PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 2, February 2007, Pages 393–395
S 0002-9939(06)08472-3
Article electronically published on August 21, 2006

COMPLEMENTED INVARIANT SUBSPACES AND INTERPOLATION SEQUENCES

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(Communicated by Joseph A. Ball)

Abstract. It is shown that the invariant subspace of the Bergman space A^p of the unit disc, generated by a finite union of Hardy interpolation sequences, is complemented in A^p.

Let $\mathbb{D}$ be the open unit disc of the complex plane $\mathbb{C}$, and let $H^p = H^p(\mathbb{D})$ denote the usual Hardy space. For $0 < p < \infty$ the Bergman space A^p consists of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_p^p = \int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty,$$

where dA is area measure on $\mathbb{C}$ normalized so that $A(\mathbb{D}) = 1$.

A closed subspace I of A^p is said to be invariant if $zf(z) \in I$ for all $f \in I$.

A sequence $Z = \{a_j\}$ of points in $\mathbb{D}$ is called an A^p zero set if there exists a non-zero function $f \in A^p$ such that f vanishes on Z, counting multiplicities. Every A^p zero set generates an invariant subspace I_Z^p consisting of all functions in A^p that vanish on Z, counting multiplicities.

Recently, Korenblum and Zhu [4] exhibited several classes of *complemented* invariant subspaces I in A^p. By definition, I is complemented in A^p if there exists a bounded projection Q from A^p onto I.

In particular, Korenblum and Zhu considered the subspaces I_Z^p for a special class of zero sequences Z. Recall that a sequence $\{a_j\}$ of *distinct* points in the disc is an A^p (H^p) interpolation sequence if for every sequence $\{w_j\} \subset \mathbb{C}$ with

$$\sum (1-|a_j|^2)^2 |w_j|^p < \infty \quad \left(\sum (1-|a_j|^2)|w_j|^p < \infty\right)$$

there exists a function $f \in A^p$ ($f \in H^p$) such that $f(a_j) = w_j$ for all j. Recall that every H^p interpolation sequence is an A^p interpolation set (see, e.g., [5]); the converse implication does not hold. Further results about the interpolation sequences can be found in the monographs [3] and [2].

Theorem 1 (Korenblum and Zhu [4])**.** *Suppose that $0 < p < \infty$ and Z is an A^p interpolation sequence. Then the invariant subspace I_Z^p is complemented in A^p.*

Received by the editors August 10, 2005 and, in revised form, August 22, 2005.
2000 *Mathematics Subject Classification.* Primary 47A15; Secondary 30D55, 30H05.
Key words and phrases. Bergman space, Hardy space, invariant subspace.
This paper was written while the author enjoyed the hospitality of the Chalmers University of Technology.

In the present paper we consider a different class of zero sets Z (with possible multiplicities).

Theorem 2. *Suppose that $0 < p < \infty$ and Z is a finite union of H^p interpolation sequences. Then the invariant subspace I^p_Z is complemented in A^p.*

Proof. Let $Z = \bigcup_{k=1}^n Z_k$, where Z_k is an H^p interpolation sequence, $k = 1, 2, \dots, n$. Observe that Z_k is an A^p interpolation sequence. Therefore, by Theorem 1, there exists a bounded linear operator (a projection) Q_k from A^p onto $I^p_{Z_k}$ such that $Q_k^2 = Q_k$.

Consider the Blaschke products

$$B_m(z) = \prod_{a_j \in Z(m)} \frac{|a_j|}{a_j} \frac{a_j - z}{1 - z\overline{a}_j}, \quad m = 1, 2, \dots, n,$$

where $Z(m) = \bigcup_{k=1}^m Z_k$ (as usual, if $a_j = 0$, then the quotient $|a_j|/a_j$ is replaced by -1). Let $B_m A^p$ denote the set of all multiples of B_m by functions in A^p. Since $Z(m)$ is a finite union of H^p interpolation sequences, we have

$$B_m A^p = I^p_{Z(m)}, \tag{1}$$

$$\|f/B_m\|_p \leq \mathrm{const}(m)\|f\|_p \text{ for every function } f \in I^p_{Z(m)} \tag{2}$$

(see, e.g., [2], Sections 4.5 and 4.6).

Given $f \in I^p_{Z(m-1)}$, $m = 2, 3, \dots, n$, put

$$T_m f = B_{m-1} Q_m (f/B_{m-1})$$

(cf. [1], where such operators are used in the H^p setting). Finally, define

$$Q = T_n T_{n-1} \dots T_2 Q_1.$$

By (1) and (2), Q is a bounded linear operator from A^p onto $I^p_{Z(n)} = I^p_Z$ such that $Q^2 = Q$. In other words, I^p_Z is complemented in A^p. □

Added in proof

A different proof of Theorem 2 can be obtained with the help of interpolation techniques developed in the following paper: Hartmann A., *Traces of certain classes of holomorphic functions on finite unions of Carleson sequences*, Glasg. Math. J. **41** (1999), no. 1, 103–114. The author is grateful to Andreas Hartmann for this reference.

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