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ON THE RESIDUAL FINITENESS AND OTHER PROPERTIES OF (RELATIVE) ONE-RELATOR GROUPS

STEPHEN J. PRIDE

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Abstract. A relative one-relator presentation has the form $\mathcal{P} = \langle \mathbf{x}, H; R\rangle$ where $\mathbf{x}$ is a set, H is a group, and R is a word on $\mathbf{x}^{\pm 1} \cup H$. We show that if the word on $\mathbf{x}^{\pm 1}$ obtained from R by deleting all the terms from H has what we call the *unique max-min property*, then the group defined by $\mathcal{P}$ is residually finite if and only if H is residually finite (Theorem 1). We apply this to obtain new results concerning the residual finiteness of (ordinary) one-relator groups (Theorem 4). We also obtain results concerning the conjugacy problem for one-relator groups (Theorem 5), and results concerning the relative asphericity of presentations of the form $\mathcal{P}$ (Theorem 6).

1. Introduction

The question of when one-relator groups are residually finite is still open.

In the torsion-free case there are well-known examples of groups which are not residually finite, namely the Baumslag-Solitar/Meskin groups [4], [15]:

$$G = \langle \mathbf{x}; U^{-1}V^{l}UV^{m}\rangle,$$

where U, V do not generate a cyclic subgroup of the free group on $\mathbf{x}$, and $|l| \neq |m|$, $|l|, |m| > 1$. On the other hand, there are some examples which are known to be residually finite. For instance, it was shown in [3] that if

$$W = UV^{-1}, \tag{1}$$

where U, V are positive words on an alphabet $\mathbf{x}$ and the exponent sum of x in UV^{-1} is 0 for each $x \in \mathbf{x}$, or if

$$W = [U, V], \tag{2}$$

where U, V are (not necessarily positive) words on $\mathbf{x}$ such that no letter $x \in \mathbf{x}$ appears in both U and V, then $G = \langle \mathbf{x}; W\rangle$ is residually finite.

In the torsion case there is the well-known open question:

Question 1 ([2], [5, Question OR1]). Is every one-relator group with torsion residually finite?

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Question 1 is known to be true when $G = \langle \mathbf{x}; W^n \rangle$ where W is a *positive* word and $n > 1$ [9] (see also [19]). In [20], Wise obtains further related results, summed up by his "Quasi-Theorem 1.3": *If W is sufficiently positive, and W^n is sufficiently small cancellation, then G is residually finite.*

A related open question is:

Question 2 ([5, Question OR6], [11, Question 8.68])**.** If a torsion-free one-relator group $G_1 = \langle \mathbf{x}; W \rangle$ is residually finite, then is $G_n = \langle \mathbf{x}; W^n \rangle$ also residually finite for $n > 1$?

(Of course, if Question 1 is true, then Question 2 is trivially true.)

It was shown in [1] that Question 2 holds true when W has the form (1) or (2).

Here, amongst other things, we tackle Question 2 by considering *relative* presentations.

A relative presentation has the form

$$\mathcal{P} = \langle \mathbf{x}, H; \mathbf{r} \rangle,$$

where H is a group and $\mathbf{r}$ is a set of expressions of the form

$$R = x_1^{\varepsilon_1} h_1 x_2^{\varepsilon_2} h_2 \dots x_r^{\varepsilon_r} h_r \ (r > 0, x_i \in \mathbf{x}, \varepsilon_i = \pm 1, \ h_i \in H, 1 \leq i \leq r). \tag{3}$$

The word

$$W = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_r^{\varepsilon_r} \quad (r > 0, x_i \in \mathbf{x}, \varepsilon_i = \pm 1, 1 \leq i \leq r) \tag{4}$$

is called the $\mathbf{x}$-*skeleton* of R. We *do not* require that the $\mathbf{x}$-skeleton be reduced or cyclically reduced. The group $G = G(\mathcal{P})$ *defined by* $\mathcal{P}$ is the quotient of $H * F$ (where F is the free group on $\mathbf{x}$) by the normal closure of the elements of $H * F$ represented by the expressions $R \in \mathbf{r}$. The composition of the canonical imbedding $H \to H * F$ with the quotient map $H * F \to G$ is called the *natural homomorphism*, denoted by $\nu : H \to G$ (or simply $H \to G$).

As is normal, we will often abuse notation and write $G = \langle \mathbf{x}, H; \mathbf{r} \rangle$, or $G \cong \langle \mathbf{x}, H; \mathbf{r} \rangle$.

When $\mathbf{r}$ consists of a single element R, then we have the *one-relator relative presentation*

$$\mathcal{P} = \langle \mathbf{x}, H; R \rangle. \tag{5}$$

Heuristically, $G = G(\mathcal{P})$ should be governed by the *shape* of the $\mathbf{x}$-skeleton of R and the algebraic properties of H.

Here we introduce the *unique max-min property* for the shape of W. (Words of the form (1) are a very special case.) For a group H, denote by $\mathcal{M}_H$ the class of one-relator relative presentations of the form (5), where W has the unique max-min property.

Theorem 1. *If P is in $\mathcal{M}_H$, then*

(i) *the natural homomorphism $H \to G(\mathcal{P})$ is injective;*

(ii) *$G(\mathcal{P})$ is residually finite if and only if H is residually finite.*

We can deduce from this

Theorem 2 (Substitution theorem)**.** *Let K be a one-relator group given by an ordinary presentation $\langle \mathbf{y}, z; S(\mathbf{y}, z) \rangle$, and let $\mathcal{P} = \langle \mathbf{x}, H; R \rangle$ be an $\mathcal{M}_H$-presentation. Then the group given by the relative presentation $\langle \mathbf{x}, \mathbf{y}, H; S(\mathbf{y}, R) \rangle$ is residually finite if and only if H and K are residually finite.*

We can give the proof of this straightaway. Consider the $\mathcal{M}_{H*K}$-presentation $\overline{\mathcal{P}} = \langle \mathbf{x}, H * K; Rz^{-1}\rangle$. By Theorem 1, $L = G(\overline{\mathcal{P}})$ is residually finite if and only if $H * K$ is residually finite, which is equivalent to requiring that both H and K are residually finite (using results discussed in [12], p. 417). Now note that

$$L \cong \langle \mathbf{x}, \mathbf{y}, z, H; S(\mathbf{y}, z), Rz^{-1}\rangle \cong \langle \mathbf{x}, \mathbf{y}, H; S(\mathbf{y}, R)\rangle.$$

In particular, taking K to be defined by $\langle z; z^n\rangle$ $(n > 1)$ we have

Theorem 3. *If $G = \langle \mathbf{x}, H; R\rangle$ is a residually finite $\mathcal{M}_H$-group, then the group $G_n = \langle \mathbf{x}, H; R^n\rangle$ $(n > 1)$ is also residually finite.*

Now take H to be a free group Φ. Then $\mathcal{M}_\Phi$-groups are one-relator groups. Since Φ is residually finite ([12], p. 116 or p. 417), we obtain the following theorem concerning the residual finiteness of one-relator groups.

Theorem 4. *Every $\mathcal{M}_\Phi$-group $G = \langle \mathbf{x}, \Phi; R\rangle$ is a residually finite one-relator group. Moreover, if $K = \langle \mathbf{y}, z; S(\mathbf{y}, z)\rangle$ is a one-relator group, then the one-relator group $\overline{K} = \langle \mathbf{x}, \mathbf{y}, \Phi; S(\mathbf{y}, R)\rangle$ is residually finite if and only if K is residually finite. In particular, $G_n = \langle \mathbf{x}, \Phi; R^n\rangle$ $(n > 1)$ is residually finite.*

The solution of the conjugacy problem for one-relator groups with *torsion* has been solved by B. B. Newman [16]. However, for the *torsion-free* case the problem is still open [5, Question O5].

Theorem 5. *Every $\mathcal{M}_\Phi$-group (Φ a finitely generated free group) has a solvable conjugacy problem. Also, such groups have a solvable power conjugacy problem.*

(Two elements c, d of a group are said to be *power conjugate* if some power of c is conjugate to some power of d.)

Other aspects of relative presentations (and in particular, one-relator relative presentations) have been studied intensively, particularly *asphericity*. Recall [6] that a relative presentation $\mathcal{P}$ is *aspherical* (more accurately, *diagrammatically* aspherical) if every spherical picture over $\mathcal{P}$ contains a dipole. Under a weaker condition on shape (the *unique min property*, or equivalently the *unique max property*) we can prove

Theorem 6. *Let $\mathcal{P}$ be a relative presentation as in* (5), *where W has the unique min property. Then $\mathcal{P}$ is aspherical.*

It then follows from [6] (see Corollary 1 of Theorem 1.1, Theorem 1.3, and Theorem 1.4) that for the group $G = G(\mathcal{P})$ we have

(i) *the natural homomorphism $H \to G$ is injective*;

(ii) *every finite subgroup of G is contained in a conjugate of H*;

(iii) *for any left $\mathbb{Z}G$-module A and any right $\mathbb{Z}G$-module B,*

$$H^n(G, A) \cong H^n(H, A),$$

$$H_n(G, B) \cong H_n(H, B)$$

for all $n \geq 3$.

2. Max-min property

Let $\mathbf{x}$ be an alphabet. A *weight function* on $\mathbf{x}$ is a function

$$\theta : \mathbf{x} \longrightarrow \mathbb{Z}$$

such that $\operatorname{Im}\theta$ generates the additive group $\mathbb{Z}$ (that is, $gcd\{\theta(x) : x \in \mathbf{x}\}$ is 1). A *strict* weight function is one for which $\theta(x) \neq 0$ for all $x \in \mathbf{x}$.

Let W be a word on $\mathbf{x}$ as in (4). Given a weight function θ, we then have the function

$$\phi = \phi_W^\theta : \{0, 1, 2, \ldots, r\} \to \mathbb{Z},$$
$$\phi(j) = \sum_{i=0}^{j} \varepsilon_i \theta(x_i),$$

where $\phi(0) = 0$ since the empty sum is taken to be 0. We will say that the weight function is *admissible* for W if $\phi(r) = 0$.

For visual purposes, it is useful to extend ϕ to a piecewise linear function $\phi : [0, r] \to \mathbb{R}$, so that the graph of ϕ in the interval $[j-1, j]$ is the straight line segment joining the points $(j-1, \phi(j-1))$, $(j, \phi(j))$ $(0 < j \leq r)$. We will informally refer to this graph as "the graph of W" (with respect to θ).

A word W as in (4) will be said to have the *unique max-min property* if for some admissible strict weight function θ, the graph of W has a unique maximum and a unique minimum. To be precise, we require that, for some admissible strict weight function and some $k, l \in \{1, 2, \ldots, r\}$, we have $\phi(j) < \phi(k)$ for all $j \in \{1, 2, \ldots, r\} - k$ and $\phi(j) > \phi(l)$ for all $j \in \{1, 2, \ldots, r\} - \{l\}$. We also require that $x_k \neq x_{k+1}$ and $x_l \neq x_{l+1}$ (subscripts modulo r). This amounts to requiring that W is "reduced at the unique maximum and minimum"; that is, $x_k^{\varepsilon_k} \neq x_{k+1}^{-\varepsilon_{k+1}}, x_l^{\varepsilon_l} \neq x_{l+1}^{-\varepsilon_{l+1}}$ (subscripts modulo r). For at the maximum and minimum we must have *either* $x_j \neq x_{j+1}$, *or* $x_j = x_{j+1}$ and $\varepsilon_j = -\varepsilon_{j+1}$ $(j = k, l)$. If the two letters occurring at the unique maximum are not disjoint from the two letters occurring at the unique minimum (i.e. $\{x_k, x_{k+1}\} \cap \{x_l, x_{l+1}\}$ is not empty), then we will say that W has the *strong* unique max-min property.

A word W as in (4) will be said to have the *unique min property* if for some strict weight function θ, the graph of W has a unique minimum (but not necessarily a unique maximum). The *unique max property* is defined similarly, but is not really of interest because replacing θ by $-\theta$ will convert this property to the unique min property.

We let $\mathcal{M}_H^{\mathbf{1}}$ (respectively $\mathcal{S}_H^{\mathbf{1}}$) denote the subclass of $\mathcal{M}_H$ consisting of relative presentations of the form (5) for which W has the unique max-min property (respectively, the strong unique max-min property) with respect to the weight function

$$\mathbf{1} : \mathbf{x} \longrightarrow \mathbb{Z} \quad x \mapsto 1 \ (x \in \mathbf{x}).$$

Lemma 1. *Every $\mathcal{M}_H$-group can be embedded into an $\mathcal{M}_H^{\mathbf{1}}$-group.*

Proof. Let $G = \langle \mathbf{x}, H; R \rangle$ with R as in (3), and suppose $W = x_1^{\varepsilon_1} x_2^{\varepsilon_1} \ldots x_r^{\varepsilon_r}$ has the unique max-min property with respect to some strict weight function $\theta : \mathbf{x} \to \mathbb{Z}$. We can assume $\theta(x) > 0$ for all x. For if $\theta(x) < 0$, then we can replace x by x^{-1}.

Let

$$\mathbf{y} = \{y : y \in \mathbf{x}, \theta(y) > 1\},$$

and let

$$\hat{\mathbf{x}} = (\mathbf{x} - \mathbf{y}) \cup \{y_1, y_2, \ldots, y_{\theta(y)} : y \in \mathbf{y}\}.$$

Let $\hat{G} = \langle \hat{\mathbf{x}}, H; \hat{R} \rangle$, where $\hat{R}$ is obtained from R by replacing each occurrence of $y^{\pm 1}$ by $(y_1 y_2 \ldots y_{\theta(y)})^{\pm 1}$ $(y \in \mathbf{y})$. It is easy to see that the $\hat{\mathbf{x}}$-skeleton $\hat{W}$ of $\hat{R}$ has the unique max-min property with respect to $\mathbf{1} : \hat{\mathbf{x}} \to \mathbb{Z}$. (The graph of $\hat{W}$ is obtained from that of W by *stretching* along the horizontal axis.) Moreover, G is embedded into $\hat{G}$, for we have the retraction ρ with section μ:

$$\hat{G} \underset{\mu}{\overset{\rho}{\rightleftarrows}} G \qquad \rho\mu = \mathrm{id}_G,$$

$$\rho : x \mapsto x \ (x \in \mathbf{x} - \mathbf{y}),\ y_1 \mapsto y, y_i \mapsto 1 \ (y \in \mathbf{y}, 1 < i \le \theta(y)),\ h \mapsto h \ (h \in H),$$
$$\mu : x \mapsto x \ (x \in \mathbf{x} - \mathbf{y}), y \mapsto y_1 y_2 \ldots y_{\theta(y)} \ (y \in \mathbf{y}), h \mapsto h \ (h \in H).$$ □

Lemma 2. *Every $\mathcal{M}_H^{\mathbf{1}}$-group can be embedded into an $\mathcal{S}_H^{\mathbf{1}}$-group.*

Proof. Let $G = \langle \mathbf{x}, H; R \rangle$, where the $\mathbf{x}$-skeleton W of R has the unique max-min property with respect to the constant function $\mathbf{1} : \mathbf{x} \to \mathbb{Z}$. Suppose the letters occurring at the unique maximum are a, b, and those occurring at the unique minimum are c, d. We can assume that $\{a, b\} \cap \{c, d\}$ is empty; otherwise, there is nothing to prove. Let $\mathbf{y} = \mathbf{x} - \{a, b, c, d\}$, and introduce a new alphabet

$$\hat{\mathbf{x}} = \{a, b, c, d, e\} \cup \{y_1, y_2 : y \in \mathbf{y}\}.$$

Let $\hat{R}$ be obtained from R as follows. For each $y \in \mathbf{y}$, replace all occurrences of $y^{\pm 1}$ by $(y_1 y_2)^{\pm 1}$, and replace all occurrences of $a^{\pm 1}$ (respectively, $b^{\pm 1}$, $c^{\pm 1}, d^{\pm 1}$) by $(ea)^{\pm 1}$ (respectively, $(be)^{\pm 1}, (ec)^{\pm 1}, (de)^{\pm 1}$). Let $\hat{G} = \langle \hat{\mathbf{x}}, H; \hat{R} \rangle$, and let $\hat{W}$ be the word obtained from $\hat{R}$ by deleting all terms from H. The graph of $\hat{W}$ under the weight function $\mathbf{1} : \hat{\mathbf{x}} \to \mathbb{Z}$ is the graph of W magnified by a factor of 2, and e occurs at the unique maximum and the unique minimum. Moreover, G is embedded into $\hat{G}$ for we have the retraction ρ with section μ:

$$\hat{G} \underset{\mu}{\overset{\rho}{\rightleftarrows}} G \qquad \rho\mu = \mathrm{id}_G,$$

$$\rho : z \mapsto z \ (z \in \{a, b, c, d\}), e \mapsto 1, y_1 \mapsto y, y_2 \mapsto 1 \ (y \in \mathbf{y}), h \mapsto h \ (h \in H),$$
$$\mu : a \mapsto ea, b \mapsto be, c \mapsto ec, d \mapsto de, y \mapsto y_1 y_2 \ (y \in \mathbf{y}),\ h \mapsto h \ (h \in H).$$ □

Remark 1. Note that in both the above proofs, we have $\mu\nu = \hat{\nu}$, where $\nu : H \to G$, $\hat{\nu} : H \to \hat{G}$ are the natural homomorphims. Thus if $\hat{\nu}$ is injective, then so is ν.

Remark 2. Note also from the proof of the above two lemmas, we get that every $\mathcal{M}_H$-group is a retract of an $\mathcal{S}_H^{\mathbf{1}}$-group.

Remark 3. The referee has brought to my attention the work of K. S. Brown [8], which is concerned with whether a homomorphism χ from a one-relator group $B = \langle \mathbf{x}; W \rangle$ ($|\mathbf{x}| \ge 2$, W as in (4) and cyclically reduced) onto $\mathbb{Z}$ has a finitely generated kernel. Such a homomorphism is induced by a weight function θ which is admissible for W. However, since θ need not be strict, it is necessary to interpret the max-min property more widely. Thus the unique maximum could be a *plateau*: i.e., for some $k \in \{1, 2, \ldots, r\}$ we could have $\phi(k) = \phi(k+1)$ and $\phi(j) < \phi(k)$ for all $j \in \{1, 2, \ldots, r\} - \{k, k+1\}$ (subscripts modulo r). Similarly, the unique minimum could be a *reverse plateau*. Then according to Brown [8], as restated in Theorem 2.2

of [13], $\ker\chi$ is finitely generated if and only if $|\mathbf{x}| = 2$, and W has the unique max-min property in the above sense with respect to the corresponding weight function. In our work we could also allow non-strict weight functions. However, for the most part this can be avoided. For example, if the unique maximum is a plateau with $x_k \neq x_{k+2}$, then we could transform it to a genuine maximum by deleting x_{k+1} from $\mathbf{x}$ and replacing H by $H * \langle x_{k+1}\rangle$. However, if the unique maximum is a plateau with $x_k = x_{k+2}$, then some of our arguments need to be modified, which we leave as an exercise for the reader.

3. A construction

By a 2-*complex of groups* we mean a connected graph of groups (in the sense of Serre [18]) with trivial edge groups, together with a set of closed paths which we call *defining paths*. (These are essentially the *generalized complexes* defined in §1 of [10], where more detail can be found. Note however, that in [10] a 2-*cell* $c(\alpha)$ consists of *all* cyclic permutations of $\alpha^{\pm 1}$ for each one of our defining paths α. We specifically *do not* add these extra paths. This makes no significant difference.)

Let $\mathcal{P}$ be as in (5), and let θ be an admissible weight function for W. There is then an induced epimorphism

$$\psi : G \to \mathbb{Z} \quad x \mapsto \theta(x)\ (x \in \mathbf{x}), h \mapsto 0\ (h \in H).$$

We can construct a 2-complex of groups

$$\tilde{\mathcal{P}} = \langle \Gamma, H_n\ (n \in \mathbb{Z});\ (n, R)\ (n \in \mathbb{Z})\rangle$$

whose fundamental group is isomorphic to the kernel K of ψ. The underlying graph Γ has vertex set $\mathbb{Z}$, edges (n, x^{ε}) $(n \in \mathbb{Z}, x \in \mathbf{x}, \varepsilon = \pm 1)$, and initial, terminal and inversion functions $\iota, \tau, ^{-1}$ given by $\iota(n, x^{\varepsilon}) = n, \tau(n, x^{\varepsilon}) = n + \varepsilon\theta(x), (n, x^{\varepsilon})^{-1} = (n+\varepsilon\theta(x), x^{-\varepsilon})$. The vertex groups are copies $H_n = \{(n, h) : h \in H\}$ of H (with the obvious multiplication $(n, h)(n, h') = (n, hh')$). We extend $\iota, \tau, ^{-1}$ to the elements of the vertex groups by defining $\iota(n, h) = n = \tau(n, h)$, $(n, h)^{-1} = (n, h^{-1})$, where h^{-1} is the inverse of h in H. We extend θ to $\mathbf{x}^{\pm 1} \cup H$ by defining $\theta(x^{-1}) = -\theta(x)$ $(x \in \mathbf{x})$, $\theta(h) = 0$ $(h \in H)$. Then for any sequence $\alpha = z_1 z_2 \ldots z_q$ with $z_i \in \mathbf{x}^{\pm 1} \cup H$ and any vertex $n \in \Gamma$, we have a path (n, α) in the graph of groups starting at n, where

$$(n, \alpha) = (n, z_1)(n + \theta(z_1), z_2)(n + \theta(z_1) + \theta(z_2), z_3) \ldots (n + \theta(z_1) + \theta(z_2) + \ldots + \theta(z_{q-1}), z_q).$$

In particular we have the (closed) paths (n, R).

There is an obvious action of $\mathbb{Z}$ on the above graph of groups, with $i \in \mathbb{Z}$ acting on vertices by $i \cdot n = i + n$ $(n \in \mathbb{Z})$, and on the edges and vertex groups by $i.(n, z) = (i + n, z)$ $(n \in \mathbb{Z}, z \in \mathbf{x}^{\pm 1} \cup H)$. This action of course extends to paths. Thus $(i, \alpha) = i.(0, \alpha)$. In particular, $(i, R) = i.(0, R)$, so $\mathbb{Z}$ acts on $\tilde{\mathcal{P}}$.

If we regard $\mathcal{P}$ as a 2-complex of groups with a single vertex o, edges x^{ε} $(x \in \mathbf{x}, \varepsilon = \pm 1)$, vertex group H, and defining path R, then we have a mapping of 2-complexes of groups

$$\rho : \tilde{\mathcal{P}} \longrightarrow \mathcal{P},$$
$$n \mapsto o,\ (n, x^{\varepsilon})\ \mapsto x^{\varepsilon},\ (n, h)\ \mapsto h,\ (n, R) \mapsto R$$

$(n \in \mathbb{Z}, x \in \mathbf{x}, \varepsilon = \pm 1, h \in H)$. This induces a homomorphism

$$\rho_* : \pi_1(\tilde{\mathcal{P}}, 0) \longrightarrow \pi_1(\mathcal{P}, o) = G$$

which is injective, and $\text{Im}\rho_* = K$. This can easily be proved by adapting the standard arguments of covering space theory for ordinary 2-complexes (see for example [17], pp. 157-159) to this relative situation.

4. Proof of Theorem 1

Since residual finiteness is closed under taking subgroups, it follows from Lemmas 1 and 2 and Remark 1 at the end of §2 that it suffices to prove Theorem 1 for $\mathcal{S}_H^{\mathbf{1}}$-groups.

We will make use of the following results:

(a) *A free product $F * B$, where F is a free group, is residually finite if and only if B is residually finite*;

(b) *An infinite cyclic extension of a finitely generated group L is residually finite if and only if L is residually finite.*

(The first of these follows from results on p. 417 of [12]; the second is a special case of Theorem 7, p. 29 of [14].)

We can assume $\mathbf{x}$ is finite. For if not let $\mathbf{x}'$ be the set of letters occurring in R. Then G is isomorphic to $G' * \Psi$ where $G' \cong \langle \mathbf{x}', H; R \rangle$, and Ψ is the free group on $\mathbf{x} - \mathbf{x}'$. So by (a) above, it is enough to work with G'.

Let G be defined by an $\mathcal{S}_H^{\mathbf{1}}$ presentation as in (5), with $e \in \mathbf{x}$ occurring at both the unique maximum and the unique minimum of the graph of W under the weight function $\theta = \mathbf{1}$. We denote the maximum and minimum values of ϕ_W by M, m, respectively. Note that $m \leq 0 \leq M$ and $m < M$.

We first deal with the trivial case when $M - m = 1$. Then up to cyclic permutation and inversion, $R = eha^{-1}h'$, where $a \in \mathbf{x} - \{e\}$, $h, h' \in H$. Thus $G = \Phi * H$, where Φ is the free group on $\mathbf{x} - \{e\}$, so the theorem holds by (a) above.

Now suppose $M - m > 1$. Let $f \in \mathbf{x} - \{e\}$.

We have the epimorphism

$$\psi : G \to \mathbb{Z} \quad x \mapsto 1 \ (x \in \mathbf{x}), h \mapsto 0 \ (h \in H).$$

Also, we have the homomorphism

$$\eta : \mathbb{Z} \to G \quad 1 \mapsto f.$$

Then $\psi\eta = \text{id}_{\mathbb{Z}}$, so G is a semidirect product $K \rtimes \mathbb{Z}$, where $K = \ker \psi$, and with the action of $n \in \mathbb{Z}$ on K being induced by conjugation by f^n.

The fundamental group of $\widetilde{\mathcal{P}}$ (at the vertex 0), as in §3, is isomorphic to K.

We will obtain a relative presentation for K by collapsing a maximal tree.

The edges $(n, f)^{\pm 1}$ form a maximal tree T in Γ. Let R_n be the word on $\{(i, x) : i \in \mathbb{Z}, x \in \mathbf{x}, x \neq f\} \cup (\bigcup_{i \in \mathbb{Z}} H_i)$ obtained from (n, R) by deleting all edges from T which occur in (n, R) and replacing all terms (i, x^{-1}) by $(i-1, x)^{-1}$ $(i \in \mathbb{Z}, x \in \mathbf{x}, x \neq f)$. Then

$$\mathcal{Q} = \langle (n, x) \ (n \in \mathbb{Z}, x \in \mathbf{x}, x \neq f)\, , *_{n \in \mathbb{Z}} H_n \, ; R_n \ (n \in \mathbb{Z}) \rangle$$

is a relative presentation for K. Moreover, since the edges in T constitute an orbit under the action of $\mathbb{Z}$ on our graph of groups, the action of $\mathbb{Z}$ on K is given by the automorphism

$$\mu : (n, x) \mapsto (n+1, x) \ (x \in \mathbf{x}, x \neq f), \ (n, h) \mapsto (n+1, h) \ (h \in H)$$

$(n \in \mathbb{Z})$.

Now consider the HNN-extension $\overline{K}$ of K given by the relative presentation

$$\begin{aligned}\overline{\mathcal{Q}} = \quad & \langle (n,x)\ (n\in\mathbb{Z}, x\in\mathbf{x}, x\neq f), *_{n\in\mathbb{Z}}H_n\, , s; R_n\ (n\in\mathbb{Z})\\ & s(n,x)s^{-1} = (n+1,x)\ (n\in\mathbb{Z}, x\in\mathbf{x}, x\neq e,f),\\ & s(n,h)s^{-1} = (n+1,h)\ (n\in\mathbb{Z}, h\in H)\rangle.\end{aligned}$$

The automorphism μ of K can be extended to an automorphism $\overline{\mu}$ of $\overline{K}$ by defining $\overline{\mu}(s) = s$. Then $G = K \rtimes_\mu \mathbb{Z}$ can be embedded into $\overline{G} = \overline{K} \rtimes_{\overline{\mu}} \mathbb{Z}$.

By our assumption, up to cyclic permutation and inversion, $(0,R)$ will have the form

$$(M-1,e)(M,h)(M-1,a)^{-1}\gamma_0((m,b)^{-1}(m,h')(m,e))^{\varepsilon}\delta_0,$$

where $h,h'\in H, \varepsilon=\pm1, a,b\in\mathbf{x}-\{e\}$, and each term (i,z) occurring in the paths γ_0,δ_0 is such that both its initial and terminal vertices lie in the range $m+1$, $m+2,\dots,M-1$. Then

$$R_0 = (M-1,e)\alpha_0(m,e)^{\varepsilon}\beta_0,$$

where α_0,β_0 do not contain any occurrence of $(i,e)^{\pm1}$ with $i\le m$ or $i\ge M-1$. More generally, for $n\in\mathbb{Z}$

$$R_n = (n+M-1,e)\alpha_n(n+m,e)^{\varepsilon}\beta_n,$$

where α_n,β_n do not contain any occurrence of $(i,e)^{\pm1}$ with $i\le n+m$ or $i\ge n+M-1$.

Let F_0 be the free group on

$$(\mathbf{x}-\{e,f\})\cup\{s,(m+1,e),(m+2,e),\dots,(M-1,e)\}.$$

Then there is a homomorphism

$$\overline{K}\to H*F_0$$

defined as follows:

$$\begin{aligned} & s\mapsto s,\\ & (n,x)\mapsto s^nxs^{-n}\ (x\in\mathbf{x}, x\neq e,f,\ n\in\mathbb{Z}),\\ & (n,h)\mapsto s^nhs^{-n}\ (h\in H, n\in\mathbb{Z}),\\ & (i,e)\mapsto(i,e)\quad(m+1\le i\le M-1),\end{aligned}$$

and inductively, for $k=0,1,2,\dots,$

$$\begin{aligned}(k+M,e)&\mapsto\beta_{k+1}^{-1}(k+1+m,e)^{-\varepsilon}\alpha_{k+1}^{-1},\\ (-k+m,e)&\mapsto(\beta_{-k}(-k+M-1,e)\alpha_{-k})^{-\varepsilon}.\end{aligned}$$

This homomorphism is actually an isomorphism. The inverse is defined by

$$\begin{aligned} x &\mapsto (0,x)\quad (x\in\mathbf{x}, x\neq e,f),\\ h &\mapsto (0,h)\quad (h\in H),\\ (i,e) &\mapsto (i,e)\quad m+1\le i\le M-1,\\ s &\mapsto s.\end{aligned}$$

Thus $\overline{G}$ is an infinite cyclic extension of the group F_0*H.

Remark 4. Note that by sending s to the generator $1\in\mathbb{Z}\subset G=K\rtimes_\mu\mathbb{Z}$, we obtain a retraction of $\overline{G}$ onto G (with section induced by the inclusion of K into $\overline{K}$).

We can now complete the proof.

Clearly, the natural homomorphism from H into $\overline{G}$ is injective (and is thus injective into G). Hence if H is not residually finite, then neither is G. It remains to show that if H is residually finite, then so is $\overline{G}$ (and thus G).

Case 1. If H is finitely generated, then the result holds straightaway by (a) and (b) above.

Case 2. Suppose that H is not finitely generated. For any homomorphism θ from H to a group H_θ we obtain an induced homomorphism from $\overline{G} = (F_0 * H) \rtimes_{\overline{\mu}} \mathbb{Z}$ to $\overline{G}_\theta = (F_0 * H_\theta) \rtimes_{\overline{\mu}} \mathbb{Z}$ which acts as θ on H and acts as the identity on F_0 and $\mathbb{Z}$. Let $g = (w_0 h_1 w_1 \dots h_q w_q).n$ be a non-trivial element of $\overline{G}$, where $q \geq 0, h_1 \dots h_q \in H - \{1\}, w_1, \dots, w_{q-1} \in F_0 - \{1\}, w_0, w_q \in F_0$, $n \in \mathbb{Z}$, and if q is 0, then either $n \neq 0$ or w_0 is non-trivial. Since residually finite groups are fully residually finite, there is a homomorphism τ from H onto a finite group H_τ such that $\tau(h_i) \neq 1$ $(i = 1, \dots, q)$. So the image of g in $\overline{G}_\tau = (F_0 * H_\tau) \rtimes_{\overline{\mu}} \mathbb{Z}$ is non-trivial, and then Case 1 applies.

5. Proof of Theorem 5

Lemma 3. *Let C be a group which is a retract of a group B. If B has solvable conjugacy (or power conjugacy) problem, then so does C.*

Proof. By assumption we have maps $B \underset{\mu}{\overset{\rho}{\rightleftarrows}} C$, $\rho\mu = \mathrm{id}_C$. Clearly if $c, d \in C$ are conjugate (respectively, power conjugate) in C, then $\mu(c), \mu(d)$ are conjugate (respectively, power conjugate) in B. Conversely if there exists $b \in B$ such that $b\mu(c)b^{-1} = \mu(d)$ (respectively, $b\mu(c)^i b^{-1} = \mu(d)^j$), then $\rho(b)c\rho(b)^{-1} = d$ (respectively, $\rho(b)c^i\rho(b)^{-1} = d^j$). Thus the result follows. $\square$

Now it is shown in [7] that infinite cyclic extensions of finitely generated free groups have a solvable conjugacy, and power conjugacy, problem. By Remarks 2 and 4, every $\mathcal{M}_\Phi$-group is a retract of such a group.

6. Proof of Theorem 6

We will assume familiarity with the terminology in §§1.2, 1.4 of [6].

As in Lemma 1, we can assume that $\theta(x) > 0$ for all x. We can extend θ to any word $U = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \dots y_s^{\varepsilon_s}$ $(s > 0, y_i \in \mathbf{x}, \varepsilon_i = \pm 1, 1 \leq i \leq s)$ by $\theta(U) = \sum_{i=0}^{s} \varepsilon_i \theta(y_i)$.

Let $\mathbb{P}$ be a based connected spherical picture (with at least one disc) over $\mathcal{P}$, with global basepoint O, and basepoint O_Δ for each disc Δ. (Note that since R is not periodic, there will be just one basepoint for each disc.) We will also choose, for each region R, a point O_{R} in the interior of R.

We can relabel $\mathbb{P}$ to obtain a picture $\tilde{\mathbb{P}}$ over $\tilde{\mathcal{P}}$ as follows:

(a) For each region R, choose a tranverse path γ_{R} from O to O_{R}, and let U_{R} (a word on $\mathbf{x}$) be the label on the path γ_{R}. Then the *potential* $q(\mathsf{R})$ of R is $\theta(U_{\mathsf{R}})$. (This is independent of the choice of path γ_{R}, since $\theta(W) = 0$.)

(b) For an arc tranversely labelled $x \in \mathbf{x}$ say, relabel it by $(q(\mathsf{R}), x)$ where R is the region where the tranverse arrow on the arc begins.

(c) For a corner of a disc, with label $h \in H$ say, relabel the corner by (q, h), where q is the potential of the region in which the corner occurs.

For a disc Δ, let q_Δ be the potential of the region containing O_Δ. Then in the relabelled picture, Δ will be labelled by the path (q_Δ, R).

Let Θ be a *minimal* disc, that is, a disc such that $q_\Theta \leq q_\Delta$ for all discs Δ. Let m be the minimum value of ϕ_W^θ, and let e be one of the two distinct letters occurring at the unique minimum. Then in the path $(0, R)$ there is a unique edge labelled (m, e). Now Θ is labelled by (q_Θ, R) in $\tilde{\mathbb{P}}$, and thus there is a unique edge labelled $(m + q_\Theta, e)$ incident with Θ. This arc must intersect another disc $\Theta^{'}$, which must also be labelled by (q_Θ, R), but with the opposite orientation. Thus we obtain a dipole in $\tilde{\mathbb{P}}$ where Θ, $\Theta^{'}$ are the discs of the dipole. Reverting to $\mathbb{P}$, this dipole in $\tilde{\mathbb{P}}$ gives rise to a dipole in $\mathbb{P}$.

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Department of Mathematics, University of Glasgow, University Gardens, Glasgow G12 8QW, Scotland, United Kingdom

E-mail address: sjp@maths.gla.ac.uk