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ON A PROBLEM OF BERNARD CHEVREAU CONCERNING THE ρ-CONTRACTIONS

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Abstract. We prove new results for the operators of class C_ρ ($\rho > 0$) on Hilbert spaces defined by B. Sz.-Nagy and C. Foiaş. The main result is an answer to a problem posed in 2006 by B. Chevreau: Let $p \geq 2$ be a natural number and $T \in \mathrm{L}(\mathcal{H})$; if there exists $\rho_0 > 0$ such that $T^p \in C_{\rho_0}$, then necessarily is $T \in \bigcup_{\rho>0} C_\rho$?

1. Introduction

We denote by $\mathcal{H}$ a Hilbert space over field $\mathbb{C}$ of all complex numbers. Let $\mathrm{L}(\mathcal{H})$ denote the algebra of all bounded operators on $\mathcal{H}$. $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. For $\rho > 0$, we denote by $C_\rho(\mathcal{H})$ the class of operators which admits a unitary ρ-dilation: an operator $T \in \mathrm{L}(\mathcal{H})$ admits a unitary ρ-dilation if there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $U \in \mathrm{L}(\mathcal{K})$ unitary such that

$$T^n = \rho P U^n|_{\mathcal{H}},\ n \geq 1,$$

where P is the orthogonal projection of $\mathcal{K}$ on $\mathcal{H}$.

If $T \in C_\rho(\mathcal{H})$, then T is also called a ρ-contraction. The classes C_ρ were introduced by B. Sz.-Nagy and C. Foiaş in [15] (see also [16]). The class $C_1(\mathcal{H})$ is precisely the class of all contractions on $\mathcal{H}$. The class $C_2(\mathcal{H})$ is precisely the class of all operators $T \in \mathrm{L}(\mathcal{H})$ which have numerical radius equal to at most one ([1]).

We remember that for an operator $T \in \mathrm{L}(\mathcal{H})$, the numerical radius is defined by

$$w(T) = \sup\{|\langle Th, h\rangle|, h \in \mathcal{H}, ||h|| = 1\}.$$

It is known ([16]) that $C_\rho(\mathcal{H}) \subset C_{\rho'}(\mathcal{H})$ if $0 < \rho < \rho' < \infty$ and $C_\rho(\mathcal{H}) \neq C_{\rho'}(\mathcal{H})$ if $\rho \neq \rho'$ and $dim(\mathcal{H}) \geq 2$.

We denote $C_\infty(\mathcal{H}) = \bigcup\limits_{\rho>0} C_\rho(\mathcal{H})$.

In February 2006, Professor Bernard Chevreau [3] posed the following:

Problem. Let $p \geq 2$ be a natural number and $T \in \mathrm{L}(\mathcal{H})$. If there exists $\rho_0 > 0$ such that $T^p \in C_{\rho_0}$, then necessarily is $T \in C_\infty(\mathcal{H})$?

The answer is important for recent development in the theory of operators of class C_ρ ([2], [4], [5]).

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We give an answer to this question. In fact, we prove that there exists $T \in \mathrm{L}(\mathbb{C}^p)$ such that $T^p = I_{\mathbb{C}^p}$ and $T \notin C_\infty(\mathbb{C}^p)$.

We use the following result given by G. Eckstein [6]:

Theorem 1.1. *Let $0 < \rho < \infty$ and $T \in C_\rho(\mathcal{H})$. Then, for any $h \in \mathcal{H}$, the sequence $\{||T^n h||\}_{n=0}^\infty$ converges.*

A short proof of this theorem was given in [11]. An another proof was given in [9].

2. The results

We proved before the following result:

Theorem 2.1. *Let $p \geq 2$ be a natural number and $0 < \rho < \infty$. If $T \in C_\rho(\mathcal{H})$ and T^p is a unitary operator, then T is unitary.*

Proof. We denote $V := T^p$. Let $h \in \mathcal{H}$ and n a natural number. We have

$$||T^{np}h|| = ||V^n h|| = ||h||$$

and

$$||T^{np+1}h|| = ||T^{np}Th|| = ||V^n Th|| = ||Th||.$$

Since $T \in C_\rho(\mathcal{H})$, it follows by Theorem 1.1 that the sequence $\{||T^n h||\}_{n=0}^\infty$ converges. It follows that the sequences

$$\{||T^{np}h||\}_{n=0}^\infty \text{ and } \{||T^{np+1}h||\}_{n=0}^\infty$$

converge to the same limit. It results in $||h|| = ||Th||$. Hence T is an isometry.

We also have $V^* = T^{*p}$. But $T^* \in C_\rho(\mathcal{H})$ since we can write

$$T^{*n}h = \rho P U^{*n} h, \ n \geq 1, \ h \in \mathcal{H}.$$

By the previous argument it follows that T^* is also an isometry. Hence T is a unitary operator. □

We give a generalization of Theorem 2.1, but in the proof we use another technique.

Let $T_1, T_2 \in \mathrm{L}(\mathcal{H})$. We say that T_1 and T_2 are *double commuting* if T_1 commutes with T_2 and with T_2^*.

We use the following result given by B. Fuglede [7]:

Theorem 2.2. *Let T_1, T_2 be bounded operators on $\mathcal{H}$, with T_2 being normal. If T_1 and T_2 commute, then T_1 and T_2 are double commuting.*

Our result is the following:

Theorem 2.3. *Let $p \geq 2$ be a natural number and $0 < \rho < \infty$. If $T \in C_\rho(\mathcal{H})$ is such that $N := T^p$ is normal and norm increasing (i.e. $N^*N \geq I_\mathcal{H}$), then T is unitary.*

Proof. Since N is norm increasing and normal, we have

$$||Nh|| \geq ||h|| \text{ and } ||N^*h|| = ||Nh|| \geq ||h||, \ h \in \mathcal{H}.$$

It follows that N is invertible and $||N^{-1}h|| \leq ||h||, \ h \in \mathcal{H}$.

We have $TN = T^{p+1}$ and $NT = T^{p+1}$; hence $N^{-1}T = TN^{-1}$. We obtain

$$T(T^{p-1}N^{-1}) = I_\mathcal{H} \text{ and } (T^{p-1}N^{-1})T = I_\mathcal{H}.$$

It follows that T is invertible and $T^{-1} = T^{p-1}N^{-1}$. Since $T \in C_\rho(\mathcal{H})$, it follows by [16] that $T^{p-1} \in C_\rho(\mathcal{H})$.

Since T^{p-1} and N^{-1} commute and N^{-1} is normal, by Theorem 2.2 it results that T^{p-1} and N^{-1} are double commuting. We use the following known result (see [8], [10], [14]):

If $T_1 \in C_{\rho_1}(\mathcal{H})$, $T_2 \in C_{\rho_2}(\mathcal{H})$ are double commuting, then $T_1T_2 \in C_{\rho_1\rho_2}(\mathcal{H})$ $(0 < \rho_1, \rho_2 < \infty)$.

We apply this result for $T_1 = T^{p-1} \in C_\rho(\mathcal{H})$ and $T_2 = N^{-1} \in C_1(\mathcal{H})$. It follows that $T^{-1} = T^{p-1}N^{-1} \in C_\rho(\mathcal{H})$. Since $T, T^{-1} \in C_\rho(\mathcal{H})$, by a well-known result of J. G. Stampfli [13] (see also [12]), it follows that T is unitary. □

We now prove the main result of this paper.

Theorem 2.4. *Let $p \geq 2$ be a natural number. Then there exists $T \in \mathrm{L}(\mathbb{C}^p)$ such that $T^p = I_{\mathbb{C}^p}$ and $T \notin C_\infty(\mathbb{C}^p)$.*

Proof. We consider T the operator on $\mathbb{C}^p$ which has the matrix

$$[T] = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & a_1 \\ a_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_p & 0 \end{pmatrix}$$

where $a_1 \cdot a_2 \cdot \dots \cdot a_p = 1$ and there exists $j \in \{1, 2, ..., p\}$ such that $a_j \neq 1$. We have

$$det[T] = (-1)^{p+1} a_1 \cdot a_2 \cdot \dots \cdot a_p \neq 0;$$

hence T is diagonalizable.

We denote by $\lambda_1, \lambda_2, ..., \lambda_p$ the roots of the equation

$$det([T] - \lambda I_p) = 0, \tag{2.1}$$

where I_p is the identity matrix of order p. Then we can we write

$$[T] = S^{-1}\Lambda S,$$

where

$$[\Lambda] = diag(\lambda_1, \lambda_2, ..., \lambda_p)$$

and S is an invertible matrix.

The equation (2.1) is equivalent to

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 & \dots & 0 & a_1 \\ a_2 & -\lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & a_3 & -\lambda & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_p & -\lambda \end{vmatrix} = 0;$$

that is,

$$(-\lambda)^p + (-1)^{p+1} a_1 a_2 \dots a_p = 0$$

or

$$\lambda^p = a_1 \dots a_p.$$

We have

$$[T]^p = S^{-1}\Lambda^p S = S^{-1} I_p S = I_p.$$

Hence $T^p = I_{\mathbb{C}^p}$.

But $T \notin C_\infty(\mathbb{C}^p)$. In a contrary case, by Theorem 2.1, it follows that T is unitary. $\square$

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