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HYPERCYCLICITY IN OMEGA

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ABSTRACT. A sequence $\mathbb{T} = (T_n)$ of operators $T_n : \mathscr{X} \to \mathscr{X}$ is said to be hypercyclic if there exists a vector $x \in \mathcal{X}$, called hypercyclic for $\mathbb{T}$, such that $\{T_n x : n \geq 0\}$ is dense. A hypercyclic subspace for $\mathbb{T}$ is a closed infinite-dimensional subspace of, except for zero, hypercyclic vectors. We prove that if $\mathbb{T}$ is a sequence of operators on ω that has a hypercyclic subspace, then there exist (i) a sequence (p_n) of one variable polynomials p_n such that $(p_n(\xi)) \in \omega$ is hypercyclic for every fixed ξ and (ii) an operator $S : \omega \to \omega$ that maps nonzero vectors onto hypercyclic vectors for $\mathbb{T}$.

We complement earlier work of several authors.

1. INTRODUCTION

Throughout this paper, $\mathscr{X}$ denotes a separated real or complex locally convex space, and by $\mathscr{L}(\mathscr{X})$ we denote the algebra of continuous linear operators on $\mathscr{X}$. A sequence $\mathbb{T} = (T_n)_{n\geq 0}$ of operators $T_n : \mathscr{X} \to \mathscr{X}$ is said to be hypercyclic (or universal) if there exists a vector $x \in \mathscr{X}$, called hypercyclic for $\mathbb{T}$, such that $\{T_n x : n \geq 0\}$ is dense. A single operator $T \in \mathscr{L}(\mathscr{X})$ is said to be hypercyclic when the sequence (T^n) of powers is hypercyclic, and, accordingly, the hypercyclic vectors for T are those for (T^n). A hypercyclic subspace for $\mathbb{T}$ (T) is an infinite-dimensional closed subspace $H \subseteq \mathscr{X}$ whose nonzero vectors are hypercyclic for $\mathbb{T}$ (T).

The space ω, that is, the space $\mathbb{K}^{\mathbb{N}}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, $\mathbb{N} \equiv \{0, 1, ...\}$) provided with the product topology, is a reflexive nuclear Fréchet space. Moreover, ω is an important, and the "simplest", example of a separable Fréchet space that does not admit any continuous norm. (Another example is the space $\mathscr{E}(\mathbb{R}^n)$ of smooth functions, standard topology.)

Let us recall some history. In 1997 Ansari [1], and independently Bernal-González [2], proved that every infinite-dimensional separable Banach space admits a hypercyclic operator. Soon after that Leon-Saavedra and Montes-Rodríguez [12] improved the result by proving that every such Banach space supports an operator with a hypercyclic subspace. (The study of hypercyclic subspaces was initiated in [4, 13].) Some year later, Bonet and Peris [7] extended the result of Ansari and Bernal-González in another direction by proving that, more generally, every infinite-dimensional separable Fréchet space supports a hypercyclic operator. However, at

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this point it was not known to what degree such Fréchet spaces carry operators with hypercyclic subspaces. In [14] we proved that every infinite-dimensional separable Fréchet space that admits a continuous norm supports an operator with a hypercyclic subspace. The same result was independently obtained by Bernal-González in [3]. The question is thus: what can be said about Fréchet spaces that do not admit any continuous norm, such as ω? This question motivated Bès and Conejero in their recent work [5], where they obtained the following result.

Theorem 1 (Bès, Conejero)**.** *For any nonconstant polynomial p, $p(B)$ has a hypercyclic subspace $H \subseteq \omega$, where $B : (\alpha_n) \mapsto (\alpha_{n+1})$ is the unweighted backward shift on ω.*

(More generally they proved that for any countable family $\{p_n\}_{n\in\mathbb{N}}$ of nonconstant polynomials, the operators $p_n(B)$, $n \in \mathbb{N}$, have a common hypercyclic subspace. That any $p(B)$ in the theorem is hypercyclic was proved by Herzog et al. in [11].) The result is somewhat surprising in view of the fact that for any $|l| > 1$, lB acting on ℓ_p $(1 \le p < \infty)$ is hypercyclic but lacks hypercyclic subspaces [13, Theorem 3.4].

Now, the proofs in [14, 3] that $\mathscr{L}(\mathscr{F})$ contains an operator with a hypercyclic subspace for any infinite-dimensional separable Fréchet space $\mathscr{F}$ with a continuous norm both rest on the following criterion.

Proposition 1 (Bonet, Martínez-Giménez, Peris)**.** *Let $\mathbb{T} = (T_n)$ be a sequence of operators $T_n \in \mathscr{L}(\mathscr{F})$ on a separable Fréchet space $\mathscr{F}$ that admits a continuous norm. Assume there exist a sequence $(n_k) \subseteq \mathbb{N}$, for which $\mathbb{T}$ satisfies the Universality Criterion* (*UC*) [6, Definition 2.1]*, and an infinite-dimensional closed subspace $E \subseteq \mathscr{F}$ such that $T_{n_k} \to 0$ pointwise on E. Then $\mathbb{T}$ has a hypercyclic subspace.*

(Proposition 1 reduced to the single operator case was also obtained in [14].) The criterion can be somewhat improved in the Banach space setting; see [10]. However, a counterexample involving ω (see [6, Remark 3.6]) shows that Proposition 1 does not extend to Fréchet spaces without a continuous norm. The beautiful proof of Proposition 1 in [6] rests on the following quite remarkable proposition from [6] (see Theorems 3.1 and 3.5).

Proposition 2 (Bonet, Martínez-Giménez, Peris)**.** *Let $\mathbb{T} = (T_n)$ be a sequence of operators $T_n \in \mathscr{L}(\mathscr{F})$ on a separable Fréchet space $\mathscr{F}$ that admits a continuous norm. If $\mathbb{T}$ satisfies the UC for some sequence $(n_k) \subseteq \mathbb{N}$, then the sequence $L_{\mathbb{T}} \equiv (L_{T_n})$ of left-multipliers $L_{T_n} : S \mapsto T_n S$ acting on $\mathscr{L}(\mathscr{F})$ has a compact SOT-hypercyclic vector $K \in \mathscr{L}(\mathscr{F})$, and any such K maps nonzero vectors onto hypercyclic vectors for $\mathbb{T}$.* (*SOT = strong operator topology*)*.*

(Recall that the SOT is the topology of pointwise convergence.) Again, Proposition 2 does not remain true if we do not assume $\mathscr{F}$ has a continuous norm [6, Example 3.2]. The history for Proposition 2 is as follows. Chan initiated in [8] the study of hypercyclitity for left-multipliers; his results in [8] are restricted to the Hilbert space setting. Later he pursued his work in [9], together with Taylor, by extending the study to Banach spaces. They obtained essentially the analogue of Proposition 2 for Banach spaces; see [9, Corollary 6].

In this note we shall complement Proposition 2 by proving the following.

Theorem 2. *Let $\mathbb{T} = (T_n)$ be a sequence of operators $T_n \in \mathscr{L}(\omega)$. Assume $\mathbb{T}$ has a hypercyclic subspace $H \subseteq \omega$. Then there is a one-to-one operator $S \in \mathscr{L}(\omega)$ with $\operatorname{Im} S \subseteq H$, and thus, S maps nonzero vectors onto hypercyclic vectors for $\mathbb{T}$.*

(Noteworthy is that Theorem 2 holds true if we replace ω by a separable Fréchet space $\mathscr{F}$ admitting a continuous norm [15], in fact, we may then, as in Proposition 2, choose S to be compact. Thus we are led to the questions: Does Theorem 2 hold for an arbitrary separable Fréchet space, and when can we choose S to be compact?)

An immediate consequence of Theorems 2 and 1 is:

Corollary 1. *For any nonconstant polynomial p, there is an operator $S \in \mathscr{L}(\omega)$ that maps nonzero vectors onto hypercyclic vectors for $p(B)$.*

By virtue of Theorem 2, and via a kernel theorem for the operator algebra $\mathscr{L}(\omega)$, we also obtain the following result.

Theorem 3. *Assume the sequence $\mathbb{T}$ in $\mathscr{L}(\omega)$ has a hypercyclic subspace. Then there exists a sequence (p_n) of polynomials p_n such that $(p_n(\xi)) \in \omega$ forms a hypercyclic vector for $\mathbb{T}$ for all $\xi \in \mathbb{K}$.*

Corollary 2. *For any nonconstant polynomial p, there exists a sequence (p_n) of polynomials such that $(p_n(\xi)) \in \omega$ forms a hypercyclic vector for $p(B)$ for all $\xi \in \mathbb{K}$.*

2. Results

We start by proving Theorem 2.

Proof of Theorem 2. Let $H \subseteq \omega$ be a hypercyclic subspace for $\mathbb{T}$. For each $n \geq 0$, let π_n denote the projector on H onto the first $n+1$ coordinates, i.e. $\pi_n(\alpha_k) \equiv (\alpha_0, ..., \alpha_n, 0, ...)$. Since H is infinite-dimensional but $\operatorname{Im} \pi_n$ finite dimensional, $\ker \pi_n \neq \{0\}$ for all n. Thus, for any $n \geq 0$ we can find an $h \in H \setminus \{0\}$ whose first $n+1$ coordinates are zero.

Now, choose $h = (h_n) \in H \setminus \{0\}$ arbitrarily, and define $n_0 \equiv \min\{n : h_n \neq 0\}$ and $e_{n_0} \equiv h$. Next we choose an $h = (h_n) \in \ker \pi_{n_0} \setminus \{0\}$ and define $n_1 \equiv \min\{n : h_n \neq 0\}$ and $e_{n_1} \equiv h$. Continuing in this way we get a strictly increasing sequence $(n_k) \subseteq \mathbb{N}$ and elements $e_{n_k} = (e_{n_k,n})_n \in H$ such that $e_{n_k,n_k} \neq 0$ and $e_{n_k,n} = 0$ for all $n < n_k$.

We are now ready to construct the required operator $S \in \mathscr{L}(\omega)$. We define $S(\alpha_k) \equiv \sum_{k \geq 0} \alpha_k e_{n_k}$. By our construction it follows that the series converges for any $\alpha = (\alpha_k) \in \omega$, so S is well defined. Hence, as a pointwise limit of operators in $\mathscr{L}(\omega)$, $S \in \mathscr{L}(\omega)$. Further, since $e_{n_k} \in H$ and H is closed, $\operatorname{Im} S \subseteq H$. Thus it remains only to prove that S is one-to-one. Again this is an easy consequence of the properties of the elements e_{n_k}. Indeed, let c_n denote the map $\omega \ni (\alpha_k) \mapsto \alpha_n$. Assume $S\alpha = \sum \alpha_k e_{n_k} = 0$. Then $0 = c_{n_0}(S\alpha) = \alpha_0 e_{n_0,n_0}$, so $\alpha_0 = 0$. This implies in turn that

$$0 = c_{n_1}(S\alpha) = \alpha_0 e_{n_0,n_1} + \alpha_1 e_{n_1,n_1} = \alpha_1 e_{n_1,n_1},$$

and so $\alpha_1 = 0$. Continuing in this way (or by induction), we conclude that $\alpha = 0$ and hence S is one-to-one. $\square$

From the proof it follows that if $\{p_n\}_{n \in \mathbb{N}}$ is a denumerable set of nonconstant polynomials, then we can find an $S \in \mathscr{L}(\omega)$ that maps nonzero vectors onto common hypercyclic vectors for the operators $p_n(B)$, $n \in \mathbb{N}$. Indeed, we only have to choose H as a common hypercyclic subspace for the operators $p_n(B)$, $n \in \mathbb{N}$, which exists by the remark following Theorem 1.

Next, in order to prove Theorem 3 it is convenient to work with the ring (and algebra) $\mathscr{W} \equiv \mathbb{K}[[z]]$ of formal power series $\sum_{n\geq 0} f_n z^n$. We equip $\mathscr{W}$ with the product topology so that $\mathscr{W} \simeq \omega$. The polynomial ring (and algebra) $\mathbb{K}[\xi]$ is denoted by $\mathscr{P}$. We put $\mathscr{W}$ and $\mathscr{P}$ into duality by $\langle f, p\rangle \equiv \sum_{n\geq 0} n! f_n p_n$. It follows that $\mathscr{W}' = \mathscr{P}$ in this way. For fixed $\xi \in \mathbb{K}$ we define $e_\xi = e_\xi(z) \equiv e^{z\xi} = \sum \xi^n z^n/n! \in \mathscr{W}$. In particular e_ξ is a unit with inverse $e_{-\xi}$. Note also that $\langle e_\xi, p\rangle = p(\xi)$ for any $p \in \mathscr{P}$, hence $\{e_\xi : \xi \in \mathbb{K}\}$ is total in $\mathscr{W}$. If $f = \sum f_n z^n \in \mathscr{W}$, $f(0)$ denotes f_0 (as usual) so that $f_n = D^n f(0)/n!$ for all $n \geq 0$.

Definition 1. By $\mathfrak{S}$ we denote the ring of formal power series $\sum_{n\geq 0} p_n(\xi) z^n$ with polynomial coefficients $p_n(\xi) \in \mathscr{P}$.

Proposition 3 (Kernel Theorem)**.** *The map $T \mapsto P(z,\xi) \equiv e_{-\xi} T e_\xi(z)$ defines a one-to-one correspondence between $\mathscr{L}(\mathscr{W})$ and $\mathfrak{S}$. We write $T = P(z, D)$ and have $P(z,D)f = \sum z^n p_n(D) f$ if $P = \sum p_n(\xi) z^n$.*

Proof. It is easily checked that $P(z,D)f \equiv \sum z^n p_n(D) f$ defines a continuous linear operator on $\mathscr{W}$ for any $P = \sum z^n p_n(\xi) \in \mathfrak{S}$. Hence we only have to prove that if $T \in \mathscr{L}(\mathscr{W})$, then there is a unique $P \in \mathfrak{S}$ with $T = P(z,D)$. So define $P(z,\xi) \equiv e_{-\xi} T e_\xi(z) = \sum z^n p_n(\xi)$. Assume for a moment that we can prove that $P \in \mathfrak{S}$, i.e., that $p_n(\xi)$, $n = 0, 1, ...$, are indeed polynomials in ξ. Then it follows that $T = P(z,D)$ since, for fixed ξ, $Te_\xi = e_\xi P(\cdot, \xi)$ and

$$P(z,D)e_\xi = \sum z^n p_n(D) e_\xi = \sum z^n p_n(\xi) e_\xi = e_\xi P(\cdot,\xi).$$

Hence $T = P(z,D)$ on the total set $\{e_\xi : \xi \in \mathbb{K}\}$, so $T = P(z,D)$. From $P(z,D)e_\xi = e_\xi P(\cdot,\xi)$ we also conclude that $P \in \mathfrak{S}$ is unique. Hence, it remains only to prove that $P(z,\xi) \equiv e_{-\xi} T e_\xi(z) \in \mathfrak{S}$. To do so it is enough to prove the following.

Sublemma. *For any $T \in \mathscr{L}(\mathscr{W})$ we have that $P(z,\xi) \equiv Te_\xi(z) \in \mathfrak{S}$.*

Proof of the Sublemma. We derive

$$\begin{aligned} Te_\xi = \sum_{n\geq 0} \frac{\xi^n}{n!} T(z^n) &= \sum_{n\geq 0} \frac{\xi^n}{n!} \sum_{m\geq 0} D^m T(z^n)(0) \frac{z^m}{m!} \\ &= \sum_{m\geq 0} \frac{z^m}{m!} \sum_{n\geq 0} D^m T(z^n)(0) \frac{\xi^n}{n!}. \end{aligned}$$

Hence we must prove that $\sum_{n\geq 0} D^m T(z^n)(0)\xi^n/n!$, $m = 0, 1, ...$, are polynomials, i.e. for fixed m, $D^m T(z^n)(0) = 0$ for all but a finite number of n. To this end we apply the duality between $\mathscr{W}$ and $\mathscr{P}$, and the fact that transpose of D^m acting on $\mathscr{W}$ and $\mathscr{P}$ is, respectively, multiplication by ξ^m and z^m, which gives

$$D^m T(z^n)(0) = \langle D^m T(z^n), 1\rangle = \langle T(z^n), \xi^m\rangle = \langle z^n, {}^tT(\xi^n)\rangle = D^n\, {}^tT(\xi^m)(0).$$

This shows the required conclusion since ${}^tT(\xi^m)$ is a polynomial. Hence the Sublemma and thus the proposition. $\square$

Theorem 4. *Let $\mathbb{T}$ be a sequence in $\mathscr{L}(\mathscr{W})$ and assume that $\mathbb{T}$ has a hypercyclic subspace. Then there is an $h = \sum p_n(\xi) z^n \in \mathfrak{S}$ such that $h(\cdot,\xi) \in \mathscr{W}$ is hypercyclic for $\mathbb{T}$ for every fixed $\xi \in \mathbb{K}$.*

Proof. By Theorem 2 and the fact that $\omega \simeq \mathscr{W}$, we can find an operator $S \in \mathscr{L}(\mathscr{W})$ that maps nonzero vectors onto hypercyclic vectors for $\mathbb{T}$. Thus $h(\cdot, \xi) \equiv Se_\xi$ is hypercyclic for any fixed ξ and, by Proposition 3 (see the Sublemma), $h \in \mathfrak{S}$. $\square$

If we translate Theorem 4 to ω we obtain Theorem 3. Indeed, consider the (linear) isomorphism $i : \omega \to \mathscr{W}$ defined by $u_n \mapsto z^n/n!$, where (u_n) denotes the canonical "unit" basis of ω. Then $i^{-1}h(\cdot, \xi) = \sum u_n n! p_n(\xi) = (n! p_n(\xi))$ from which Theorem 3 follows. We conclude by also discussing what the Kernel Theorem, Proposition 3, corresponds to for ω in this way. We equip ω with the algebra structure induced by $\mathscr{W}$ so that i becomes an algebra isomorphism. This means that the algebra on ω is defined by $u_n \cdot u_m = \binom{m+n}{n} u_{n+m}$. Next we observe that the backward shift $B \in \mathscr{L}(\omega)$ corresponds to the differentiation operator D on $\mathscr{W}$, i.e. $iB = Di$ and thus $ip(B) = p(D)i$ for any polynomial p. (Parenthetically this shows, in view of Theorem 1, that $p(D) \in \mathscr{L}(\mathscr{W})$ has a hypercyclic subspace for any nonconstant $p \in \mathscr{P}$.) In view of all this we conclude: To every $T \in \mathscr{L}(\omega)$ there is a unique sequence (p_n) of polynomials such that $Tf = \sum u_n p_n(B) f = \sum u_n \cdot p_n(B) f$ and, conversely, any such sequence (p_n) defines an operator $T \in \mathscr{L}(\omega)$ by $f \mapsto \sum u_n p_n(B) f$.

References

[1] S. Ansari, *Existence of hypercyclic operators on topological vector spaces*, J. Funct. Anal. **148** (1997), 384-390. MR1469346 (98h:47028a)

[2] L. Bernal-González, *On hypercyclic operators in Banach spaces*, Proc. Amer. Math. Soc. **127** (1999), 1003-1010. MR1476119 (99f:47010)

[3] L. Bernal-González, *Hypercyclic subspaces in Fréchet spaces*, Proc. Amer. Math. Soc., to appear.

[4] L. Bernal-González and A. Montes-Rodríguez, *Non-finite dimensional closed vector spaces of universal functions for composition operators*, J. Approx. Theory **82** (1995), 375-391. MR1348728 (96f:30034)

[5] J. Bès and A. Conejero, *Hypercyclic subspaces in omega*, J. Math. Anal. Appl., to appear.

[6] J. Bonet, F. Martinez and A. Peris, *Universal and chaotic multipliers on spaces of operators*, J. Math. Anal. Appl. **297** (2004), 599-611. MR2088683 (2005g:47006)

[7] J. Bonet and A. Peris, *Hypercyclic operators on non-normable Fréchet spaces*, J. Funct. Anal. **159** (1998), 587-595. MR1658096 (99k:47044)

[8] K. C. Chan, *Hypercyclicity of the operator algebra for a separable Hilbert space*, J. Operator Theory **42** (1999), 231-244. MR1716973 (2000i:47066)

[9] K. C. Chan and R. D. Taylor, *Hypercyclic subspaces of a Banach space*, Integr. equ. oper. theory **41** (2001), 381-388. MR1857797 (2002g:47011)

[10] M. González, F. Leon-Saavedra and A. Montes-Rodríguez, *Semi-Fredholm Theory: Hypercyclic and Supercyclic Subspaces*, Proc. London Math. Soc. **81** (3) (2000), 169-189. MR1757050 (2001g:47013)

[11] G. Herzog and R. Lemmert, *Über Endomorphismen mit den dichten Bahnen*, Math. Z **213** (1993), 473-477. MR1227494 (94h:47052)

[12] F. Leon-Saavedra and A. Montes-Rodríguez, *Linear Structure of Hypercyclic Vectors*, J. Funct. Anal. **148** (1997), 524-545. MR1469352 (98h:47028b)

[13] A. Montes-Rodríguez, *Banach spaces of hypercyclic vectors*, Michigan Math. J. **43** (1996), 419-436. MR1420585 (98g:47027)

[14] H. Petersson, *Hypercyclic subspaces for Fréchet space operators*, J. Math. Anal. Appl. **319** (2006), 764-782.

[15] H. Petersson, *Complemented hypercyclic subspaces*, to appear in Houston J. Math.

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