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# ALMOST COMPLEX RIGIDITY OF THE COMPLEX PROJECTIVE PLANE

BENJAMIN MCKAY

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Abstract. An isomorphism of symplectically tame smooth pseudocomplex structures on the complex projective plane which is a homeomorphism and differentiable of full rank at two points is smooth.

## 1. Introduction

A longstanding concern of many mathematicians is to understand global features of elliptic partial differential equations using global geometry and weak local estimates. This paper employs global symplectic geometry and weak local estimates to prove smoothness of isomorphisms of pseudocomplex structures on the complex projective plane.

A *pseudocomplex structure* on a 4-dimensional manifold $M$ is a choice, for any smooth complex-valued local coordinate system $z, w : M \to \mathbb{C}$, of a system of partial differential equations

$$\frac{\partial w}{\partial \bar{z}} = F\left(z, \bar{z}, w, \bar{w}, \frac{\partial w}{\partial z}, \frac{\partial \bar{w}}{\partial \bar{z}}\right)$$

(with $F$ a smooth function), so that under smooth changes of coordinates, the partial differential equations are equivalent, i.e. have the same local solutions. An example: every almost complex structure has Cauchy-Riemann equations for pseudoholomorphic curves, giving a pseudocomplex structure. If $E$ is the pseudocomplex structure, the local solutions of the partial differential equations are smooth surfaces in the manifold $M$, called *$E$-curves*. The concept is due to Gromov [3], p. 342. McKay [4, 5] analyzed $E$-curves locally and globally.

A *morphism* of pseudocomplex structures $E_0$ on $M_0$ and $E_1$ on $M_1$ is a map $M_0 \to M_1$ carrying $E_0$-curves to $E_1$-curves. A pseudocomplex structure $E$ on a 4-dimensional manifold $M$ is *tamed* by a symplectic structure $\omega$ on $M$ if $\omega > 0$ on all $E$-curves; it is *tame* if it is tamed by some symplectic structure.

**Theorem 1** (Main theorem)**.** *An isomorphism of smooth tame pseudocomplex structures on the complex projective plane which is a homeomorphism, and differentiable of full rank at two points, is smooth.*

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**Corollary 1.** *A continuous biholomorphic map between smooth tame almost complex structures on the complex projective plane, which is differentiable of full rank at two points, is smooth.*

**Corollary 2.** *A biLipschitz isomorphism of smooth tame pseudocomplex structures on the complex projective plane is smooth.*

The proof is purely geometric, using elementary projective plane geometry coupled with arguments from Gromov [3]. By comparison, the strongest known result for almost complex 4-dimensional manifolds is due to Coupet, Gaussier and Sukhov [1], proving smoothness assuming continuous differentiability (although their result extends to a pseudoconvex boundary).

## 2. Generalities on pseudocomplex structures

See McKay [4, 5] for an extensive discussion of pseudocomplex structures. Let us give a slightly more geometric definition of pseudocomplex structures, following Gromov.

**Definition 1.** A *pseudocomplex structure* on a four-dimensional manifold $M$ is a choice of a smooth immersed submanifold $E \subset \widetilde{\mathrm{Gr}}\,(2, TM)$ inside the bundle of oriented 2-planes in the tangent spaces of $M$, so that the map $E \to M$ is a submersion, and so that the requirement that a surface $C \subset M$ have tangent planes belonging to $E$ be equivalent in local coordinates to a determined elliptic system of partial differential equations.

As Gromov points out, and the previously cited articles [4, 5] prove, the ellipticity requirement can be expressed neatly in terms of the canonical $(2, 2)$ signature conformal structure of the Grassmannians $\widetilde{\mathrm{Gr}}\,(2, T_m M) = \widetilde{\mathrm{Gr}}\,(2, \mathbb{R}^4)$ (invariant under $\mathrm{GL}\,(4, \mathbb{R})$) as the requirement that the fibers $E_m \subset \widetilde{\mathrm{Gr}}\,(2, T_m M)$ of $E$ are definite surfaces in that conformal structure, i.e. nowhere tangent to null directions. We will refer to a manifold $M$ with pseudocomplex structure $E$ as a pseudocomplex manifold.

**Definition 2.** An oriented immersed surface $C \subset M$ in a pseudocomplex manifold is called an *$E$-curve* if its tangent spaces belong to $E$.

There are two natural notions of morphism of a pseudocomplex structure: we could require, as we have chosen to do, only that a morphism $(M_0, E_0) \to (M_1, E_1)$ take $E_0$-curves to $E_1$-curves, or we could require that it also preserve orientations. We will ignore the orientations. For example, the automorphism group of the usual complex structure on $\mathbb{CP}^2$ includes conjugate holomorphic maps, given in local affine charts by complex conjugation.

**Theorem 2** (McKay [4, 5])**.** *Let $E \subset \widetilde{\mathrm{Gr}}\,(2, TM)$ be a pseudocomplex structure. Pick a point $e \in E$ and let $m \in M$ be its image under $E \to \widetilde{\mathrm{Gr}}\,(2, TM) \to M$. To each point $e \in E$ there is associated smoothly a choice of* osculating complex structure *$J_e : T_m M \to T_m M$. The 2-plane $e$ is a $J_e$-complex line. In particular, if $C$ is an $E$-curve, then at each point $m \in C$, $e = T_m C$ is invariant under $J_e$. In particular, $C$ inherits the structure of a Riemann surface. The osculating complex structure is uniquely determined by the requirement that the linearization of the elliptic equation for $E$-curves at any $E$-curve $C$ is of the form*

$$\bar{\partial}\sigma = \tau\,(\sigma)$$

*where $\sigma$ is a section of the normal bundle of $C$, $\tau$ is a $J_e$-conjugate linear map, and $J_e$ is used to fix the meaning of $\bar{\partial}$. (This $\tau$ is construced from the torsion functions $T_2, T_3$ of McKay* [5]. *We will refer to a solution $\sigma$ of this equation as a* pseudoholomorphic *normal vector field.) This linearized equation on sections of the normal bundle makes the $E$-curve $C$ into a Riemann surface, and the normal bundle into a holomorphic line bundle.*

*Proof.* Most of this is proven in McKay [5]. The remarks on the linearization can be easily checked from the structure equations in that paper. Let's first recall the notation. To each choice of pseudocomplex structure $E \subset \widetilde{\mathrm{Gr}}\,(2, TM)$, in that paper a bundle $B \to E$ is constructed, which is a principal $G$-subbundle of the frame bundle $FE$, where $FE \to E$ is the bundle of choices of bases for tangent spaces of $E$, and the group $G$ is the group of linear transformations of the form

$$g = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & ac^{-1} \end{pmatrix}$$

where $a, b, c, d, e$ are complex numbers. This group is identified with a subgroup of $\mathrm{GL}\,(3, \mathbb{C}) \subset \mathrm{GL}\,(6, \mathbb{R})$, and acts on bases for tangent spaces of $E$ accordingly. On the bundle $B$, we constructed explicitly a collection of complex-valued 1-forms $\theta, \omega, \pi$, which transform under $G$-action on $B$ by (if we write $r_g$ for right action of $g \in G$ on $B$):

$$r_g^* \begin{pmatrix} \theta \\ \omega \\ \pi \end{pmatrix} = g^{-1} \begin{pmatrix} \theta \\ \omega \\ \pi \end{pmatrix}.$$

We then differentiate these 1-forms, and demonstrate the existence (and nonuniqueness) of 1-forms $\alpha, \beta, \gamma, \delta, \epsilon$ and existence and uniqueness of 1-forms $\sigma, \tau_1, \tau_2, \tau_3$ for which

$$d \begin{pmatrix} \theta \\ \omega \\ \pi \end{pmatrix} = - \begin{pmatrix} \alpha & 0 & 0 \\ \beta & c & 0 \\ \delta & \epsilon & \alpha - \gamma \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \omega \\ \pi \end{pmatrix} - \pi \wedge \begin{pmatrix} \omega \\ \sigma \\ 0 \end{pmatrix} + \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \wedge \bar{\theta}$$

(note the similarity between the $\alpha, \beta$, etc. and the $a, b$, etc.), with the further equations

$$\sigma = S_1 \bar{\theta} + S_2 \bar{\omega}$$

and

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} T_2 & T_3 \\ U_2 & U_3 \\ V_2 & V_3 \end{pmatrix} \begin{pmatrix} \bar{\omega} \\ \bar{\pi} \end{pmatrix}.$$

If a surface $C \subset M$ is an $E$-curve, then when we pull back the bundle $B \to E$ over the lift $C \to E$, we obtain the equations $\theta = 0$, $\pi = p\omega$, for some function $p$ on the pull-back bundle. We can adapt frames, i.e. define a uniquely determined smooth principal subbundle, by imposing the equation $p = 0$.

A one-parameter family of $E$-curves, say with parameter $t$, will lift to $E$ so that on the pull-back bundle of $B$, $\theta = s\,dt$. This $s$ function on the bundle of adapted frames over the total space of the family of $E$-curves varies under the representation of the structure group (as the reader can check) which is the defining representation of the normal bundle. We linearize just by taking the exterior derivative, and then setting $t = 0$. We can always adapt frames (as in that paper) to arrange

$\pi = 0$, so that we find by taking the exterior derivative: $\bar{\partial} s + \tau_1 \bar{s} = 0$. Check that this equation descends from the bundle of adapted frames down to the $E$-curve itself. The complex structure of the $E$-curve is determined by the characteristic variety of this equation—the characteristic variety is precisely the union of the holomorphic and conjugate holomorphic tangent bundles. The orientation of the $E$-curve picks out the holomorphic tangent bundle. By following Duistermaat [2], we find the complex structure $J_e$ completely determined, and indeed the structure of holomorphic line bundle on the normal bundle, from the form of the differential equation above for pseudoholomorphic sections of the normal bundle. Note that the torsion prevents the first-order deformations of the $E$-curve from being identified with the holomorphic sections of this line bundle. Nonetheless, as Duistermaat points out, we can carry out Riemann–Roch/Chern class theory of these $\bar{\partial}\sigma + \tau(\sigma) = 0$ equations just as for Cauchy–Riemann equations, since the behaviour near zeroes of $\sigma$ is identical. □

**Definition 3.** A pseudocomplex structure $E \subset \widetilde{\mathrm{Gr}}(2, TM)$ is *proper* if its fibers $E_m \subset \widetilde{\mathrm{Gr}}(2, T_m M)$ are compact (in other words, if the map $E \to M$ is a proper map).

All pseudocomplex structures will be assumed henceforth to be proper. Improper ones seem to be of no interest.

## 3. Duality of pseudocomplex structures

We need to recall some results from McKay [4, 5] concerning tame pseudocomplex structures on the complex projective plane. (As always, we are assuming that our pseudocomplex structures are proper.)

**Theorem 3** (Taubes [7])**.** *There is a unique symplectic structure on the complex projective plane, up to symplectomorphism and scaling by a constant.*

**Theorem 4** (Gromov [3], McKay [4])**.** *Pick $E$ to be any tame pseudocomplex structure on $M = \mathbb{CP}^2$. Define an $E$-line to be a smooth immersed sphere which is an $E$-curve and lives in the homology class generating $H_2\left(\mathbb{CP}^2\right)$. Every $E$-line is embedded. Any pair of $E$-lines meet transversely at a unique point. (The proof is a Chern class argument, similar to those below.) Deformations of $E$-lines are unobstructed (proof: same elliptic theory as for the usual complex structure, following Fredholm). Let $M^* = M^*(E)$ be the set of $E$-lines. Then $M^*$ is a smooth manifold, diffeomorphic to the complex projective plane. (We will see below the technique for proving this: affine coordinates.) There is a unique $E$-line tangent to any given 2-plane in $M$ which belongs to $E \subset \widetilde{\mathrm{Gr}}(2, TM)$. The map $E \to M^*$, given by taking a 2-plane to the $E$-line tangent to it, is a smooth fiber bundle. Write the map $E \to M$ as $\pi : E \to M$. Consider the 4-plane field $\Theta$ on $E$ determined by the equations $\Theta_e = \pi'(e)^{-1} e$. Write the map $E \to M^*$ as $\pi^* : E \to M^*$. The map*

$$E \to \widetilde{\mathrm{Gr}}(2, TM^*)$$

*given by taking each 2-plane $e \in E$ to the 2-plane $(\pi^*)'(e) \cdot \Theta_e \subset TM^*$ is a pseudocomplex structure on $M^*$. An $E$-line of $M^*$ is precisely a set of points of $M^*$ consisting of just those $E$-lines in $M$ passing through a point of $M$, i.e. points of $M$ are lines of $M^*$ and vice versa.*

## 4. Affine coordinates

Fix a smooth tame pseudocomplex structure $E$ on $\mathbb{CP}^2$. Consider two distinct $E$-lines in $\mathbb{CP}^2$, say $X$ and $Y$. They must meet at a single point transversely. Now pick any points $\infty_X$ in $X$ and not in $Y$, and $\infty_Y$ in $Y$ and not in $X$. Given any point $p$ of the projective plane which is not one of these two points, draw the line through $\infty_X$ and $p$, and find that it strikes $Y$ at a single point $y(p) \in Y$. Similarly the line through $\infty_X$ and $p$ strikes $Y$ at a single point $x(p) \in X$. In this way, we smoothly map $\alpha_{XY} : \mathbb{CP}^2 \setminus \{\infty_X, \infty_Y\} \to X \times Y$.

**Lemma 1.** *Let $\infty$ be the line through $\infty_X$ and $\infty_Y$. This map $\alpha_{XY}$, called an* affine chart*, is a local diffeomorphism*

$$\alpha_{XY} : \mathbb{CP}^2 \setminus \infty \to X \times Y.$$

*Proof.* We can differentiate this map, because the infinitesimal motions of a line are governed by pseudoholomorphic sections of the normal bundle (holomorphic in the osculating almost complex structure), and the deformation theory is unobstructed by Chern class calculation. We need to show that if we move the point $p$ infinitesimally, i.e. with a nonzero tangent vector $v \in T_p\mathbb{CP}^2$, then one of the points $x(P), y(P)$ must move by a nonzero tangent vector. Let $L_X$ be the line through $\infty_Y$ and $p$, and let $L_Y$ be the line through $\infty_X$ and $p$. We form the pseudoholomorphic normal vector fields $A_X$ on $L_X$, and $A_Y$ on $L_Y$, determined by requiring that $A_X = 0$ at $\infty_Y$ and $A_X(p) = v$ modulo $T_pL_X$, and similarly $A_Y(p) = v$ modulo $T_pL_Y$. Existence of $A_X$ comes again from Riemann–Roch, following Duistermaat. Imagine that $A_X$ vanishes at $x(p)$ and that $A_Y$ vanishes at $y(p)$. By Chern class count, if $A_X$ vanishes anywhere other than $\infty_X$, it vanishes everywhere. But it vanishes at $x(P)$, so it must vanish everywhere, so $v$ is tangent to $L_X$. Similarly, $v$ is tangent to $L_Y$. So $L_X = L_Y$, and the line through $p$ and $\infty_X$ must contain $\infty_Y$. □

Similarly, we can take a *dual affine chart*, defined most simply by taking the same construction as above, and assigning to each line $Z$ not equal to $X$ or $Y$ its point $x(Z) \in X$ of intersection with $X$, and its point $y(Z) \in Y$ of intersection with $Y$.

**Lemma 2.** *This map, called a* dual affine chart*, is a local diffeomorphism*

$$\hat{\alpha}_{XY} : \mathbb{CP}^{2*} \setminus \infty^* \to X \times Y$$

*where $\infty^*$ is the set of lines not striking the point $X \cap Y$.*

*Proof.* This is the dual statement to the previous lemma. □

Using these charts, we derive the smoothness of the double fibration:

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ M & & M^* \end{tikzcd}$$

(see McKay [4]) taking a pointed $E$-line to either a point, via the left leg, or a line via the right.

## 5. Proof of the main theorem

Take two smooth tame pseudocomplex structures $E_0$ and $E_1$ on $M_0 = M_1 = \mathbb{CP}^2$, and a continuous map $\phi : M_0 \to M_1$ which identifies their curves. Suppose that the map is differentiable at two points $x_0, y_0 \in M_0$ and its derivative is an invertible linear map. Let $M_0^*$ and $M_1^*$ be the dual projective planes in the two pseudocomplex structures. Using dual affine charts, we see that the map $\phi$ induces a continuous map identifying the dual planes: call it $\phi^* : M_0^* \to M_1^*$. Then we take the lines $x_0^*, y_0^*$ in $M_0^*$ dual to the points $x_0, y_0$, and the lines $x_1^*, y_1^*$ in $M_1^*$ dual to the points $x_1 = \phi(x_0), y_1 = \phi(y_0)$. The points of the line $x_0^*$ are precisely the $E_0$-lines through $x_0$, which are smoothly identified with their 2-planes in $T_{x_0}M_0$, forming the fiber of $E_0$ over $x_0$. Under the map $\phi$, we map

$$\phi'(x_0) : T_{x_0}M_0 \to T_{x_1}M_1,$$

which is a linear map between $T_{x_0}M_0$ and $T_{x_1}M_1$, so analytic. Under this analytic identification of tangent spaces, we get an analytic identification of Grassmanians of 2-planes in those tangent spaces, and therefore a smooth identification of the fibers of $E_0$ and $E_1$ over $x_0$ and $x_1$ (these fibers are smooth submanifolds of Grassmannians), so a smooth identification of the points of the lines $x_0^*$ and $x_1^*$. Similar remarks hold for $y_0^*$ and $y_1^*$. Finally, we use the coordinate axis construction to produce smooth affine coordinates on the dual projective planes, which must be matched up by the map $\phi^* : M_0^* \to M_1^*$, forcing $\phi^*$ to be smooth on the open set where the coordinates are defined, i.e. away from some chosen points on the two lines. We can change the choice of those points, and obtain global smoothness. The smoothness of the map $\phi^*$ allows us to repeat the above argument on the dual planes, so that $\phi$ is also smooth.

Recalling the concept of a smooth projective plane from Salzmann et al. [6], the proof actually gives:

**Theorem 5.** *A homeomorphic isomorphism of any smooth projective planes, differentiable of full rank at two points, is smooth.*

We obtain our theorem as a consequence of the result that smooth tame pseudocomplex structures on $\mathbb{CP}^2$ are topological projective planes, and that the topological projective plane structure is invariant under homeomorphic isomorphism. This holds because the image of an $E_0$-line is a sphere in the generating homology class, and an $E_1$-curve, so an $E_1$-line.

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School of Mathematical Sciences, University College Cork, Cork, Ireland
*E-mail address*: `B.McKay@UCC.ie`