

## A COMPACTIFICATION OF THE MODULI SPACE OF POLYNOMIALS

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ABSTRACT. In this paper, we introduce a compactification of the moduli space of polynomial maps with a fixed degree  $n$  ( $\geq 2$ ) such that the map from it to  $\mathbb{P}^{n-1}(\mathbb{C})$  defined by using the elementary symmetric functions of multipliers at fixed points is a continuous surjection.

### 1. INTRODUCTION AND MAIN THEOREMS

Let  $n \geq 2$  and  $\text{Poly}_n$  be the set of all polynomial maps of  $\mathbb{C}$  to itself with degree  $n$ . We say that two maps  $p_1$  and  $p_2$  in  $\text{Poly}_n$  are *affine conjugate* if there exists a biholomorphic automorphism  $g$  of  $\mathbb{C}$  such that  $g \circ p_1 \circ g^{-1} = p_2$ . The *moduli space* of polynomial maps with degree  $n$  is the set of all affine conjugacy classes of maps in  $\text{Poly}_n$ , and is denoted by  $\text{MPoly}_n$ . Here, recall that a natural complex orbifold structure can be introduced on  $\text{MPoly}_n$ .

*Remark 1.1.* These spaces were investigated by Branner and Hubbard [2], [3], and Milnor [12] in the case of degree 3, and then by the first author in general cases (cf. [9]).

Several kinds of compactification of  $\text{MPoly}_n$  have been considered. One is given as the closure in the GIT compactification of the moduli space of rational maps, which is defined in [14]. (See also [4].) DeMarco and McMullen introduced one by using tree representations ([5]), and also DeMarco discussed another one in [4]. Here, we show that the extended moduli space introduced by the first author in [8] (also cf. [9], Introduction) gives a natural compactification of  $\text{MPoly}_n$  with a suitable topology.

**Definition 1.2** (Extended moduli spaces). The *extended moduli space*  $\widehat{M}_n$  of degree  $n$  is the set of all collections  $X = \{\langle p_j \rangle\}_{j=1}^J$  of the affine conjugacy classes  $\langle p_j \rangle$  of polynomial maps  $p_j$  with degree  $n_j^X$  ( $\geq 2$ ) such that  $N_X = \sum_{j=1}^J n_j^X \leq n$ . Here we include the empty collection in the extended moduli space.

By identifying the affine conjugacy class  $\langle p \rangle$  of  $p \in \text{Poly}_n$  with the collection  $\{\langle p \rangle\}$  consisting of this class only, we consider  $\text{MPoly}_n$  as a subset of  $\widehat{M}_n$ .

Here, by definition, an element  $X$  of  $\widehat{M}_n$  is an unordered set of affine conjugacy classes. But for the sake of convenience, we fix an order temporarily, and represent

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$X$  as  $\{p_j\}_{j=1}^J$ . Then, we can associate  $X$  with a collection of sets of numbers as follows. For  $X = \{p_j\}_{j=1}^J$ , assume that the fixed points of  $p_j$ , counted including multiplicities, are represented by numbers

$$n_1^X + \cdots + n_{j-1}^X + 1, \dots, n_1^X + \cdots + n_j^X$$

for every  $j$ . We call the partition of  $\{1, \dots, n\}$  into

$$\begin{aligned} E_0^X &= \{N_X + 1, \dots, n\}, \\ E_1^X &= \{1, \dots, n_1^X\}, \\ &\dots, \\ E_J^X &= \{n_1^X + \cdots + n_{J-1}^X + 1, \dots, N_X\} \end{aligned}$$

the *partition associated to*  $X = \{p_j\}_{j=1}^J$ . Here, if  $X$  is the empty collection, then we set  $J = 0$  and  $E_0^X = \{1, \dots, n\}$ .

**Definition 1.3** (Natural topology on the extended moduli space). Fix a point  $X = \{p_j\}_{j=1}^J$  of  $\widehat{M}_n$ , and let  $n_j^X$ ,  $N_X$ , and  $\{E_0^X, \dots, E_J^X\}$  be as above.

For every  $\epsilon > 0$ , let  $U_\epsilon(X)$  be the set of all points in  $\widehat{M}_n$ , i.e. of all collections  $Y = \{q_h\}_{h=1}^H$  of the affine conjugacy classes of polynomial maps  $q_h$  with degree  $n_h^Y$  such that  $N_Y = \sum_{h=1}^H n_h^Y \leq n$ , which satisfy the following conditions:

- (1) (*Marking*) There is a family of mutually disjoint subsets  $P_1, \dots, P_H$  of  $\{1, \dots, J\}$  such that  $\bigcup_{h=1}^H P_h = \{1, \dots, J\}$  and

$$n_h^Y \geq N_h^X = \sum_{j \in P_h} n_j^X$$

for every  $h = 1, \dots, H$  (which implies that  $N_Y \geq N_X$ ). Here, some of  $P_h$  may be empty and then we set  $N_h^X = 0$ . (In particular,  $H$  may be greater than  $J$ .)

Furthermore, if  $N_X > 0$ , then there is an injection

$$\iota : \{1, \dots, N_X\} \rightarrow \{1, \dots, N_Y\}$$

such that the image of  $\bigcup_{j \in P_h} E_j^X$  by  $\iota$  is contained in  $E_h^Y$  for every  $h$ , where  $\{E_0^Y, \dots, E_H^Y\}$  is the partition associated to  $Y = \{q_h\}_{h=1}^H$ .

- (2) (*Convergence*) For every  $k$  with  $1 \leq k \leq N_X$ , denote by  $p_{j(k)}$  and  $q_{h(k)}$  the polynomial maps in  $\{p_j\}_{j=1}^J$  and  $\{q_h\}_{h=1}^H$  having fixed points represented by  $k$  and  $\iota(k)$ , respectively. We normalize  $p_{j(k)}$  and  $q_{h(k)}$  so that these fixed points are 0 and all coefficients of  $z^r$  with  $r \neq 1$  have the absolute values not greater than 1 and at least one of them is 1. Then, among polynomial maps obtained from  $p_{j(k)}$  and  $q_{h(k)}$  by this normalization, we can find polynomial maps  $\tilde{p}_k$  and  $\tilde{q}_k$ , respectively, such that

$$|\tilde{p}_k - \tilde{q}_k| < \epsilon$$

on  $\{z \in \mathbb{C} \mid |z| < 1/\epsilon\}$ .

- (3) (*Degeneration*) For every  $l$  with  $1 \leq l \leq N_Y$  not in the image of  $\iota$ , let  $q_{h(l)}$  be the polynomial map in  $\{q_h\}_{h=1}^H$  having a fixed point represented by  $l$ , and  $m_l(Y)$  be the multiplier of  $q_{h(l)}$  at the fixed point. Then we have

$$|m_l(Y)| > \frac{1}{\epsilon}.$$

Finally, take

$$\{U_\epsilon(X) \mid \epsilon > 0, X \in \widehat{M}_n\}$$

as basic open sets of  $\widehat{M}_n$ , and we have a topology on  $\widehat{M}_n$ .

*Remark 1.4.* If  $J = 0$ , i.e.  $X$  is the empty collection, then we assume that  $N_X = 0$  and  $P_h = \emptyset$  for every  $h$ , and hence conditions (1) and (2) hold trivially.

**Proposition 1.5.** *The set  $\text{MPoly}_n$  is open in  $\widehat{M}_n$ .*

*Proof.* Fix  $X \in \text{MPoly}_n$ . Then  $X$  consists of a single affine conjugacy class of a polynomial map  $p$  of degree  $n$ . Fix  $\epsilon > 0$  arbitrarily. Then the condition (1) implies that every point of  $U_\epsilon(X)$  again consists of a single affine conjugacy class of a polynomial  $p$  of degree  $n$ . Thus  $U_\epsilon(X) \subset \text{MPoly}_n$ .  $\square$

**Proposition 1.6.** *Points  $Y_m \in \text{MPoly}_n$  tend to the empty collection if and only if the multipliers  $m_h(Y_m)$  tend to  $\infty$  as  $m$  tend to  $\infty$  for every  $h$  ( $= 1, \dots, n$ ).*

*Proof.* By definition,  $Y_m \in \text{MPoly}_n$  tend to the empty collection if and only if, for every  $\epsilon > 0$ , there is an  $m_0$  such that

$$|m_h(Y_m)| > \frac{1}{\epsilon}$$

for every  $h$  and every  $m \geq m_0$ , which is clearly equivalent to the fact that the multipliers  $m_h(Y_m)$  tend to  $\infty$  as  $m$  tend to  $\infty$  for every  $h$ .  $\square$

**Corollary 1.7.** *The points  $Y_m = \{\langle q_m \rangle\} \in \widehat{M}_n$  with  $q_m(z) = z^n + z - m^n$  tend to the empty collection as  $m$  tends to  $\infty$ .*

Now the main theorem of this paper is the following one.

**Theorem 1.8.** *The extended moduli space  $\widehat{M}_n$  of degree  $n$  equipped with the topology defined above is compact, and  $\text{MPoly}_n$  is dense in it.*

The proof will be given in §2.

Next, we define a map

$$\widehat{\Psi}_n : \widehat{M}_n \rightarrow \mathbb{P}^{n-1}(\mathbb{C}).$$

**Definition 1.9** (Extended multiplier representation). Let  $X = \{\langle p_j \rangle\}_{j=1}^J$  be an element of  $\widehat{M}_n$ , and let  $n_j^X$  and  $N_X$  be as before. Let  $\sigma_k$  ( $k = 1, \dots, N_X$ ) be the elementary symmetric functions of multipliers  $\{m_k\}_{k=1}^{N_X}$  at finite fixed points of all  $p_j$  counted including multiplicities:

$$\begin{aligned} \sigma_1 &= m_1 + m_2 + \dots + m_{N_X}, \\ \dots, \\ \sigma_r &= \sum_{j_1 < j_2 < \dots < j_r} m_{j_1} m_{j_2} \dots m_{j_r}, \\ \dots, \\ \sigma_{N_X} &= m_1 m_2 \dots m_{N_X}. \end{aligned}$$

We define the map  $\widehat{\Psi}_n$  by sending  $X$  to the point of  $\mathbb{P}^{n-1}(\mathbb{C})$  represented by

$$\underbrace{(0 : \dots : 0)}_{n-N_X} : 1 : \sigma_1 : \dots : \sigma_{N_X-2} : \sigma_{N_X}.$$

Finally, we map the empty collection to the point represented by

$$\underbrace{(0 : \cdots : 0 : 1)}_{n-1}.$$

Here it is well-known that for every polynomial with degree  $n$  the unique linear relation between the elementary symmetric functions  $\{\sigma_k\}_{k=1}^n$  of multipliers at finite fixed points is

$$n + \sum_{k=1}^n (-1)^k (n - k) \sigma_k = 0,$$

which is called Fatou’s index theorem. This relation still holds in the circumstance as in the above definition when we replace  $n$  by  $N_X$ . This is the reason why we delete  $\sigma_{N_X-1}$  in the definition of  $\widehat{\Psi}_n$ .

Now, we conclude the following theorem.

**Theorem 1.10** (Cf. [8], Theorem 3.2). *The map*

$$\widehat{\Psi}_n : \widehat{M}_n \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$$

*defined as above is a continuous surjection.*

Here recall that  $\widehat{\Psi}_n$  restricted to  $\text{MPoly}_n$  can be considered as a map into  $\mathbb{C}^{n-1}$ , which is never surjective if  $n \geq 4$  ([9], Theorem 2). Also we have the following corollary of [9], Theorem 6.

**Theorem 1.11.** *For a generic point  $x$  of  $\mathbb{P}^{n-1}(\mathbb{C})$ , the preimage  $\widehat{\Psi}_n^{-1}(x)$  consists of exactly  $(n - 2)!$  points of  $\widehat{M}_n$ .*

The proofs of Theorems 1.10 and 1.11 will be given in §3.

*Remark 1.12.* 1) The construction of our compactification resembles that of the Deligne-Mumford one due to Bers for the moduli space of compact Riemann surfaces ([1] and cf. [11]). Also see [14], the Remark after Corollary 1.6.

2) There is another deformation space concerning  $\text{Poly}_n$ . We say that two maps  $p_1$  and  $p_2$  in  $\text{Poly}_n$  determine the same *isomorphism class* if there is a biholomorphic automorphism  $g$  of  $\mathbb{C}$  such that  $p_2 = p_1 \circ g$ . The (*classical*) *Hurwitz space*  $H_{0,n}[n]$  of genus 0 and degree  $n$  with type  $(n)$  is the set consisting of all isomorphism classes of polynomial maps of degree  $n$  in general position. Recall that a natural compactification of the Hurwitz space was introduced in [7] and [13]. (Also cf. [6].)

## 2. PROOF OF THEOREM 1.8

First, we show that  $\text{MPoly}_n$  is dense in  $\widehat{M}_n$ . By Corollary 1.7, we start with a point  $X = \{\langle p_j \rangle\}_{j=1}^J$  in  $\widehat{M}_n$  which is not the empty collection, i.e. with  $J \geq 1$ .

First, assume that  $J = 1$  but  $(2 \leq) n_1 = n_1^X < n$ . Here, we may write  $p_1$  as

$$p_1(z) = z + \prod_{\mu=1}^{n_1} (z - a_\mu).$$

Set

$$p_1 \#^r(z) = z + \frac{r - z}{r} \prod_{\mu=1}^{n_1} (z - a_\mu).$$

Then  $p_1 \#^r$  are polynomials with degree  $n_1 + 1$  converging to  $p_1$  uniformly on compact sets in  $\mathbb{C}$  as  $r$  tend to  $\infty$ . Also, since  $n_1 \geq 2$ , the multipliers at  $r$  tend to  $\infty$  as do  $r$ . Thus the following lemma is clear.

**Lemma 2.1.**  $Y_r = \{ \langle p_1 \#^r \rangle \}$  converge to  $X$  as  $r$  tend to  $\infty$ .

If  $J > 1$ , then we also consider  $p_2$  and set  $n_2 = n_2^X (\geq 2)$ . We may write  $p_2$  as

$$p_2(z) = z + \prod_{\nu=1}^{n_2} (-z + b_\nu).$$

We set

$$p_1 \#^r p_2(z) = z + \frac{1}{r^{(n_2-1)n_2}} \prod_{\mu=1}^{n_1} (z - a_\mu) \prod_{\nu=1}^{n_2} (-z + r^{n_2-1} + r^{n_2-n_1} b_\nu).$$

Then  $p_1 \#^r p_2$  are polynomials with degree  $n_1 + n_2$  converging to  $p_1$  uniformly on compact sets in  $\mathbb{C}$  as  $r$  tend to  $\infty$ . Also, we have

**Lemma 2.2.**  $Y_r = \{ \langle p_1 \#^r p_2 \rangle, \langle p_3 \rangle, \dots, \langle p_J \rangle \}$  converge to  $X$  as  $r$  tend to  $+\infty$ .

*Proof.* The polynomials  $p_1 \#^r p_2(z)$  are conjugate to

$$z + \frac{1}{r^{(n_1-1)n_1}} \prod_{\mu=1}^{n_1} (z + r^{n_1-1} - r^{n_1-n_2} a_\mu) \prod_{\nu=1}^{n_2} (-z + b_\nu)$$

by the affine automorphisms

$$V_r(z) = \frac{z - r^{n_2-1}}{r^{n_2-n_1}},$$

which converge to  $p_2$  uniformly on compact sets in  $\mathbb{C}$  as  $r$  tend to  $\infty$ . Thus the assertion follows. □

*Proof of denseness.* Take a sufficiently large  $r$ , and we have an element  $\{ \langle p_1 \#^r \rangle \}$  or  $\{ \langle p_1 \#^r p_2 \rangle, \langle p_3 \rangle, \dots, \langle p_J \rangle \}$  of  $\widehat{M}_n$  which is arbitrarily near to  $X$  by Lemmas 2.1 and 2.2 and which satisfies that the degree of  $p_1 \#^r$  is  $n_1 + 1$  or (the degree of  $p_1 \#^r p_2$  is  $n_1 + n_2$  but) the number of the conjugacy classes decreases from  $J$  to  $J - 1$ , respectively.

Repeating such an approximation by a finite number of times, we have an element of  $MPoly_n$  arbitrarily near to  $X$ . Thus we conclude that  $MPoly_n$  is dense in  $\widehat{M}_n$ . □

**Example 2.3.** The points  $Y_m = \{ \langle q_m \rangle \} \in \widehat{M}_n$  ( $n \geq 3$ ) with  $q_m(z) = z^3 - mz^2 + z$  converge to  $X = \{ \langle q \rangle \}$  with  $q(z) = z^2 + z$  as  $m$  tend to  $\infty$ .

Indeed, the fixed points of  $q_m$  are 0 (double) and  $m$ , with multipliers 1 and  $m^2 + 1$ , respectively. Associate the numbers 1, 2 to 0 and normalize  $q_m$  as before with respect to 0, and we have  $\tilde{q}_1[m](z) = m^{-2}z^3 + z^2 + z$ , which converge to  $q(z)$  uniformly on compact sets in  $\mathbb{C}$ .

**Example 2.4.** The points  $Y_m = \{ \langle q_m \rangle \} \in \widehat{M}_n$  ( $n \geq 4$ ) with  $q_m(z) = z^4 - 2mz^3 + m^2z^2 + z$  converge to  $\{ \langle q \rangle, \langle q \rangle \}$  with  $q(z) = z^2 + z$  as  $m$  tend to  $\infty$ .

Indeed, the fixed points of  $q_m$  are 0 (double) and  $m$  (double) with multiplier 1. We associate the number 1, 2 to 0 and 3, 4 to  $m$ . Normalize  $q_m$  as before with respect to 0 and  $m$ , and get

$\tilde{q}_1[m](z) = m^{-6}z^4 - 2m^{-3}z^3 + z^2 + z$  and  $\tilde{q}_3[m](z) = m^{-6}z^4 + 2m^{-3}z^3 + z^2 + z$ , respectively, both of which converge to  $q(z)$  uniformly on compact sets in  $\mathbb{C}$ .

Next, the space  $\widehat{M}_n$  equipped with the topology defined above satisfies the second countability axiom. (Actually, the topology has a countable base

$$\{U_{\epsilon_u}(X_v) \mid u, v \in \mathbb{Z}\},$$

where  $\epsilon_u$  moves over all positive rational numbers and  $X_v$  over all collections of the affine conjugacy classes of polynomial maps with rational coefficients.) Hence to finish the proof of Theorem 1.8, it is enough to show sequential compactness of  $\widehat{M}_n$ .

Let  $\{Y_l\}_{l=1}^\infty$  be an arbitrary sequence of distinct points in  $\widehat{M}_n$ . Here taking a subsequence if necessary, we may assume that every  $Y_l$  consists of the same number of affine conjugacy classes, say  $\{q_h[l]\}_{h=1}^H$ , and that the degree of  $q_h[l]$  is a constant  $n_h$  for every  $l$ . Set  $N = \sum_{h=1}^H n_h$ . We may also assume that all fixed points of all  $q_h[l]$  are numbered so that the fixed point  $z_k[l]$  represented by  $k$  is a fixed point of  $q_{h(k)}[l]$  with  $h(k)$  independent of  $l$  for every  $k$  ( $= 1, \dots, N$ ).

Let  $m_k[l]$  be the multipliers of  $q_{h(k)}[l]$  at the fixed point  $z_k[l]$ . Here, again taking a subsequence and renumbering the fixed points if necessary, we may assume that  $m_k[l]$  converge to some value  $m_k$  in  $\mathbb{P}^1(\mathbb{C})$  as  $l$  tend to  $\infty$  for every  $k$ , and that  $m_k = \infty$  if and only if  $k$  satisfies  $N' < k \leq N$  with some  $N' \leq N$ . Here, if  $N' = 0$ , then as in the proof of Proposition 1.6,  $Y_l$  converge to the empty collection as  $l$  tend to  $\infty$ .

If  $N' > 0$ , then  $m_1[l]$  converge to a finite  $m_1$  as  $l$  tend to  $\infty$ . Let  $\tilde{q}_1[l]$  be a polynomial map obtained from  $q_{h(1)}[l]$  by the normalization as before with respect to  $z_1[l]$  for every  $l$ . Then taking a subsequence if necessary, we may assume that  $\tilde{q}_1[l]$  converge to a polynomial  $\tilde{p}_1$  uniformly on compact sets in  $\mathbb{C}$  as  $l$  tend to  $\infty$ . Here from the definition, the degree of  $\tilde{p}_1$  is not less than 2, and the multiplier at the fixed point 0 is  $m_1$ .

Let  $k$  satisfy that  $1 < k \leq N'$ ,  $q_{h(k)}[l] = q_{h(1)}[l]$  and the fixed points  $\tilde{z}_k[l]$  of  $\tilde{q}_1[l]$  corresponding to  $z_k[l]$  are bounded. Then, we may further assume that  $\tilde{z}_k[l]$  converge to a fixed point  $\tilde{z}_k$  of  $\tilde{p}_1$ , which is represented also by  $k$ , as  $l$  tend to  $\infty$ . Let  $\text{Fix}_1$  be the set of all numbers representing fixed points of  $\tilde{p}_1$  including multiplicities, and we have the following lemma.

**Lemma 2.5.** *For every  $k \in \text{Fix}_1$ , we normalize  $q_{h(1)}[l]$  and  $\tilde{p}_1$  as before with respect to  $z_k[l]$  and  $\tilde{z}_k$ , respectively. Then among polynomial maps obtained by the normalization, we can find polynomial maps  $\tilde{q}_k[l]$  and  $\tilde{p}_k$ , respectively, such that  $\tilde{q}_k[l]$  converge to  $\tilde{p}_k$  locally uniformly on  $\mathbb{C}$  as  $l$  tend to  $\infty$ .*

*Proof.* If  $k > 1$  and  $k \in \text{Fix}_1$ , take the conjugates of  $\tilde{q}_1[l]$  by the affine automorphisms  $U_l(z) = z - \tilde{z}_k[l]$ . Then we have polynomials with bounded coefficients which converge to the conjugate of  $\tilde{p}_1$  by  $U(z) = z - \tilde{z}_k$ , which implies the assertion.  $\square$

Now let  $k_2 (> 1)$  be the least number not contained in  $\text{Fix}_1$ . Then starting from  $q_{h(k_2)}[l]$ , we also have a polynomial map  $\tilde{p}_{k_2}$  satisfying the same assertion as in Lemma 2.5 with  $k_2$  instead of 1.

*Proof of compactness.* Repeating the above arguments a finite number of times, we finally get a set of polynomial maps  $\{\tilde{p}_k\}_{k=1}^{N'}$ . Let

$$\{\text{Fix}_1, \text{Fix}_{k_2}, \dots, \text{Fix}_{k_K}\}$$

be the corresponding partition of  $\{1, \dots, N'\}$ , and set

$$X = \{\langle \tilde{p}_1 \rangle, \langle \tilde{p}_{k_2} \rangle, \dots, \langle \tilde{p}_{k_K} \rangle\}.$$

Then from the construction,  $Y_l$  converge to  $X$  in  $\widehat{M}_n$  as  $l$  tend to  $\infty$ . □

### 3. PROOFS OF THEOREMS 1.10 AND 1.11

First we show the following lemma.

**Lemma 3.1.**  $\widehat{\Psi}_n$  is continuous.

*Proof.* If the multipliers  $m_k(X)$  at the fixed point represented by  $k$  are bounded for all  $k$ , then by condition (2),  $m_k(X)$  are continuous. Since we define  $\widehat{\Psi}_n$  by using the elementary symmetric functions of these multipliers, the assertion follows.

The case that some  $m_k(X)$  are  $\infty$  or tend to  $\infty$  is similar. For the sake of simplicity, we give a proof only for the case that  $m_1(X)$  are finite but tend to  $\infty$ , while the others are bounded, as  $X$  tend to  $X_\infty$ . In this case,  $\widehat{\Psi}_n(X)$  can also be represented as

$$\left( \frac{1}{m_1(X)} : 1 + \frac{\sigma'_1(X)}{m_1(X)} : \cdots : \sigma'_n(X) \right)$$

with  $\sigma'_1(X) = \sum_{k=2}^n m_k(X), \cdots, \sigma'_n(X) = \prod_{k=2}^n m_k(X)$ , which converge to  $\widehat{\Psi}_n(X_\infty)$  from the definition. □

*Proof of Theorem 1.10.* Since  $\widehat{\Psi}_n$  is continuous by Lemma 3.1 and  $\widehat{M}_n$  is compact by Theorem 1.8, the image  $\widehat{\Psi}_n(\widehat{M}_n)$  is also compact in  $\mathbb{P}^{n-1}(\mathbb{C})$ . Hence to prove Theorem 1.10, it remains only to show that the image  $\widehat{\Psi}_n(\widehat{M}_n)$  is dense. But by Lemma 3.2 below,  $\widehat{\Psi}_n(\text{MPoly}_n)$  is already dense in  $\mathbb{P}^{n-1}(\mathbb{C})$ . □

**Lemma 3.2.**  $\widehat{\Psi}_n(\text{MPoly}_n)$  is dense in  $\mathbb{P}^{n-1}(\mathbb{C})$ .

*Proof.* Let  $\Omega$  be the set of all points  $(s_1, \cdots, s_{n-2}, s_n) \in \mathbb{C}^{n-1}$  such that there is a solution  $\{m_1, \cdots, m_n\}$  of the equation system

$$\begin{aligned} s_1 &= m_1 + m_2 + \cdots + m_n, \\ \cdots, \\ s_r &= \sum_{j_1 < j_2 < \cdots < j_r} m_{j_1} m_{j_2} \cdots m_{j_r}, \\ \cdots, \\ s_n &= m_1 m_2 \cdots m_n, \end{aligned}$$

with  $m_j \neq 1$  for every  $j$  and  $\sum_{j \in S} \frac{1}{1-m_j} \neq 0$  for every proper subset  $S$  of  $\{1, \cdots, n\}$ . Here, the unique linear relation between the elementary symmetric functions corresponds to  $\sum_{j=1}^n \frac{1}{1-m_j} = 0$ . This equation between  $\{m_j\}_{j=1}^n$  is independent of the equations  $m_j = 1$  and  $\sum_{j \in S} \frac{1}{1-m_j} = 0$  for every proper subset  $S$  of  $\{1, \cdots, n\}$ . Since the map sending  $(m_1, \cdots, m_n)$  to  $(s_1, \cdots, s_n)$  is a finite-sheeted holomorphic branched covering projection of  $\mathbb{C}^n$  onto itself, we see that  $\Omega$  is a generic set in  $\mathbb{C}^{n-1}$ .

On the other hand, [9], Theorem 4, implies that  $\widehat{\Psi}_n(\text{MPoly}_n)$  contains  $\Omega$ , which shows the assertion. □

*Proof of Theorem 1.11.* Every element of  $\widehat{M}_n - \text{MPoly}_n$  consists of either a single affine conjugacy class of a polynomial map with degree less than  $n$  or of more than one class. Hence the set  $\widehat{M}_n - \text{MPoly}_n$  is, at most, complex  $n - 2$  dimensional, which implies that  $\mathbb{C}^{n-1} - \widehat{\Psi}_n(\widehat{M}_n - \text{MPoly}_n)$  is a generic set in  $\mathbb{P}^{n-1}(\mathbb{C})$ . Thus it is enough to show that for a generic point in  $\mathbb{C}^{n-1}$  the preimage of the point by

$\widehat{\Psi}_n$  contains exactly  $(n-2)!$  points counted including multiplicities. But this is [9], Theorem 6.  $\square$

As closing remarks, we discuss more details in the case that  $n = 2, 3$ , and 4.

First,  $\widehat{M}_2 - \text{MPoly}_2$  consists of the empty collection only. Thus  $\widehat{M}_2$  is the one-point compactification of  $\text{MPoly}_2$ , and  $\widehat{M}_2$  is homeomorphic to  $\mathbb{P}^1(\mathbb{C})$ .

Next,  $\widehat{M}_3 - \text{MPoly}_3$  consists of the empty collection and a set which can be identified with  $\text{MPoly}_2$ . Hence  $\widehat{M}_3$  is homeomorphic to  $\mathbb{P}^2(\mathbb{C})$  and  $\widehat{\Psi}_3$  gives a homeomorphism. Compare with [5], Theorem 11.1.

Now,  $\widehat{M}_4 - \text{MPoly}_4$  consists of the empty collection and sets

$$E_2, E_3, E_{2,2}$$

which can be identified with  $\text{MPoly}_2$ ,  $\text{MPoly}_3$ , and the symmetric product of  $\text{MPoly}_2$  with itself, respectively. Also we can describe the map  $\widehat{\Psi}_4$  completely as follows (cf. [10], Theorem 3):

- (1) If  $x \in \mathbb{P}^3(\mathbb{C})$  is represented by  $(1 : 4 : 6 : 1)$ , then  $\widehat{\Psi}_4^{-1}(x)$  consists of infinite number of points.
- (2) If  $x$  is represented by either
  - (a)  $(0 : s_1 : s_2 : s_4)$  or
  - (b)  $(1 : s_1 : s_2 : s_4) (\neq (1 : 4 : 6 : 1))$  satisfying the equation

$$\begin{aligned} &54s_1^5 + (-81s_2 - 27s_4 - 135)s_1^4 + (36s_2^2 - 144s_2 - 1008)s_1^3 \\ &+ (-4s_2^3 + 360s_2^2 + (144s_4 + 2976)s_2 + 576s_4 + 4192)s_1^2 \\ &+ (-160s_2^3 - 2176s_2^2 + (-384s_4 - 6400)s_2 - 1280s_4 - 5376)s_1 \\ &+ 16s_2^4 + 448s_2^3 + (-128s_4 + 2176)s_2^2 + (256s_4 + 3840)s_2 \\ &+ 256s_4^2 + 768s_4 + 2304 = 0, \end{aligned}$$

then  $\widehat{\Psi}_4^{-1}(x)$  consists of a single point.

- (3) Otherwise  $\widehat{\Psi}_4^{-1}(x)$  consists of two points.

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#### REFERENCES

1. L. Bers, *On the space of Riemann surfaces with nodes*, Bull. AMS. **80** (1974), 1219–1222. MR0361165 (50:13611)
2. B. Branner and J. H. Hubbard, *The iteration of cubic polynomials, Part I: The global topology of parameter space*, Acta Math. **160** (1988), 143–206. MR945011 (90d:30073)
3. ———, *The iteration of cubic polynomials, Part II: Patterns and parapatterns*, Acta Math. **169** (1992), 229–325. MR1194004 (94d:30044)
4. L. DeMarco, *Finiteness for degenerate polynomials*, arXiv: math/0608800v1.
5. L. DeMarco and C. McMullen, *Trees and the dynamics of polynomials*, arXiv: math/0608759v1.
6. S. Diaz, *On the Natanzon-Turaev compactification of the Hurwitz space*, Proc. AMS **130** (2001), 613–618. MR1866008 (2002h:14042)
7. S. Diaz and D. Edidin, *Towards the homology of Hurwitz spaces*, J. Diff. Geom. **43** (1996), 66–98. MR1424420 (98e:14028)
8. M. Fujimura, *Projective moduli space for the polynomials*, Dynam. Conti. Discrete Impul. Systems **13** (2006), 787–801. MR2273342 (2007h:37061)
9. ———, *The moduli space of rational maps and surjectivity of multiplier representation*, Comp. Meth. Funct. Th. **7** (2007), 345–360.

10. M. Fujimura and K. Nishizawa, *Some dynamical loci of quartic polynomials*, J. Japan Soc. Symb. Alg. Compt. **11** (2005), 57–68.
11. Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller spaces*, Springer (Tokyo), 1992. MR1215481 (94b:32031)
12. J. Milnor, *Remarks on iterated cubic maps*, Experiment. Math. **1** (1992), 5–24. MR1181083 (94c:58096)
13. S. M. Natanzon and V. Turaev, *A compactification of the Hurwitz spaces*, Topology **38** (1999), 889–914. MR1679803 (2000b:57004)
14. J. Silverman, *The space of rational maps on  $\mathbb{P}^1$* , Duke Math. J. **94** (1998), 41–77. MR1635900 (2000m:14010)

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