gamma)$ is ∂ -incompressible. Suppose $Q = P - N(\Gamma)$ is ∂ -compressible and let E be a ∂ -compressing disk for Q with $\partial E = \alpha \cup \beta$, $\alpha \subset Q$, $\beta \subset \overline{\partial N(\Gamma)}$ and $\partial\alpha = \partial\beta$. If $\partial\alpha$ lies in the same component of ∂Q , then E is a good disk based at a boundary component of $P - N(\Gamma)$, which contradicts the hypothesis of Claim 1. If the endpoints of α lie in different components of ∂Q , then α and E yield an upper or a lower triple, a contradiction to our assumption at the beginning. Thus $Q = P - N(\Gamma)$ must be ∂ -incompressible and the claim holds. \square

Claim 2. If, among all good loops and compressing simple closed curves in P , an innermost one is a compressing simple closed curve, then the union of the associated compressing disk and a planar surface it cuts off from P is an essential meridional planar surface.

Proof of Claim 2. Let θ be a compressing simple closed curve in $P - N(\Gamma)$ that is innermost among all good loops and compressing simple closed curves. Let $\Delta \subset S^3 - N(\Gamma)$ be the embedded disk bounded by θ . Since θ is innermost, θ cuts off a planar surface Q' from $P - N(\Gamma)$ such that Q' does not contain any good loop or compressing simple closed curve (except for θ). Let $Q = Q' \cup \Delta$. Since Q' does not contain any compressing simple closed curve after isotopy, we may assume $\text{int}(\Delta) \cap Q' = \emptyset$ and Q is properly embedded in $S^3 - N(\Gamma)$. Now by the proof of Claim 1, Q must be an essential meridional planar surface. \square

Claim 3. If, among all good loops and compressing simple closed curves in P , an innermost one is a good loop, then the union of the associated good disk and the planar surface the loop bounds in P is an essential almost meridional planar surface.

Proof of Claim 3. Let θ be a good loop and P_1 the planar surface that θ cuts off from $P - N(\Gamma)$. Suppose P_1 does not contain any nontrivial good loop nor compressing simple closed curve.

Next we show that there are no ∂ -compressing disks for P_1 . Suppose there is a ∂ -compressing disk E' for P_1 and suppose $\partial E' = \alpha' \cup \beta'$, where $\alpha' = E' \cap P_1$ is a properly embedded nontrivial arc in P_1 with $\partial\alpha' = \partial\beta' \subset \overline{\partial N(\Gamma)} \cap \partial P_1$ and $\beta' \subset \overline{\partial N(\Gamma)}$. Similarly to the proof of Claim 1, if the endpoints of α' lie in the same boundary component of P_1 , then since $E' \cap P_1 = \alpha'$, E' is a good disk, which contradicts that the good loop θ is innermost. Suppose the endpoints of α' lie in different boundary components of P_1 . Then there is a component of $\overline{\partial N(\Gamma)} \cap P_1$, denoted by γ' , which totally lies in P_1 and contains an endpoint of α' . After shrinking $N(\Gamma)$ to Γ , γ' becomes a point v' in $\Gamma \cap P$ and α' becomes an arc with endpoints in different punctures of $\Gamma \cap P$. Since $E' \cap P_1 = \alpha'$, (v', α', E') is an upper or a lower triple, contradicting our assumption at the beginning of the proof. Thus we may assume there is no such ∂ -compressing disk for P_1 .

Let D be the good disk corresponding to the good loop θ . Since P_1 contains no compressing simple closed curve after isotopy, we may assume $D \cap P_1$ contains no closed curve. If $\text{int}(D) \cap P_1 \neq \emptyset$, then $D \cap P_1$ is a collection of arcs and the subdisk of D cut off by an outermost arc is a ∂ -compressing disk for P_1 , contradicting our conclusion above. Thus $\text{int}(D) \cap P_1 = \emptyset$ and $D \cup P_1$ is properly embedded in $S^3 - N(\Gamma)$. Since P_1 contains no compressing simple closed curve nor ∂ -compressing disk as above, $D \cup P_1$ is an essential almost meridional planar surface \square

These three claims imply that if Γ does not admit any upper or lower triple, then $S^3 - N(\Gamma)$ contains an essential almost meridional planar surface. Hence by Lemma 4, Theorem 2 holds. \square

If thin position for a knot is also bridge position, then it is easy to see that the punctured bridge sphere is strongly irreducible and ∂ -strongly irreducible in the knot complement. Theorem 3 can be viewed as an extension of this observation to graphs.

Proof of Theorem 3. Since the theorem is about $S^3 - N(\Gamma)$, we may assume Γ is trivalent. As in the proof of Theorem 2, we may assume $\overline{\partial N(\Gamma)}$ is incompressible in $S^3 - N(\Gamma)$. By Theorem 2, either $S^3 - N(\Gamma)$ contains a desired essential almost meridional planar surface or thin position for Γ is also bridge position. Suppose thin position for Γ is also bridge position and consider the bridge 2-sphere S for Γ . Then $P = S - N(\Gamma)$ is a planar surface in $S^3 - N(\Gamma)$, and our goal is to show

that either P is strongly irreducible and ∂ -strongly irreducible or we can construct a nonseparating incompressible almost meridional planar surface in $S^3 - N(\Gamma)$.

We say a properly embedded tree in a 3-ball is unknotted if it can be isotoped (relative to the boundary) into a tree in the boundary 2-sphere. The bridge 2-sphere S divides S^3 into an upper 3-ball B_u and a lower 3-ball B_l . If $\Gamma \cap B_u$ is connected, since S is a bridge sphere, by inductively pushing down the maxima, we can isotope $\Gamma \cap B_u$ into ∂B_u , and hence $\Gamma \cap B_u$ is an unknotted tree in B_u . If $\Gamma \cap B_u$ is not connected, we can push down the maxima in one component of $\Gamma \cap B_u$, passing all other maxima. Then it is easy to see that we can find a disk properly embedded in B_u that cuts off a 3-ball E from B_u such that E only contains this component of $\Gamma \cap B_u$. This implies that $S - \Gamma$ is incompressible in B_u if and only if $\Gamma \cap B_u$ is connected. Moreover, if $S - \Gamma$ is compressible in B_u , we can maximally compress $S - \Gamma$ in B_u and obtain a collection of mutually disjoint 3-balls, E_1, \dots, E_n , such that each $\Gamma_i = \Gamma \cap E_i$ is a connected and unknotted tree in E_i . Note that if one collapses all the edges of Γ_i that are not incident to ∂E_i into a single vertex, then Γ_i becomes a cone over the points $\Gamma_i \cap \partial E_i$; see [4, Figure 11]. Moreover, $\partial E_i - N(\Gamma)$ is parallel to $E_i \cap \partial \overline{N(\Gamma)}$ in E_i .

We have the following three cases.

Case 1. P is compressible on one side but incompressible on the other side.

Suppose P is compressible in B_u but incompressible in B_l . Then $\Gamma \cap B_l$ is a connected unknotted tree. Now we consider B_u . By the discussion above, we can maximally compress $S - \Gamma$ in B_u and obtain a collection of mutually disjoint 3-balls E_1, \dots, E_n , and each $\Gamma_i = \Gamma \cap E_i$ is a connected and unknotted tree. So we can find an arc α properly embedded in $\partial E_i - N(\Gamma)$ such that the two endpoints of α lie in different boundary components of $\partial E_i - N(\Gamma)$. Since $\partial E_i - N(\Gamma)$ is parallel to $E_i \cap \partial \overline{N(\Gamma)}$, α is parallel to an arc β_u in $E_i \cap \partial \overline{N(\Gamma)}$. By the construction of E_i , we may view α as an arc in P , and there is a disk D_u properly embedded in $B_u - N(\Gamma)$ with $\partial D_u = \alpha \cup \beta_u$, $\partial \alpha = \partial \beta_u$, and $\beta_u \subset B_u \cap \partial \overline{N(\Gamma)}$. Moreover, since $\Gamma \cap B_l$ is a connected unknotted tree, $\partial B_l - N(\Gamma)$ is parallel to $B_l \cap \partial \overline{N(\Gamma)}$ and α is parallel to an arc β_l in $B_l \cap \partial \overline{N(\Gamma)}$. Thus $\beta_u \cup \beta_l$ is a nonseparating simple closed curve in $\partial \overline{N(\Gamma)}$ bounding a disk in $S^3 - N(\Gamma)$. This contradicts our assumption at the beginning that $S^3 - N(\Gamma)$ has incompressible boundary.

Case 2. P is incompressible on both sides.

This case is similar to Case 1. In this case, $\Gamma \cap B_u$ and $\Gamma \cap B_l$ are unknotted trees in B_u and B_l respectively. Let α be an arc properly embedded in P with endpoints in different components of ∂P . Then as in the proof of Case 1, α is parallel to an arc β_u in $B_u \cap \partial \overline{N(\Gamma)}$ and an arc β_l in $B_l \cap \partial \overline{N(\Gamma)}$, and $\beta_u \cup \beta_l$ is a nonseparating simple closed curve in $\partial \overline{N(\Gamma)}$ bounding a disk in $S^3 - N(\Gamma)$. This again contradicts our assumption that $S^3 - N(\Gamma)$ has incompressible boundary.

Case 3. P is compressible on both sides.

We first show that P is strongly irreducible. If there are compressing disks D_u and D_l for $S - \Gamma$ in B_u and B_l respectively with $D_u \cap D_l = \emptyset$, then D_u and D_l together with subdisks of S bound two disjoint 3-balls E_u and E_l in B_u and B_l respectively. Then we can push all the maxima in the E_u into B_l and push all the minima in E_l into B_u . Since maxima are pushed below minima, this operation

reduces the width and contradicts that Γ is in thin position. Hence $S - \Gamma$ and P must be strongly irreducible.

Next we show that P is also ∂ -strongly irreducible. Suppose P is not ∂ -strongly irreducible. Then there is a ∂ -compressing disk on one side of P disjoint from a compressing or ∂ -compressing disk on the other side. Suppose Δ_u is a ∂ -compressing disk for P in B_u and Δ_l is either a compressing or a ∂ -compressing disk for P in B_l with $\Delta_u \cap \Delta_l = \emptyset$. Suppose $\partial\Delta_u = \alpha_u \cup \beta_u$ with $\alpha_u \subset P$, $\beta_u \subset B_u \cap \partial N(\Gamma)$ and $\partial\alpha_u = \partial\beta_u$. Next we show that one can choose Δ_u so that the two endpoints of α_u lie in different components of ∂P .

Suppose $\partial\alpha_u$ lies in the same component γ of ∂P . Then α_u must be separating in P , and suppose α_u cuts P into two planar subsurfaces P_1 and P_2 . Since $\Delta_l \cap P$ is either a simple closed curve or a simple arc and $\Delta_u \cap \Delta_l = \emptyset$, we may suppose $\Delta_l \cap P \subset P_2$. Let Q be the component of $B_u \cap \partial N(\Gamma)$ that contains γ . Then β_u is an arc properly embedded in the planar surface Q with $\partial\beta_u \subset \gamma$. Hence β_u is separating in Q . Since $\Gamma \cap B_u$ is unknotted, by the discussion on Γ_i and E_i above, at least one component of ∂Q , say γ' , lies in $\text{int}(P_1)$. We can find an arc β'_u properly embedded in Q with one endpoint in the arc $\gamma \cap \partial P_1$ and the other endpoint in γ' and $\beta'_u \cap \beta_u = \emptyset$. Similarly, by the discussion on Γ_i and E_i above, β'_u is parallel to an arc α'_u properly embedded in P_1 with $\partial\alpha'_u = \partial\beta'_u$. Thus $\alpha'_u \cup \beta'_u$ bounds another ∂ -compressing disk Δ'_u for P in B_u . The two endpoints of α'_u lie in different components of ∂P . As $\Delta_l \cap P \subset P_2$ and $\alpha'_u \subset P_1$, $\Delta'_u \cap \Delta_l = \emptyset$. Therefore after replacing Δ_u by Δ'_u if necessary, we may assume that the two endpoints of α_u lie in different components of ∂P . Similarly, if Δ_l is a ∂ -compressing disk, we may also assume that the endpoints of α_l lie in different components of ∂P .

Now we shrink $N(\Gamma)$ to Γ and, to simplify notation, we still use Δ_u and Δ_l to denote the disk after the operation. So α_u is an arc in S whose endpoints are different punctures of $S \cap \Gamma$. Since Γ is in bridge position, after some edge slides as shown in Figure 1(c) and [4, Figure 11], we may assume that Δ_u is a triangle and that β_u contains exactly one vertex of Γ , which is a λ -vertex. Similarly, if Δ_l is also a ∂ -compressing disk, we may also assume that Δ_l is a triangle whose boundary contains one Y -vertex.

If Δ_l is a compressing disk, then Δ_l and a subdisk of S bound a 3-ball $E_l \subset B_l$. Since $\Delta_u \cap \Delta_l = \emptyset$, we may choose E_l so that $\Delta_u \cap E_l = \emptyset$. We can push Δ_u and the corresponding λ -vertex below S and push all the minima in E_l above S . Since a maximum (the λ -vertex) is pushed below a minimum, this operation reduces the width of Γ and contradicts that Γ is in thin position. So we may suppose Δ_l is also a ∂ -compressing disk and assume Δ_l is a triangle as above. Let $\alpha_l = \Delta_l \cap S$. After collapsing $N(\Gamma)$ back to Γ , α_u and α_l may have common endpoints, but $\text{int}(\alpha_u) \cap \text{int}(\alpha_l) = \emptyset$.

If $\partial\alpha_u \cap \partial\alpha_l = \emptyset$, we can push Δ_u and the corresponding λ -vertex below S while pushing Δ_l and the corresponding Y -vertex above S . This reduces the width of Γ , a contradiction. If α_u and α_l share exactly one endpoint, then the operation as shown in Figure 1(d) and [4, Figure 12(a,b)] reduces the width of Γ .

So it remains to consider the case where $\partial\alpha_u = \partial\alpha_l$; see Figure 1(b) and [4, Figure 12(c)]. Since $S^3 - N(\Gamma)$ has incompressible boundary, α_u and α_l are not isotopic in $S - \Gamma$. So $\alpha_u \cup \alpha_l$ bounds a disk D in S and $\text{int}(D) \cap \Gamma \neq \emptyset$. We consider the disk $\Delta = \Delta_u \cup D \cup \Delta_l$. Let Z be the set of punctures in $\text{int}(D) \cap \Gamma$. The punctured disk $\Delta - Z$ corresponds to an almost meridional planar surface Q

properly embedded in $S^3 - N(\Gamma)$. By our construction, Q is nonseparating. Next we show that Q is incompressible. Suppose Q is compressible. Then there must be a curve δ in $\Delta - Z$ that bounds a compressing disk for $\Delta - Z$. After isotopy, we may assume $\delta \subset \text{int}(D) - \Gamma$. Letting D_δ be the compressing disk bounded by δ , we may assume $|D_\delta \cap S|$ is minimal among all compressing disks bounded by δ . Let δ' be a component of $D_\delta \cap S$ that is innermost in D_δ . Let D'_δ be the subdisk of D_δ bounded by δ' . Then D'_δ is a compressing disk for $S - \Gamma$ in either B_u or B_l . Moreover D'_δ is disjoint from both Δ_u and Δ_l . So similarly to the case that Δ_l is a compressing disk above, we can push D'_δ and either Δ_u or Δ_l to reduce the width of Γ . Thus Q must be incompressible in $S^3 - N(\Gamma)$ and part (3) of Theorem 3 holds. \square

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