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ON A DESINGULARIZATION OF THE MODULI SPACE OF NONCOMMUTATIVE TORI

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Abstract. It is shown that the moduli space of the noncommutative tori $\mathbb{A}_\theta$ admits a natural desingularization by the group *Ext* $(\mathbb{A}_\theta, \mathbb{A}_\theta)$. Namely, we prove that the moduli space of pairs $(\mathbb{A}_\theta, Ext\ (\mathbb{A}_\theta, \mathbb{A}_\theta))$ is homeomorphic to a punctured two-dimensional sphere. The proof is based on a correspondence (a covariant functor) between the complex and noncommutative tori.

1. Introduction

A. Let $0 < \theta < 1$ be an irrational number, whose regular continued fraction has the form $\theta = [a_0, a_1, a_2, \dots]$. Consider an AF-algebra $\mathbb{A}_\theta$ given by the Bratteli diagram in Figure 1. The a_i indicate the number of edges in the upper row of

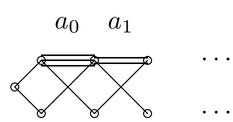

Figure 1. The Bratteli diagram of the AF-algebra $\mathbb{A}_\theta$.

the diagram. With a moderate abuse of the terminology, we shall call $\mathbb{A}_\theta$ a *noncommutative torus*. (Note that a standard definition of the noncommutative torus – a universal C^*-algebra generated by the unitaries u, v satisfying the commutation relation $vu = e^{2\pi i}uv$ – is not an AF-algebra. However, the two objects are isomorphic at the level of their dimension groups [8], [9].)

B. Recall that the noncommutative tori $\mathbb{A}_\theta, \mathbb{A}_{\theta'}$ are said to be *stably isomorphic*, whenever $\mathbb{A}_\theta \otimes \mathcal{K} \cong \mathbb{A}_{\theta'} \otimes \mathcal{K}$, where $\mathcal{K}$ is the AF-algebra of the compact operators. It is well known that the AF-algebras $\mathbb{A}_\theta, \mathbb{A}_{\theta'}$ are stably isomorphic if and only if $\theta' \equiv \theta \ mod\ SL(2, \mathbb{Z})$, i.e. $\theta' = (a\theta + b) \ / \ (c\theta + d)$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ [3]. It is easy to see that the stable isomorphism is an equivalence relation, which splits the set $\{\mathbb{A}_\theta \mid 0 < \theta < 1, \ \theta \in \mathbb{R} - \mathbb{Q}\}$ into the disjoint equivalence classes. By $\mathcal{M}$ we shall understand a collection of such classes, or the

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"moduli space" of the noncommutative tori. An examination of $\mathcal{M}$ as a topological space (with the topology induced by $\mathbb{R}$) shows that the points of $\mathcal{M}$ have no disjoint neighborhoods, since each orbit $\{\theta' \in \mathbb{R} \mid \theta' \equiv \theta \ mod \ SL(2,\mathbb{Z})\}$ is dense in the real line $\mathbb{R}$. A question arises as to how to "desingularize" the (non-Hausdorff) moduli space $\mathcal{M}$.

C. Let A, B be a pair of the C^*-algebras. Recall that an extension of A by B is a C^*-algebra E filling the short exact sequence $0 \to B \to E \to A \to 0$ of the C^*-algebras. If A is a separable nuclear C^*-algebra, the $Ext\ (A,B)$ is an additive abelian group, whose group operation is a sum of the two extensions. The $Ext\ (A,B)$ is a homotopy invariant in both variables. The extensions E_1, E_2 are said to be *stably equivalent* if there exists an isomorphism $\psi : E_1 \otimes \mathcal{K} \cong E_2 \otimes \mathcal{K}$, such that $\psi \circ \alpha_1(B \otimes \mathcal{K}) = \alpha_2(B \otimes \mathcal{K})$, where $\alpha_i : B \to E_i, i = 1, 2$ [1]. We shall further restrict to the case $A = B = \mathbb{A}_\theta$ and study the stable equivalence classes of the group $Ext\ (\mathbb{A}_\theta, \mathbb{A}_\theta)$. Using the classification results of D. Handelman [5], it will develop that the group $Ext\ (\mathbb{A}_\theta, \mathbb{A}_\theta) \cong Hom\ (K_0(\mathbb{A}_\theta), \mathbb{R}) \cong \mathbb{R}$. Moreover, the $Ext\ (\mathbb{A}_\theta, \mathbb{A}_\theta)/$ stable equivalence $\cong \mathbb{R}/\mathbb{Z}$.

D. An objective of the note is to show that the moduli of the pairs $(\mathbb{A}_\theta, Ext\ (\mathbb{A}_\theta, \mathbb{A}_\theta))$ under the stable equivalence is no longer a non-Hausdorff topological space, but a two-dimensional orbifold (a punctured sphere). To prove this result we shall use the Teichmüller space of a torus (a space of the complex structures on the torus) [6]. Namely, Hubbard and Masur established a homeomorphism between the Teichmüller space T_g of a surface of genus $g \geq 1$ and the space of quadratic differentials on it. We shall use the homeomorphism to extend the action of the modular group $SL(2,\mathbb{Z})$ from the upper half-plane $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\} \cong T_1$ to the space $(\mathbb{A}_\theta, Ext\ (\mathbb{A}_\theta, \mathbb{A}_\theta))$. Denote by $\widetilde{\mathcal{M}}$ the set of pairs $(\mathbb{A}_\theta, Ext\ (\mathbb{A}_\theta, \mathbb{A}_\theta))$ modulo the stable equivalence. One obtains the following (natural) desingularization of the moduli space of the noncommutative tori.

Theorem 1. *$\widetilde{\mathcal{M}}$ is a punctured two-dimensional sphere.*

2. Proof

We shall split the proof into two lemmas. The background material is mostly standard, and we shall recall in passing some important notation and ideas.

Lemma 1. *$\widetilde{\mathcal{M}}$ is a two-dimensional orbifold.*

Proof of Lemma 1. We shall use a standard dictionary existing between the AF-algebras and their dimension groups [2]. Instead of dealing with the AF-algebra $\mathbb{A}_\theta$, we shall work with its dimension group $G_\theta = (G, G^+)$, where $G \cong \mathbb{Z}^2$ is the lattice and $G^+ = \{(x,y) \in \mathbb{Z}^2 \mid x + \theta y \geq 0\}$ is a positive cone of the lattice. The G_θ is the additive abelian group with an order, which defines the AF-algebra $\mathbb{A}_\theta$ up to a stable isomorphism.

Under the dictionary, the extension problem for the AF-algebra $\mathbb{A}_\theta$ translates as an extension problem for the dimension groups $G_\theta \to E \to G_\theta$ (we omit the zeros in the exact sequence). An important result of Handelman establishes the intrinsic classification of the extensions of the simple dimension group by a simple dimension group; see Theorem III.5 of [5]. Let us recall the classification as it is exposed in [4, Theorem 17.5 and Corollary 17.7]. We shall adopt the same notation as in the cited work.

Let H be a dense subgroup of the real line $\mathbb{R}$ and K a nonzero dimension group. Let E be the abelian group $H \oplus K$, and let $\tau : H \to E$ and $\pi : E \to K$ be a natural injection and projection maps. Assume that $f : K \to \mathbb{R}$ is a homomorphism of the dimension groups.[1] Then: (i) E is a dimension group with the positive cone

$$E_f^+ = \{(0,0)\} \,\cup\, \{(x,y) \in E \mid y \geq 0 \text{ and } x + f(y) > 0\}, \tag{1}$$

which gives an extension $H \xrightarrow{\tau} (E, E_f^+) \xrightarrow{\pi} K$ of H by K; (ii) if $f, f' : K \to \mathbb{R}$ are the group homomorphisms, then the extensions $E_f, E_{f'}$ are equivalent if and only if $(f - f')(K) \subseteq H$.

We have to specialize the above theorem to the case $H = K = G_\theta$. It is immediate from (i) that $E \cong \mathbb{Z}^4$. Note that the group homomorphisms $f : G_\theta \to \mathbb{R}$ are bijective with the reals $\mathbb{R}$. Indeed, we have to find all the linear maps $f : \mathbb{R}^2 \to \mathbb{R}$, such that $Ker\ f = x + \theta y$. (The last equation follows from the condition $f(G^+) > 0$.) Such maps have the form $f_t(p) = (p,t),\ p,t \in \mathbb{R}^2$, where (p,t) is the dot product of the two vectors. Let $t = (t_1, t_2)$. Then $f_t(-\theta y, y) = t_1(-\theta y) + t_2 y = y(t_2 - t_1\theta) = 0$ for all $y \in \mathbb{R}$. Therefore, $t_2 = \theta t_1$ and $f_t(x,y) = t_1 x + \theta t_1 y = t_1(x + \theta y),\ t_1 \in \mathbb{R}$. Thus, all linear maps $f : \mathbb{R}^2 \to \mathbb{R}$ with $Ker\ f = x + \theta y$ are bijective with the reals $t_1 \in \mathbb{R}$. In other words, $Ext\ (\mathbb{A}_\theta, \mathbb{A}_\theta)$ and $\mathbb{R}$ are isomorphic as additive abelian groups.

Let us find when the two extensions E, E' are equivalent. Since $H = G_\theta$ is a subgroup of $\mathbb{R}$, one can write $H = \mathbb{Z} + \theta\mathbb{Z}$. Let t, t' be the real numbers corresponding to the homomorphisms f, f'. Then $f(G_\theta) = t(\mathbb{Z} + \theta\mathbb{Z})$ and $f'(G_\theta) = t'(\mathbb{Z} + \theta\mathbb{Z})$. The condition $(f - f')(K) \subseteq H$ of the item (ii) will take the form $(t - t')(\mathbb{Z} + \theta\mathbb{Z}) \subseteq \mathbb{Z} + \theta\mathbb{Z}$. One gets immediately that $t = t' + n, n \in \mathbb{Z}$ as a necessary and sufficient condition for the last inclusion. In other words, the extensions E, E' are equivalent if and only if $t' = t\ mod\ \mathbb{Z}$. Thus, the equivalence classes of $Ext\ (\mathbb{A}_\theta, \mathbb{A}_\theta)$ are bijective with the factor space $\mathbb{R}/\mathbb{Z}$ (a unit interval).

To finish the proof of Lemma 1, let us extend the domain of definition of θ from the interval $(0,1)$ to the real line $\mathbb{R}$ by allowing a_0 to take on any integer value. In this way, one can identify the pairs $(\mathbb{A}_\theta, Ext\ (\mathbb{A}_\theta, \mathbb{A}_\theta))$ with the points of $\mathbb{R}^2$ equipped with the usual Euclidean topology. We have seen that the points $(\theta, t) \sim (\theta', t') \in \mathbb{R}^2$ are equivalent if and only if $\theta' \equiv \theta\ mod\ SL(2,\mathbb{Z})$ and $t' \equiv t\ mod\ \mathbb{Z}$. Note that the action of the modular group on the second coordinate is always free. Therefore, the points x, y of the space $\widetilde{\mathcal{M}} \cong \mathbb{R}^2/\sim$ admit the disjoint neighborhoods defined, e.g., by the open balls of radius $1/3$ centered in x and y, respectively. The balls are locally homeomorphic to the Euclidean plane, and therefore $\widetilde{\mathcal{M}}$ is a two-dimensional orbifold. $\square$

Lemma 1 gives a (partial) desingularization of the space $\mathcal{M}$. Indeed, we have seen that the group $SL(2,\mathbb{Z}) \times \mathbb{Z}$ acts in the plane $(\mathbb{A}_\theta, Ext_t\ (\mathbb{A}_\theta, \mathbb{A}_\theta))$ by the formula $(\theta, t) \mapsto (\frac{a\theta+b}{c\theta+d}, t + n)$, where $ad - bc = 1$ and $a, b, c, d, n \in \mathbb{Z}$. However, the last formula does not specify the action on the parameter plane (θ, t) of the modular group $SL(2,\mathbb{Z})$ alone, since the function $n = n(a,b,c,d)$ is unknown. To find how the integer n depends on the integers a, b, c, d, we would need a special construction which involves a correspondence (a covariant functor) between the complex and noncommutative tori. Such a construction will be given in the next paragraph and is encapsulated in the following lemma.

[1] That is, f preserves the positive cone of K and $\mathbb{R}$: $f(K^+) > 0$.

Lemma 2. *There exists a homeomorphism $h : \widetilde{\mathcal{M}} \to \mathbb{H} \ / \ SL(2,\mathbb{Z})$, where $\mathbb{H} = \{x+iy \in \mathbb{C} \mid y > 0\}$ is the Lobachevsky plane endowed with a hyperbolic metric.*

Proof of Lemma 2. Let X be a topological surface of genus $g \geq 0$. The Teichmüller space T_g of X consists of the equivalence classes of the complex structures on X. The space T_g is an open ball of the (real) dimension $6g-6$ if $g \geq 2$ and $2g$ if $g = 0, 1$. By $Mod\ X$ we designate a group of the orientation-preserving diffeomorphisms of X modulo the trivial ones. The points $S, S' \in T_g$ are equivalent if there exists a conformal map $f \in Mod\ X$ such that $S' = f(S)$. The moduli of conformal equivalence is denoted by $\mathcal{M}_g = T_g / \ Mod\ X$. The space $\mathcal{M}_g$ is a (classical) moduli space, whose definition dates back to Riemann.

Let $S \in T_g$ be a Riemann surface thought of as a point in the Teichmüller space, and let $H^0(S, \Omega^{\otimes 2})$ be the space of the holomorphic quadratic forms on S. The fundamental theorem of Hubbard and Masur says that there exists a homeomorphism $h_S : H^0(S, \Omega^{\otimes 2}) \to T_g$ [6, p. 224]. The space $H^0(S, \Omega^{\otimes 2})$ is a real vector space of dimension $6g-6$, where $g \geq 2$. It has been shown in the above cited work that $H^0(S, \Omega^{\otimes 2}) \cong Hom\ (H_1(\tilde{X}, \tilde{\Gamma})^-; \mathbb{R})$ defined by the formula

$$\omega \longmapsto \left(\gamma \mapsto Im \int_\gamma \omega \right), \tag{2}$$

where $H_1(\tilde{X}, \tilde{\Gamma})^-$ is the odd part in the homology of a double cover $\tilde{X}$ of X ramified at the zeroes $\tilde{\Gamma}$ of the odd multiplicity of the quadratic form [6, p. 232]. (The symbols and formulas will simplify as we come to the complex torus – our principal case.) It has been proved that $H_1(\tilde{X}, \tilde{\Gamma})^- \cong \mathbb{Z}^{6g-6}$.

Let $X = T^2$, i.e. $g = 1$. In this case each quadratic differential form is the square of a holomorphic abelian form (a one-form), i.e. $H^0(S, \Omega^{\otimes 2}) = H^0(S, \Omega)$. Therefore $\tilde{X} = X = T^2$, $\tilde{\Gamma} = \emptyset$ and $H_1(\tilde{X}, \tilde{\Gamma})^- = H_1(T^2) \cong \mathbb{Z}^2$. In other words, one gets a homeomorphism $h_S : Hom\ (\mathbb{Z}^2, \mathbb{R}) \to T_1$. As we have seen earlier, $Hom\ (\mathbb{Z}^2, \mathbb{R}) = \{t_1\mathbb{Z} + t_2\mathbb{Z} \mid t_1, t_2 \in \mathbb{R}\} = \{t(\mathbb{Z} + \theta\mathbb{Z}) \mid \theta, t \in \mathbb{R}\}$, where $t = t_1, \theta = t_2/t_1$. On the other hand, the Teichmüller space $T_1 \cong \mathbb{H}$, where $\mathbb{H} = \{\tau = x+iy \in \mathbb{C} \mid y > 0\}$ is a (Lobachevsky) upper half-plane and τ is a modulus of the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ [10, pp. 6-14]. Thus, we have a homeomorphism $h_S : (\mathbb{A}_\theta, Ext_t\ (\mathbb{A}_\theta, \mathbb{A}_\theta)) \to \mathbb{H}$.

Let us show that h_S is equivariant in the first coordinate with respect to the action of $Mod\ (T^2) \cong SL(2,\mathbb{Z})$; i.e., $\tau' \equiv \tau\ mod\ SL(2,\mathbb{Z})$ if and only if $\theta' \equiv \theta\ mod\ SL(2,\mathbb{Z})$. Indeed, since $Hom\ (H_1(T^2); \mathbb{R}) \cong \mathbb{H}$ the modular group $SL(2,\mathbb{Z})$ acts on the right-hand side by the formula $\tau \mapsto (a\tau + b)/(c\tau + d)$ and on the left-hand side by a linear transformation $p_1 \mapsto ap_1 + bp_2,\ p_2 \mapsto cp_1 + dp_2$, where $p = (p_1, p_2) \in H_1(T^2)$ and $ad - bc = 1$. The $f_p(t) = p_1t_1 + p_2t_2$ will become $f_p(t') = t_1(ap_1 + bp_2) + t_2(cp_1 + dp_2) = p_1t_1' + p_2t_2'$, where $t_1' = at_1 + ct_2$ and $t_2' = bt_1 + dt_2$. Therefore $\theta = t_2/\ t_1$ goes to $\theta' = t_2'/t_1' = (b + d\theta)/(a + c\theta)$ and $\theta' \equiv \theta\ mod\ SL(2,\mathbb{Z})$. (The 'only if' part of the statement is obtained likewise by an inversion of the formulas.)

Recall that the Lobachevsky plane $\mathbb{H} = \{x+iy \in \mathbb{C} \mid y > 0\}$ carries a hyperbolic metric $ds = |dz|/y$ such that $SL(2,\mathbb{Z})$ acts on it by the isometries (linear-fractional transformations). The tesselation of $\mathbb{H}$ by the fundamental regions is shown in Figure 2. Let $\tau' = \frac{a\tau+b}{c\tau+d} = T(\tau),\ \tau \in \mathbb{H}$. The number $n = n(a,b,c,d) \in \mathbb{Z}$ we shall call a *height* of the transformation $T \in SL(2,\mathbb{Z})$ if n is equal to the number of intersections of the vertical segment $Im\ (\tau' - \tau)$ issued from τ with the lines of tiling $\mathbb{H}/SL(2,\mathbb{Z})$. (In other words, n shows how many fundamental regions apart

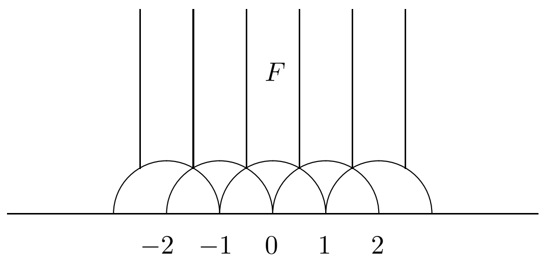

FIGURE 2. $\mathbb{H}$ and the fundamental region F.

from τ and τ' are in the vertical direction.) We leave it to the reader to verify that the height n does not depend on a particular choice of τ in the fundamental region or the fundamental region itself being a function of the transformation T only.

Let us now define an action of the modular group on $(\mathbb{A}_\theta, Ext_t\ (\mathbb{A}_\theta, \mathbb{A}_\theta))$. The action is given by the formula $(\theta, t) \mapsto (\frac{a\theta+b}{c\theta+d}, t+n)$, where $n = n(a,b,c,d)$ is the height of the transformation $T = T(a,b,c,d)$. Under the homeomorphism h_S, the tesselation of $\mathbb{H}$ maps into a tesselation of the plane (θ, t). As we have shown earlier, the action of the modular group $SL(2,\mathbb{Z})$ on $\mathbb{H}$ is equivariant with the action on (θ, t). On the other hand, it is known that $\mathbb{H}/SL(2,\mathbb{Z})$ is a punctured two-dimensional sphere [10, p. 15]. Lemma 2 and Theorem 1 follow. □

3. Remarks

Let $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be a complex torus and $h_S(E_\tau) = (\mathbb{A}_\theta, Ext_t\ (\mathbb{A}_\theta, \mathbb{A}_\theta))$ its image under the homeomorphism h_S. Let us call the respective reals $\theta = \theta(\tau)$ and $t = t(\tau)$ a *projective curvature* and an *area* of the complex torus E_τ. The projective curvature of the complex tori with a nontrivial group of endomorphisms (complex multiplication) is a quadratic irrationality. In the latter case, the noncommutative torus is said to have a *real multiplication*. The noncommutative tori with real multiplication can be used to construct the abelian extensions of the real quadratic number fields, as was suggested by Yu. Manin [7]. It seems challenging at this point to write a formula for the projective curvature and the area as a function of the complex modulus τ. It is likely that the functions will be of the class C^0.

Problem 1. Find a formula (if any) for the functions $\theta(\tau)$ and $t(\tau)$.

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