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COMPLETE MANIFOLDS WITH NONNEGATIVE CURVATURE OPERATOR

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ABSTRACT. In this short note, as a simple application of the strong result proved recently by Böhm and Wilking, we give a classification on closed manifolds with 2-nonnegative curvature operator. Moreover, by the new invariant cone constructions of Böhm and Wilking, we show that any complete Riemannian manifold (with dimension ≥ 3) whose curvature operator is bounded and satisfies the pinching condition $R \geq \delta \frac{\mathrm{tr}(R)}{2n(n-1)} \mathrm{I} > 0$, for some $\delta > 0$, must be compact. This provides an intrinsic analogue of a result of Hamilton on convex hypersurfaces.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold. The curvature operator of (M, g) lies in the subspace $S^2_B(\bigwedge^2 TM)$ of $S^2(\bigwedge^2 TM)$ cut out by the Bianchi identity. The decomposition $S^2_B(\bigwedge^2 TM) = \langle \mathrm{I} \rangle \oplus \langle \mathrm{Ric}_0 \rangle \oplus \langle \mathrm{W} \rangle$ splits the space of algebraic curvature operators into $O(n)$-invariant orthogonal irreducible subspaces. For an orthonormal basis ϕ_α (say $\phi_\alpha = e_i \wedge e_j$) of $\bigwedge^2 TM$ (which can be identified with $so(n)$), the Lie bracket is given in terms of

$$[\phi_\alpha, \phi_\beta] = c_{\alpha\beta\gamma}\phi_\gamma.$$

It is easy to check, by simple linear algebra, that $\langle [\phi, \psi], \omega \rangle = -\langle [\omega, \psi], \phi \rangle$. Here $\langle A, B \rangle = -\frac{1}{2}\,\mathrm{tr}(AB)$. This immediately implies that $c_{\alpha\beta\gamma}$ is anti-symmetric. If $A, B \in S^2(\bigwedge^2 TM)$ one can define

$$(A\#B)_{\alpha\beta} = \frac{1}{2} c_{\alpha\gamma\eta} c_{\beta\delta\theta} A_{\gamma\delta} B_{\eta\theta}.$$

It is easy to see that $A\#B$ is symmetric too. Also from the anti-symmetry of $c_{\alpha\beta\gamma}$, $A\#B = B\#A$.

In [BW], a remarkable algebraic identity was proved on how a linear transformation of $S^2_B(\bigwedge^2 TM)$ changes the quadratic form $Q(R) = R^2 + R^\#$. Böhm and Wilking then constructed a continuous *pinching family of invariant closed convex cones.* Using this construction they confirmed a conjecture of Hamilton stating that *on a compact manifold the normalized Ricci flow evolves a Riemannian metric with 2-positive curvature operator to a limit metric with constant sectional curvature.*

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Hence it gives a complete topological classification of compact manifolds with positive 2-positive curvature operator. In this short note, based on the strong result and the techniques of [BW], we give the classification for manifolds with 2-nonnegative curvature operators and an application of their invariant cone constructions to the compactness of Riemannian manifolds with pinching curvature operator.

2. A strong maximum principle

Let $(M, g(t))$ be a complete solution to Ricci flow such that there exists a constant A and the curvature tensor of $g(t)$ satisfies $|R_{ijkl}|^2(x,t) \le A$, for all $(x,t) \in M \times [0,T]$. In [H1], Hamilton proved that under the evolving normal frame the curvature tensor satisfies the following evolution equation.

Proposition 2.1 (Hamilton)**.**

$$\left(\frac{\partial}{\partial t} - \Delta\right) R = 2\left(R^2 + R^{\#}\right), \tag{2.1}$$

where $R^{\#} = R \# R$.

The following was observed for compact manifolds in [Chen, H3]. We spell out the argument for the noncompact case for the sake of the completeness.

Proposition 2.2. *The convex cone of the* 2*-nonnegative curvature operator is preserved under the Ricci flow.*

Proof. Let I be the identity of $S^2_B(\bigwedge^2 TM)$, which can be identified with the induced metric on $\bigwedge^2 TM$ (as a section of $\bigwedge^2 TM \otimes \bigwedge^2 TM$). We also denote the identity map of TM by id. With respect to the evolving normal frame we have that $\nabla \mathrm{I} = 0$ and $\frac{\partial}{\partial t}\mathrm{I} = 0$. Let $\psi(x,t) > 0$ be the fast growth function constructed in Lemma 1.1 of [NT1] satisfying $\frac{\partial}{\partial t}\psi - \Delta\psi \ge C_1\psi$. Here C_1 can be chosen as arbitrarily large as we wish. We shall consider $\tilde{R} = R + \epsilon\psi\,\mathrm{I}$ and show that $\tilde{R}$ is 2-positive for every (sufficiently small) ϵ. If not, by the boundedness of R and growth of ψ we know that it can only fail somewhere finite. Assume that t_0 is the first time $\tilde{R}$ fails to be 2-positive and it happens at some point x_0. Also assume an orthonormal basis ω_α (it may not be in the form of $e_i \wedge e_j$ as ϕ_α) such that $\tilde{R}$ is diagonal (so is R) with eigenvalues $\mu_1 \le \mu_2 \le \cdots \le \mu_N$, where $N = \frac{n(n-1)}{2}$. Parallel translate ω_α to a neighborhood of (x_0, t_0), and let $\tilde{R}_{\alpha\alpha} = \langle R(\omega_\alpha), \omega_\alpha\rangle$. Then at (x_0,t_0) we have, by the maximum principle, that

$$\begin{aligned}
0 &\ge \left(\frac{\partial}{\partial t} - \Delta\right)\left(\tilde{R}_{11} + \tilde{R}_{22}\right)\\
&\ge (R^2 + R^{\#})_{11} + (R^2 + R^{\#})_{22} + 2\epsilon C_1\psi\\
&= (\tilde{R}^2 + \tilde{R}^{\#})_{11} + (\tilde{R}^2 + \tilde{R}^{\#})_{22} + 2\epsilon C_1\psi\\
&\quad + \left(R^2 + R^{\#} - \tilde{R}^2 - \tilde{R}^{\#}\right)_{11} + \left(R^2 + R^{\#} - \tilde{R}^2 - \tilde{R}^{\#}\right)_{22}\\
&= \mu_1^2 + \mu_2^2 + \sum (c^2_{1\beta\gamma} + c^2_{2\beta\gamma})\mu_\beta\mu_\gamma + 2\epsilon C_1\psi\\
&\quad -\epsilon\psi\left((2\mathrm{Ric}\wedge \mathrm{id} + (n-1)\epsilon\psi\,\mathrm{I})_{11} + (2\mathrm{Ric}\wedge \mathrm{id} + (n-1)\epsilon\psi\,\mathrm{I})_{22}\right).
\end{aligned}$$

In the previous equation we have used Lemma 2.1 of [BW], which asserts that $R + R\#\,\mathrm{I} = \mathrm{Ric}\wedge \mathrm{id}$ (the use is not really necessary). Since $\mu_1 + \mu_2 \ge 0$ and $\mu_\gamma \ge 0$

for all $\gamma \geq 2$,

$$\sum(c^2_{1\beta\gamma}+c^2_{2\beta\gamma})\mu_\beta\mu_\gamma = 2\sum_{\gamma\geq 3}(c^2_{12\gamma}+c^2_{21\gamma})(\mu_1+\mu_2)\mu_\gamma + \sum_{\beta,\gamma\geq 3}(c^2_{1\beta\gamma}+c^2_{2\beta\gamma})\mu_\beta\mu_\gamma \geq 0.$$

Note also that at (x_0,t_0) we have that $\mu_{11}+\mu_{22}=0$, which implies that $R_{11}+R_{22}=-2\epsilon\psi$; then $2\epsilon\psi(x_0,t_0)\leq 2A$. Hence at (x_0,t_0) we have that

$$\begin{aligned} 0 &\geq \left(\frac{\partial}{\partial t}-\Delta\right)\left(\tilde{R}_{11}+\tilde{R}_{22}\right) \\ &\geq \mu_1^2+\mu_2^2+2\epsilon C_1\psi - 200nA\epsilon\psi. \end{aligned} \tag{2.2}$$

This is a contradiction if we choose $C_1 > 100nA$. □

By choosing the barrier function more carefully as in [NT2, N1] (see for example Theorem 2.1 of [N1]), we can have the following strong maximum principle.

Corollary 2.3. *Assume that $R(g(0))$ is 2-nonnegative and 2-positive somewhere. Then there exists $f(x,0)\geq \frac{1}{2}(\mu_1+\mu_2)(x,0)$, which satisfies $f(x,t)>0$ for $t>0$ and*

$$(\mu_1+\mu_2)(x,t)\geq f(x,t).$$

In particular, if $R(g(0))$ is 2-nonnegative and $(\mu_1+\mu_2)(x_0,t_0)=0$ for some $t_0\geq 0$, then $(\mu_1+\mu_2)(x,t)\equiv 0$ for all (x,t) with $t\leq t_0$. Moreover, $\mu_1(x,t)=\mu_2(x,t)=0$ for all (x,t) with $t\leq t_0$ and

$$\mathcal{N}_2(x,t)=span\{\omega_1,\omega_2\}$$

is a distribution on M which is invariant under the parallel translation. In particular, $(M,g(t))$, for $t\leq t_0$, has nonnegative curvature operator.

Proof. For the existence of f and the lower estimate of $\mu_1+\mu_2$ please see [NT2], Theorem 2.1. If $\mu_1+\mu_2$ achieves its minimum at (x_0,t_0), we can deduce from the lower estimate that $(\mu_1+\mu_2)(x,t)\equiv 0$ for all (x,t) with $t<t_0$ (which implies that it also holds at $t=t_0$ by continuity). Apply (2.2) with $\epsilon=0$. We can conclude that $M_i(x,t)\equiv 0$ for $i=1,2$. Hence $R\geq 0$. Please see Corollary 2.1 of [NT2] for the details on the part that $\mathcal{N}_2(x,t)$ is parallel. □

The above result implies the following classification of closed 2-nonnegative manifolds.

Corollary 2.4. *Assume that $R(g(0))$ is 2-nonnegative. Then for $t>0$, either the curvature operator $R(g(t))$ is 2-positive, or $R(g(t))\geq 0$. Hence Suppose (M^n,g_0) is a closed Riemannian manifold with 2-nonnegative curvature operator. Let $\tilde{g}(t)$ be the lift to the universal cover $\tilde{M}$ of the solution $g(t)$ to the Ricci flow with $g(0)=g_0$. Then for any $t>0$ we have that either $\left(\tilde{M}^n,\tilde{g}(t)\right)$ is a closed manifold with 2-positive curvature operator or it is isometric to the product of the following:*

1. *Euclidean space,*
2. *closed symmetric space,*
3. *closed Riemannian manifold with positive curvature operator,*
4. *closed Kähler manifold with positive curvature operator on real $(1,1)$-forms.*

Proof. It follows from the above corollary and Hamilton's classification result on the solutions with nonnegative curvature operator. See for example [CLN], Theorem 7.34. □

Topologically, it is now known, by [BW], that simply-connected 2-positive manifolds is sphere, and the Kähler manifold in the last case is biholomorphic to the complex projective space by the earlier result of Mori-Siu-Yau. The fact that the curvature operator of the evolving metrics becomes either 2-positive or nonnegative has been observed in [Chen]. However, in [Chen] there is no clear statement of the strong maximum principle, namely Corollary 2.3, on which the observation relies. If evoking Theorem 2.3 of [N1], the splitting result on solutions of Ricci flow on a complete Riemannian manifold with nonnegative curvature operator, we can write a similar statement even when M is not assume to be compact. However, in this case the Euclidean factor is only topological (not isometric). Also we do not know if a complete noncompact 2-positive Riemannian manifold is diffeomorphic to $\mathbb{R}^n$ or not.

3. Manifolds with pinched curvature

In [H2] Hamilton proved that any convex hypersurface (with dimension ≥ 3) in Euclidean space with second fundamental form $h_{ij} \geq \delta \frac{\mathrm{tr}(h)}{n}$ id must be compact. In [CZ], using the pre-established estimates of [Hu] and [Sh2], Chen and Zhu proved the following weak version of the above-mentioned Hamilton's result in terms of curvature operators. Namely, they proved that *if a complete Riemannian manifold (M^n, g) with bounded and (ϵ, δ_n)-pinched curvature operator (with $n \geq 3$) in the sense that*

$$|R_{\mathrm{W}}|^2 + |R_{\mathrm{Ric}_0}|^2 \leq \delta_n (1-\epsilon)^2 |R_I|^2 = \delta_n (1-\epsilon)^2 \frac{2}{n(n-1)} \mathrm{Scal}(R)^2$$

for $\epsilon > 0$, $\delta_3 > 0, \delta_4 = \frac{1}{5}, \delta_5 = \frac{1}{10}$ and $\delta_n = \frac{2}{(n-2)(n+1)}$, where R_{W}, R_{Ric_0} and R_{I} denote the Weyl curvature tensor, traceless Ricci part and the scalar curvature part, then M must be compact. The strong pinching condition was the one originally assumed in [Hu] to obtain various estimates and the smooth convergence result. It was also shown in [Hu] that it implies that $R \geq \epsilon R_I$. In [N3] the first author showed that the above result of Chen-Zhu can be shown by the blow-up analysis of [H3] and some nonexistence results on gradient steady and expanding solitons obtained in [N3]. (The detailed proof of these nonexistence results was submitted to the 2004 ICCM proceedings a while ago. See also the book [CLN].) With the help of a family of invariant cones constructed in [BW], we can now prove the following general result.

Theorem 3.1. *Let (M^n, g_0) be a complete Riemannian manifold with $n \geq 3$. Assume that the curvature operator of M is uniformly bounded ($|R_{ijkl}|(x) \leq A$) and satisfies that*

$$R \geq \delta R_{\mathrm{I}} > 0 \tag{3.1}$$

for some $\delta > 0$. Then (M, g_0) must be compact.

Recall that $R_{\mathrm{I}} = \frac{1}{n(n-1)} \mathrm{Scal}(R)\,\mathrm{I}$, where I is the identity of $S^2_B(so(n))$. Hence $R_{\mathrm{I}} > 0$ is equivalent to $\mathrm{Scal}(R) > 0$, and the above result is a natural analogue of the Hamilton result for hypersurfaces.

Proof. Let $g(t)$ be the solution to the Ricci flow with initial metric g_0 constructed by [Sh1]. First we show that if M is noncompact, $g(t)$ can be extended to a long-time solution defined on $M \times [0, \infty)$. In order to do that we first show that for sufficiently small $b > 0$, $R(g_0)$ lies inside the invariant cone constructed by Lemma 3.4 of [BW]. Recall from [BW] the linear transformation

$$\mathrm{l}_{a,b} : R \to R + 2(n-1)aR_{\mathrm{I}} + (n-2)bR_{\mathrm{Ric}_0}.$$

More precisely

$$\begin{aligned} \mathrm{l}_{a,b}(R) &= R + 2a\bar{\lambda}\,\mathrm{I} + 2b\,\mathrm{id}\wedge\mathrm{Ric}_0(R) \\ &= (1+2(n-1)a)R_{\mathrm{I}} + (1+(n-2)b)R_{\mathrm{Ric}_0} + R_{\mathrm{W}}. \end{aligned}$$

It is easy to see that $\mathrm{l}_{a,b}(S^2_B(so(n))) \subset S^2_B(so(n))$ and is invertible if $a \neq -\frac{1}{2(n-1)}$ and $b \neq -\frac{1}{n-2}$. Using this linear map and Theorem 2 of [BW], a pinching family of invariant convex cones are constructed. In particular, as one step of the construction, it was shown that

Lemma 3.2 (Böhm-Wilking). *For $b \in [0, \frac{1}{2}]$, let*

$$a = \frac{(n-2)b^2 + 2b}{2 + 2(n-2)b^2} \ \text{ and } p = \frac{(n-2)b^2}{1+(n-2)b^2}.$$

Then the set $\mathrm{l}_{a,b}(C(b))$, where

$$C(b) = \left\{ R \in S^2_B(so(n)) \,|\, R \geq 0,\ \mathrm{Ric} \geq p(b)\frac{\mathrm{tr(Ric)}}{n} \right\},$$

is invariant under the vector fields $Q(R)$. In fact for $b \in (0, \frac{1}{2}]$ it is transverse to the boundary of the set at all boundary points $R \neq 0$.

We claim that there exists $b > 0$ sufficiently small such that $R(g_0) \in \mathrm{l}_{a,b}(C(b))$, which is equivalent to the fact that $\mathrm{l}^{-1}_{a,b}(R(g_0)) \in C(b)$. For simplicity let $\tilde{R} = R(g_0)$, $\bar{\lambda}(\tilde{R}) = \frac{\mathrm{Scal}(\tilde{R})}{n}$ and $\mathrm{l} = \mathrm{l}_{a,b}$. Direct computation shows that

$$R := \mathrm{l}^{-1}(\tilde{R}) = \tilde{R}_{\mathrm{W}} + \frac{1}{1+2(n-1)a}\tilde{R}_{\mathrm{I}} + \frac{1}{1+(n-2)b}\tilde{R}_{\mathrm{Ric}_0},$$

which implies that

$$\mathrm{Ric}(\mathrm{l}^{-1}(\tilde{R})) = \frac{\bar{\lambda}(\tilde{R})}{1+2(n-1)a}\,\mathrm{id} + \frac{1}{1+(n-2)b}\mathrm{Ric}_0(\tilde{R})$$

and

$$\bar{\lambda}(R) := \frac{\mathrm{tr}(\mathrm{Ric}(\mathrm{l}^{-1}(\tilde{R})))}{n} = \bar{\lambda}(\tilde{R})\left(1 - \frac{2(n-1)a}{1+2(n-1)a}\right).$$

Let $\tilde{\lambda}_i$ be the eigenvalues of $\mathrm{Ric}_0(\tilde{R})$. Then by the assumption (3.1) we have that

$$\tilde{\lambda}_i + \bar{\lambda}(\tilde{R}) \geq \delta\bar{\lambda}(\tilde{R}). \tag{3.2}$$

Clearly we also have that

$$\tilde{\lambda}_i + \bar{\lambda}(\tilde{R}) \leq n\bar{\lambda}(\tilde{R}). \tag{3.3}$$

We first check that R satisfies the Ricci pinching condition. In fact if λ_i are the eigenvalues of $\mathrm{Ric}_0(R)$, from the above formulae we have that

$$\begin{aligned} -\lambda_i &= -\frac{1}{1+(n-2)b}\tilde{\lambda}_i \\ &\leq \frac{1-\delta}{1+(n-2)b}\bar{\lambda}(\tilde{R}) \\ &= (1-\delta)\frac{1+2(n-1)a}{1+(n-2)b}\bar{\lambda}(R). \end{aligned}$$

Then there exist $\delta_1 > 0$ and b_0 such that for all $b \in [0, b_0]$, $-\lambda_i \leq (1-\delta_1)\bar{\lambda}(R)$. Then we can find $b_1 \leq b_0$ such that for any $b \in [0, b_1]$, $p(b) \leq \delta_1$. Hence $R = \mathrm{l}_{a,b}^{-1}(\tilde{R})$ satisfies the pinching condition of $C(b)$. Now we check that $R = \mathrm{l}_{a,b}^{-1}(\tilde{R}) \geq 0$. Rewrite

$$R = \tilde{R} - \frac{2(n-1)a}{1+2(n-1)a}\tilde{R}_{\mathrm{I}} - \frac{(n-2)b}{1+(n-2)b}\tilde{R}_{\mathrm{Ric}_0}.$$

Noting that $a \to 0$ as $b \to 0$, we can find b_2 such that for any $b \in [0, b_2]$ we have that

$$R \geq \frac{\delta}{2}\tilde{R}_{\mathrm{I}} - \frac{(n-2)b}{1+(n-2)b}\tilde{R}_{\mathrm{Ric}_0}.$$

But the eigenvalue (with respect to $e_i \wedge e_j$, where $\{e_i\}$ is a basis of TM consisting of eigenvectors of $\mathrm{Ric}_0(\tilde{R})$) of the right-hand side operator can be computed as

$$\frac{\delta}{2}\frac{\bar{\lambda}(\tilde{R})}{n-1} - \frac{b}{1+(n-2)b}\left(\tilde{\lambda}_i + \tilde{\lambda}_j\right).$$

Using (3.3), the above can be bounded from below by

$$\bar{\lambda}(\tilde{R})\left(\frac{\delta}{2(n-1)} - \frac{2(n-1)b}{1+(n-2)b}\right) > 0$$

if b is close to 0. This shows that there exists $b_3 > 0$ such that for any $b \in (0, b_3]$, $R(g_0) \in \mathrm{l}_{a,b}(C(b))$.

Now the virtue of the proof of Theorem 5.1 in [BW], along with the short time existence result of [Sh1], shows that the Ricci flow has long-time solution. Otherwise, by Theorem 16.2 of [H3], we would end up with a blow-up solution, which is nonflat, noncompact, but whose curvature operator $R = R_{\mathrm{I}}$. In view of Schur's theorem, this is a contradiction. Note that $R(g_0) \in \mathrm{l}_{a,b}(C(b))$ allows us to apply the generalized pinching set construction (Theorem 4.1) from [BW], and since the evolving metric has positive curvature operator and the manifold is assumed to be noncompact, the injectivity radius always has a lower bound in terms of the size of the curvature. All these ingredients allow us to perform Hamilton's blow-up analysis [H3] (Theorem 16.2).

We continue to show that the extra assumption that M is noncompact will lead us to a contradiction by performing the singularity analysis of [H3] as $t \to \infty$. Note that for all t, $R(g(t))$ will stay in the cone $\mathrm{l}_{a,b}(C(b))$ for some fixed (but sufficiently small) b, by the tensor maximum principle, which can be verified in the same way as Proposition 2.2. Now we claim that the curvature of $g(t)$ satisfies that

$$\mathrm{Ric} \geq p\frac{\mathrm{tr(Ric)}}{n}\,\mathrm{id} \tag{3.4}$$

for some $p > 0$. Let $R^* = R(g(t))$. First, by Lemma 3.2 we know that $R(g(t)) \in \mathrm{l}_{a,b}(C(b))$ for some fixed small b. Thus we can find $R \in C(b)$ such that $\mathrm{l}_{a,b}(R) = R^*$. Now let $\bar{\lambda} = \frac{\mathrm{tr}(\mathrm{Ric}(R))}{n}$ and λ_i be the eigenvalues of $\mathrm{Ric}_0(R)$. By the assumption we have that $-\lambda_i \le (1-p)\bar{\lambda}$. Now we compute the Ricci curvature and its trace for R^*. By the definition of $\mathrm{l}_{a,b}$ we have that

$$\mathrm{Ric}(R^*) = \mathrm{Ric} + 2(n-1)a\bar{\lambda}\,\mathrm{id} + (n-2)b\mathrm{Ric}_0$$

and

$$\bar{\lambda}^* := \frac{\mathrm{tr}(\mathrm{Ric}(R^*))}{n} = \bar{\lambda}(1 + 2(n-1)a).$$

Letting λ_i^* be the eigenvalue of R^* we have that $\bar{\lambda}^* + \lambda_i^* = (1 + 2(n-1)a)\bar{\lambda} + (1 + (n-2)b)\lambda_i$. Therefore

$$\begin{aligned} -\lambda_i^* &= -(1 + (n-2)b)\lambda_i \\ &\le (1-p)(1 + (n-2)b)\bar{\lambda} \\ &= (1-p)\frac{1 + (n-2)b}{1 + 2(n-1)a}\bar{\lambda}^* \\ &\le (1-p)\bar{\lambda}^*. \end{aligned}$$

Here we have used the fact that $1 + 2(n-1)a = 1 + (n-1)\frac{(n-2)b^2 + 2b}{1 + (n-2)b} > 1 + (n-2)b$. This completes the proof of the claim (3.4).

Since for all $g(t)$, its Ricci curvature satisfies (3.4), this holds up on the blow-down/blow-up solutions, which after passing to its universal cover, are either a nonflat gradient steady soliton or a nonflat gradient expanding soliton, with nonnegative curvature operator, by results from [H3] (Theorem 16.5, Corollary 16.6). (See also [N2], Theorem 4.2 and [CZ].) This contradicts Corollary 3.1 of [N3]. $\square$

4. Discussions

In [W], the topology of so-called p-positive manifolds was studied. In view of the result of Böhm-Wilking and a result of Schoen-Yau [ScY] stating that any noncompact complete 3-manifold with positive Ricci curvature must be diffeomorphic to $\mathbb{R}^3$, it is reasonable to speculate that any noncompact complete Riemannian manifold M with 2-positive curvature operator must be diffeomorphic to $\mathbb{R}^n$. Professor Wilking informed us that one can show that M is aspherical with cyclic fundamental group. (See also [W].)

In [N3] we speculated that any complete Riemannian manifolds with positive pinched Ricci curvature must be compact. Theorem 3.1 confirms it under a stronger assumption on the curvature operator. The problem in full generality still remains unsettled.

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References

[BW] C. Böhm and B. Wilking, *Manifolds with positive curvature operator are space form*, preprint.

[Chen] H. Chen, *Pointwise* $\frac{1}{4}$*-pinched* 4*-manifolds*, Ann. Global Anal. Geom. **9**(1991), no. 2, 161–176. MR1136125 (93b:53028)

[CZ] B.-L. Chen and X.-P. Zhu, *Complete Riemannian manifolds with pointwise pinched curvature* Invent. Math. **140**(2000) no. 2, 423–452. MR1757002 (2001g:53118)

[CLN] B. Chow, P. Lu and L. Ni, *Hamilton's Ricci flow,* Graduate Studies in Mathematics, AMS Press, 2006, 608 pp.

[H1] R. Hamilton, *Four-manifolds with positive curvature operator* , J. Differenital. Geom. **24** (1986), 153–179. MR0862046 (87m:53055)

[H2] R. S. Hamilton, *Convex hypersurfaces with pinched second fundamental form,* Comm. Anal. Geom. **2**(1994), no. 1, 167–172. MR1312684 (95m:53078)

[H3] R. S. Hamilton, *Formation of singularities in the Ricci flow,* Surveys in Differential Geom. **2**(1995), 7–136. MR1375255 (97e:53075)

[Hu] G. Huisken, *Ricci deformation of the metric on a Riemannian manifold,* J. Differential Geom. **21**(1985), no. 1, 47–62. MR0806701 (86k:53059)

[N1] L. Ni, *Ricci flow and nonnegativity of sectional curvature*, Math. Res. Lett. **11**(2004), no. 5-6, 883–904. MR2106247 (2005m:53123)

[N2] L. Ni, *Monotonicity and Kähler-Ricci flow. Geometric evolution equations,* 149–165, Contemp. Math., **367**, Amer. Math. Soc., Providence, RI, 2005. MR2115758 (2006a:53082)

[N3] L. Ni, *Ancient solutions to Kähler-Ricci flow*, Math. Res. Lett. **12**(2005), no. 5-6, 633–653. MR2189227 (2006i:53097)

[NT1] L. Ni and L.-F. Tam, *Plurisubharmonic functions and the Kähler-Ricci flow,* Amer. J. Math. **125**(2003), no. 3, 623–654. MR1981036 (2004c:53101)

[NT2] L. Ni and L.-F. Tam, *Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature*, J. Differential Geom. **64**(2003), no. 3, 457–524. MR2032112 (2005a:32023)

[ScY] R. Schoen and S.-T. Yau, *Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature,* Seminar on Differential Geometry, pp. 209–228, Ann. of Math. Stud., **102**, Princeton Univ. Press, Princeton, N.J., 1982. MR0645740 (83k:53060)

[Sh1] W. X. Shi, *Deforming the metric on complete Riemannian manifolds,* J. Differential Geom. **30**(1989), 223–301. MR1001277 (90i:58202)

[Sh2] W. X. Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds,* J. Differential Geom. **30**(1989), 303–394. MR1010165 (90f:53080)

[W] H. Wu, *Manifolds of partially positive curvature,* Indiana Univ. Math. J. **36**(1987), no. 3, 525–548. MR0905609 (88k:53068)

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