PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 12, Pages 3627–3630
S 0002-9939(05)07938-4
Article electronically published on June 3, 2005

CYCLIC VECTORS IN THE FOCK SPACE OVER THE COMPLEX PLANE

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(Communicated by Joseph A. Ball)

Abstract. In this paper, we characterize the cyclic vectors in the Fock space over the complex plane. We prove that a function $f(z)$ is cyclic in the Fock space if and only if $f(z)$ is a nonvanishing function in $L^2_a(\mathbb{C})$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk of the complex plane $\mathbb{C}$. We denote the polynomial ring on $\mathbb{C}$ by $\mathcal{C}$, and the space of all entire functions by $Hol(\mathbb{C})$. Let X be a Banach space of holomorphic functions on a domain Ω in $\mathbb{C}$. For a subset E of X, let $\overline{E}$ be the norm closure of E in X. In this paper, a function f in X is said to be cyclic if $f\mathcal{C} \subset X$ and $\overline{f\mathcal{C}} = X$. In the classical Hardy space $H^2(\mathbb{D})$, it is well known that a function in $H^2(\mathbb{D})$ is cyclic if and only if it is $H^2(\mathbb{D})$-outer; see [5]. Also in the Bergman space $L^2_a(\mathbb{D})$, it is known that f is cyclic if and only if f is $L^2_a(\mathbb{D})$-outer; see [7].

The Fock space, or the so-called Segal-Bargmann space, is the space of all μ-square-integrable entire functions on $\mathbb{C}$, where

$$d\mu(z) = \exp\left(-\frac{|z|^2}{2}\right) dA(z)/2\pi$$

is the Gaussian measure on $\mathbb{C}$ and dA is the ordinary Lebesgue measure. We denote this space by $L^2_a(\mathbb{C}) = L^2_a(\mathbb{C}, \mu)$. Then $L^2_a(\mathbb{C})$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-\frac{|z|^2}{2}} dA(z)$$

for $f, g \in L^2_a(\mathbb{C})$, the reproducing kernel functions are given by $K_\lambda(z) = e^{\frac{\bar{\lambda}z}{2}}$, $\lambda \in \mathbb{C}$, and the polynomial ring $\mathcal{C}$ is dense in $L^2_a(\mathbb{C})$; see [2]. We denote by k_λ the normalized reproducing kernel at $\lambda \in \mathbb{C}$, that is, $k_\lambda = \frac{K_\lambda}{\|K_\lambda\|}$. It is significant to know which functions in $L^2_a(\mathbb{C})$ are cyclic. Obviously a cyclic vector f is an entire function without zeros in $\mathbb{C}$, so that we can write $f = e^h$ for some $h \in Hol(\mathbb{C})$. In this paper, we study for which $h \in Hol(\mathbb{C})$, e^h is cyclic in $L^2_a(\mathbb{C})$. In the Fock space $L^2_a(\mathbb{C})$, there is no multiplier of $L^2_a(\mathbb{C})$ except constant functions; see [6]. So the usual multipication operater M_z is not defined on $L^2_a(\mathbb{C})$. But M_z is defined

Received by the editors July 21, 2004 and, in revised form, August 17, 2004.
2000 *Mathematics Subject Classification.* Primary 46J15, 46H25, 47A16.
Key words and phrases. Fock space, entire function, cyclic vector.

on the dense subspace $\mathcal{C}$ of $L^2_a(\mathbb{C})$. In this meaning, we may say that the definition of the cyclicity in $L^2_a(\mathbb{C})$ can be considered as cyclic vectors for the densely defined operator M_z. The following is our main theorem.

Theorem 1.1. *Let $h \in Hol(\mathbb{C})$. Then the following are equivalent:*

(i) $e^h \in L^2_a(\mathbb{C})$.
(ii) $h = \alpha z^2 + \beta z + \gamma$ *for* $\alpha, \beta, \gamma \in \mathbb{C}$ *with* $|\alpha| < \frac{1}{4}$.
(iii) e^h *is cyclic in* $L^2_a(\mathbb{C})$.

It is known that there are nonvanishing functions in $H^2(\mathbb{D})$ and $L^2_a(\mathbb{D})$ which are not cyclic in the respective spaces. But the situation is quite different in the Fock space.

Corollary 1.2. *Every nonvanishing function in $L^2_a(\mathbb{C})$ is cyclic.*

2. Proof of the main theorem

The proof of Theorem 1.1 consists of two steps.

Step 1. Let $h \in Hol(\mathbb{C})$. Then $e^h \in L^2_a(\mathbb{C})$ if and only if $h(z) = \alpha z^2 + \beta z + \gamma$ with $|\alpha| < \frac{1}{4}$. Moreover if $e^h \in L^2_a(\mathbb{C})$, then $e^h\mathcal{C} \subset L^2_a(\mathbb{C})$.

Proof. Suppose that $e^h \in L^2_a(\mathbb{C})$. First, we shall prove that $h \in \mathcal{C}$ and $\deg h \leq 2$, where $\deg h$ denotes the polynomial degree of h. In [3], Chen, Guo, and Hou proved that $\lim_{|\lambda|\to\infty} \langle f, k_\lambda \rangle = 0$ for every $f \in L^2_a(\mathbb{C})$, so that $\Re h(\lambda) < \frac{|\lambda|^2}{4}$ for every $\lambda \in \mathbb{C}$ with sufficiently large modulus. Then obviously e^h is an entire function of finite class $p = 1$, and the order ρ of e^h is

$$\begin{aligned}\rho &= \limsup_{r\to\infty} \frac{\log\left[\max_{|\lambda|=r} \Re h(z)\right]}{\log r} \\ &\leq \limsup_{r\to\infty} \frac{\log \frac{r^2}{4}}{\log r} \\ &= 2.\end{aligned}$$

By [1, Corollary 4.5.11], we get that $h \in \mathcal{C}$ and $\deg h \leq 2$.

Now, we show that $e^{\alpha z^2+\beta z+\gamma} \notin L^2_a(\mathbb{C})$ for $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha| \geq \frac{1}{4}$. It is enough to prove that $e^{(az-b)^2} \notin L^2_a(\mathbb{C})$ for $a, b \in \mathbb{C}$ with $|a| \geq \frac{1}{2}$. Without loss of generality, we may assume that $a \geq \frac{1}{2}$. Write $b = r(\cos\theta + i\sin\theta)$, $r \geq 0$. For $t \in \mathbb{R}$, we have

$$\left|\left\langle e^{(az-b)^2}, k_t \right\rangle\right| = \frac{1}{\|K_t\|}\left|\left\langle e^{(az-b)^2}, K_t \right\rangle\right| = \left|e^{b^2}\right| e^{(a^2-\frac{1}{4})t^2-2atr\cos\theta}.$$

If $a > \frac{1}{2}$, then we have $\lim_{t\to\infty}\left|\left\langle e^{(az-b)^2}, k_t \right\rangle\right| = \infty$. If $a = \frac{1}{2}$, then we have $\lim_{t\to\infty}\left|\left\langle e^{(az-b)^2}, k_t \right\rangle\right| = \infty$ provided $\cos\theta < 0$, $\lim_{t\to-\infty}\left|\left\langle e^{(az-b)^2}, k_t \right\rangle\right| = \infty$ provided $\cos\theta > 0$, and $\left|\left\langle e^{(az-b)^2}, k_t \right\rangle\right| = 1$ for every $t \in \mathbb{R}$ provided $\cos\theta = 0$. Since $\lim_{|\lambda|\to\infty}\langle f, k_\lambda\rangle = 0$ for every $f \in L^2_a(\mathbb{C})$, we get that $e^{(az-b)^2} \notin L^2_a(\mathbb{C})$.

Finally, fix α, β, γ with $|\alpha| < \frac{1}{4}$ and put $h(z) = \alpha z^2 + \beta z + \gamma$. Let ϵ be a number satisfying

$$0 < \epsilon < \frac{1}{2}\left(\frac{1}{4} - |\alpha|\right). \tag{1}$$

For large $|z|$, $|p(z)| \le e^{\epsilon|z|^2}$ and $|\beta z+\gamma| \le \epsilon|z|^2$. Thus

$$\begin{aligned}\left|p(z)e^{h(z)}\right|^2 &= |p(z)|^2 e^{2\Re h(z)}\\ &\le e^{2\epsilon|z|^2}e^{2(|\alpha|+\epsilon)|z|^2}\\ &= e^{2(2\epsilon+|\alpha|)|z^2|}.\end{aligned}$$

Then

$$\left|p(z)e^{h(z)}\right|^2 e^{-\frac{|z|^2}{2}} \le e^{-\delta|z|^2}$$

for large $|z|$, where $\delta = \frac{1}{2} - 2(2\epsilon+|\alpha|)$. By (1), we have that $\delta > 0$. Therefore there exists a positive constant C such that

$$\int_{\mathbb{C}} \left|p(z)e^{h(z)}\right|^2 e^{-\frac{|z|^2}{2}} dA(z)/2\pi \le C\int_{\mathbb{C}} e^{-\delta|z|^2} dA(z)/2\pi.$$

Since the last integral is finite, we get that $e^{h(z)}\mathcal{C} \subset L^2_a(\mathbb{C})$. This completes the proof. □

If f is cyclic, then obviously f is a nonvanishing function. So it remains only to prove that $e^{h(z)}$ is cyclic in $L^2_a(\mathbb{C})$ for $h(z) = \alpha z^2 + \beta z + \gamma$ with $|\alpha| < \frac{1}{4}$.

Step 2. If $h(z) = \alpha z^2 + \beta z + \gamma$ with $|\alpha| < \frac{1}{4}$, then $e^{h(z)}$ is cyclic in $L^2_a(\mathbb{C})$.

Proof. By Step 1, $e^{h(z)}\mathcal{C} \subset L^2_a(\mathbb{C})$. Let N be a positive integer satisfying

$$(1+\frac{1}{N})|\alpha| < \frac{1}{4}. \tag{2}$$

Put $p_n(z) = \sum_{k=0}^{n} \frac{\{-\frac{1}{N}(\alpha z^2+\beta z+\gamma)\}^k}{k!}$. Let l be a nonnegative integer. Since

$$|p_n(z)| \le \sum_{k=0}^{\infty} \frac{\{\frac{1}{N}(|\alpha||z|^2+|\beta||z|+|\gamma|)\}^k}{k!} = e^{\frac{1}{N}(|\alpha||z|^2+|\beta||z|+|\gamma|)},$$

we have

$$\begin{aligned}&\left|z^l p_n(z)e^{\alpha z^2+\beta z+\gamma} - z^l e^{-\frac{1}{N}(\alpha z^2+\beta z+\gamma)}e^{\alpha z^2+\beta z+\gamma}\right|\\ &\quad\le |z|^l\left(e^{(1+\frac{1}{N})(|\alpha||z|^2+|\beta||z|+|\gamma|)} + e^{(1-\frac{1}{N})(|\alpha||z|^2+|\beta||z|+|\gamma|)}\right)\\ &\quad\le 2|z|^l e^{(1+\frac{1}{N})(|\alpha||z|^2+|\beta||z|+|\gamma|)}.\end{aligned}$$

Hence

$$\begin{aligned}&\left|z^l p_n(z)e^{\alpha z^2+\beta z+\gamma} - z^l e^{-\frac{1}{N}(\alpha z^2+\beta z+\gamma)}e^{\alpha z^2+\beta z+\gamma}\right|^2 e^{-\frac{|z|^2}{2}}\\ &\quad\le 4|z|^{2l}e^{\{2(1+\frac{1}{N})|\alpha|-\frac{1}{2}\}|z|^2+2(1+\frac{1}{N})|\beta||z|+2(1+\frac{1}{N})|\gamma|}.\end{aligned}$$

By (2), we have that $2(1+\frac{1}{N})|\alpha| - \frac{1}{2} < 0$, so that the last function is integrable with respect to dA. Since $p_n(z) \to e^{-\frac{1}{N}(\alpha z^2+\beta z+\gamma)}$ pointwise on $\mathbb{C}$ as $n \to \infty$, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}&\int_{\mathbb{C}} \left|z^l p_n(z)e^{\alpha z^2+\beta z+\gamma} - z^l e^{-\frac{1}{N}(\alpha z^2+\beta z+\gamma)}e^{\alpha z^2+\beta z+\gamma}\right|^2 e^{-\frac{|z|^2}{2}} dA(z)/2\pi\\ &\qquad\qquad\qquad\qquad \to 0 \text{ as } n\to\infty.\end{aligned}$$

Therefore we get

$$e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)}\mathcal{C} \subset \overline{e^{\alpha z^2+\beta z+\gamma}\mathcal{C}}.$$

Applying this method again for $e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)}$, we have

$$\int_{\mathbb{C}} \left| z^l p_n(z) e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)} - z^l e^{-\frac{1}{N}(\alpha z^2+\beta z+\gamma)} e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)} \right|^2 e^{-\frac{|z|^2}{2}} dA(z) \to 0 \text{ as } n \to \infty,$$

so that

$$e^{(1-\frac{2}{N})(\alpha z^2+\beta z+\gamma)}\mathcal{C} \subset \overline{e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)}\mathcal{C}}.$$

Continuing this way, we get

$$\mathcal{C} \subset \overline{e^{\frac{1}{N}(\alpha z^2+\beta z+\gamma)}\mathcal{C}} \subset \cdots \subset \overline{e^{(1-\frac{1}{N})(\alpha z^2+\beta z+\gamma)}\mathcal{C}} \subset \overline{e^{\alpha z^2+\beta z+\gamma}\mathcal{C}}.$$

Thus we get $\overline{e^{\alpha z^2+\beta z+\gamma}\mathcal{C}} = L^2_a(\mathbb{C})$. □

Acknowledgement

The author thanks the referee for improving the proof.

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