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THE HAUSMANN-WEINBERGER 4–MANIFOLD INVARIANT OF ABELIAN GROUPS

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Abstract. The Hausmann-Weinberger invariant of a group G is the minimal Euler characteristic of a closed orientable 4–manifold M with fundamental group G. We compute this invariant for finitely generated free abelian groups and estimate the invariant for all finitely generated abelian groups.

1. Introduction

For any finitely presented group G there exists a closed oriented 4–manifold M with $\pi_1(M) = G$. Hausmann and Weinberger [3] defined the integer-valued invariant $q(G)$ to be the least Euler characteristic among all such M. The explicit construction of a 4–manifold with $\pi_1(M) = G$, based on a presentation of G, yields an upper bound on $q(G)$. As pointed out in [3], the isomorphism $H_1(M) \to H_1(G)$, the surjection $H_2(M) \to H_2(G)$, and Poincaré duality yield a lower bound. Together these bounds are

$$2 - 2\beta_1(G) + \beta_2(G) \leq q(G) \leq 2 - 2\mathrm{def}(G), \tag{1.1}$$

where $\mathrm{def}(G)$ is the deficiency of G, the maximum possible difference $g - r$ where the g is the number of generators and r the number of relations in a presentation of G, and $\beta_i(G)$ denotes the ith Betti number of G (with rational coefficients).

Since [3], advances have been made in the study of this invariant, most notably through the methods of l^2–homology. For instance, in [1, 2] Eckmann proves that for infinite amenable groups G, $q(G) \geq 0$. Lück [9] extended this to all groups G with $b_1(G) = 0$, where b_1 denotes the first l^2–Betti number. Other work includes [5] and especially the paper by Kotschick [7] in which Problem 5.2 asks for the explicit value of $q(\mathbf{Z}^n)$. The general problem of computing $q(G)$ appears as Problem 4.59 in Kirby's problem list, [6].

Despite these past efforts, the Hausmann-Weinberger invariant remains uncomputed for some of the most elementary groups. In [3] it is observed that $q(\mathbf{Z}^n)$ is given by 2, 0, 0, 2, and 0, for $n = 0, 1, 2, 3$, and 4, respectively. (For the case of $n = 3$, [3] refers to an unpublished argument of Kreck. Proofs appear in [1, 7].) If Γ_g denotes the fundamental group of a surface of genus g, [7] computes $q(\Gamma_g)$

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and $q(\Gamma_{g_1} \times \Gamma_{g_2})$. If a closed 4-manifold X is aspherical, then $\chi(X) = q(\pi_1(X))$. Beyond this, few explicit values of $q(G)$ have been calculated. Our main theorem is the following.

Theorem 1. *With the exceptions of $q(\mathbf{Z}^3) = 2$ and $q(\mathbf{Z}^5) = 6$, $q(\mathbf{Z}^n)$ is given by*

$$q(\mathbf{Z}^n) = \begin{cases} (n-1)(n-4)/2, & \text{if } n \equiv 0 \text{ or } n \equiv 1 \mod 4; \\ (n-2)(n-3)/2, & \text{if } n \equiv 2 \text{ or } n \equiv 3 \mod 4. \end{cases}$$

For contrast, the bounds (1.1) give only that

$$(n-1)(n-4)/2 \leq q(\mathbf{Z}^n) \leq (n-1)(n-2).$$

It is straightforward to check that an alternative way to state Theorem 1 is that for $n \neq 3, 5$, the lower bound of (1.1) is attained when the binomial coefficient $C(n, 2)$ is even, and if $C(n, 2)$ is odd, the value is one more than the lower bound of (1.1).

The calculations of $q(\mathbf{Z}^n)$ can be used to estimate (and in some cases calculate) $q(G)$ for other groups. We examine the problem for finitely generated abelian groups and prove the following. In the statement, ϵ_n equals zero or one according to whether $C(n, 2)$ is even or odd, respectively.

Theorem 2. *Let $G = \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k \oplus \mathbf{Z}^n$ with $d_i | d_{i+1}$ and $d_i > 1$. Suppose that $k \geq 1$ and $k + n \neq 3, 4, 5$ or 6. Then*

$$0 \leq q(G) - \big(1 - n + C(n + k - 1, 2)\big) \leq \min\{|n - 1| + \epsilon_{n+k-1}, k + \epsilon_{n+k}\}.$$

Moreover, $q(\mathbf{Z}/d) = 2$, $q(\mathbf{Z}/d_1 \oplus \mathbf{Z}/d_2) = 2$, and if $C(k, 2)$ is even and $k \neq 5$, then

$$q(\mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k \oplus \mathbf{Z}) = C(k, 2).$$

In closing this introduction we mention results concerning the evaluation of $q(P)$ where P is a perfect group. Here the bounds given by (1.1) are

$$2 + \beta_2(P) \leq q(P) \leq 2 - 2\mathrm{def}(G).$$

In [3] perfect groups P are constructed with $\beta_2(P) = 0$ but $q(P) > 2$. Hillman [4] constructed perfect groups of deficiency -1 with $q(P) = 2$, and the second author [8] extended this to find a perfect group P with arbitrarily large negative deficiency and $q(P) = 2$.

2. Notation and basic results

A slightly different invariant, $h(G)$, can be defined to be the minimum value of $\beta_2(M)$ among all oriented closed 4-manifolds M with $\pi_1(M) = G$. We abbreviate $h(\mathbf{Z}^n) = h(n)$. Clearly $q(G) = 2 - 2\beta_1(G) + h(G)$; so the invariants are basically equivalent. It is more convenient here to work in terms of h. The bounds (1.1) on $q(\mathbf{Z}^n)$ translate to the bounds

$$C(n, 2) \leq h(n) \leq 2C(n, 2).$$

We introduce the following auxiliary function:

$$\epsilon_n = \begin{cases} 0, & \text{if } C(n, 2) \text{ is even;} \\ 1, & \text{if } C(n, 2) \text{ is odd.} \end{cases}$$

Since $C(n, 2)$ is even if and only if $n \equiv 0$ or $1 \mod 4$, Theorem 1 can be restated as follows.

Theorem 1. *With the exceptions of $h(3) = 6$ and $h(5) = 14$, $h(n) = C(n, 2) + \epsilon_n$ for all n.*

Basic examples of 4–manifolds will be built from products of surfaces. For n even we will denote by F_n the closed orientable surface of genus $n/2$.

3. Bounds on $h(n)$

Theorem 3. *If $h(n) = C(n, 2)$, then $C(n, 2)$ must be even. Thus $h(n) \geq C(n, 2) + \epsilon_n$.*

Proof. If $\phi\colon \pi_1(M) \to \mathbf{Z}^n$, we have $\phi_*\colon H_*(M) \to H_*(\mathbf{Z}^n)$. Dually there is the map of cohomology rings $\phi^*\colon H^*(\mathbf{Z}^n) \to H^*(M)$. Notice that $H^*(\mathbf{Z}^n)$ is an exterior algebra on the generators $e_1, \cdots, e_n \in H^1(\mathbf{Z}^n)$.

Suppose $\phi\colon \pi_1(M) \to \mathbf{Z}^n$ is an isomorphism and $\beta_2(M) = C(n, 2)$. Then the map $\phi_2\colon H_2(M) \to H_2(\mathbf{Z}^n)$ is a surjection from $\mathbf{Z}^{C(n,2)}$ to $\mathbf{Z}^{C(n,2)}$, and hence is an isomorphism. It follows that $\phi^2\colon H^2(\mathbf{Z}^n) \to H^2(M)$ is also an isomorphism.

Since $(e_ie_j)^2 = 0$, $H^2(M)$ has a basis for which all squares are zero. It follows that the intersection form of M is even. But even unimodular forms are of even rank. □

Theorem 4. *$h(3) \geq 6$ and $h(5) \geq 14$.*

Proof. In general, if $\phi\colon \pi_1(M) \to \mathbf{Z}^n$ is an isomorphism, then $\phi^2\colon H^2(\mathbf{Z}^n) \to H^2(M)$ is injective.

In the case that $n = 3$, all products of two elements in $H^2(\mathbf{Z}^3)$ are 0 (since $H^4(\mathbf{Z}^3) = 0$). So the intersection form on $H^2(M)$ vanishes on a rank 3 submodule, implying that this (nonsingular) form must have rank at least 6.

In the case that $n = 5$, we have the map $H^4(\mathbf{Z}^5) \to H^4(M) \cong \mathbf{Z}$. Any such map is given by multiplying with an element $D \in H^1(\mathbf{Z}^5)$. After a change of basis, D can be taken to be a multiple of a generator, say e_1. From this it follows that the intersection form of $H^2(M)$ vanishes on the 7–dimensional submodule generated by the images of the set of elements in $H^2(\mathbf{Z}^5)$, $\{e_{12}, e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{25}\}$ (where $e_{ij} = e_ie_j$). To see this, observe that the only possible nontrivial products of two of these are $\pm e_{1234}$, $\pm e_{1235}$ and $\pm e_{1245}$, each of which is killed upon multiplying by e_1. Since the nonsingular intersection form on $H^2(M)$ has a 7–dimensional isotropic subspace, it must be of rank at least 14. □

4. Algebraic and geometric 4–reductions

The following algebraic construction will be used repeatedly in constructing our desired 4–manifolds.

Definition 5. A *4–reduction of a group G by a 4–tuple of elements* $[w_1, w_2, w_3, w_4]$, $w_i \in G$, is the quotient of G by the normal subgroup generated by the 6 commutators, $[w_i, w_j], i < j$. This quotient is denoted $G/[w_1, w_2, w_3, w_4]$.

More generally, we say a group G *can be 4-reduced to the group H using the 4–tuples* $\{[w_{1k}, w_{2k}, w_{3k}, w_{4k}]\}_{k=1}^{\ell}$ if H is isomorphic to the quotient of G by the normal subgroup generated by the 6ℓ commutators $[w_{ik}, w_{jk}],\ i < j,\ k = 1, \cdots, \ell$.

The geometric motivation for this definition comes from the following theorem.

Theorem 6. *If X is a 4–manifold and $\{w_1, w_2, w_3, w_4\} \subset \pi_1(X)$, then there is a 4–manifold X' with $\pi_1(X') = \pi_1(X)/[w_1, w_2, w_3, w_4]$ and $\beta_2(X') = \beta_2(X) + 6$.*

Before proving this we make the following simple observation.

Lemma 7. *If a 4–manifold X' is constructed from a compact 4–manifold X via surgery along a curve α, then $\beta_2(X') = \beta_2(X)$ if α is of infinite order in $H_1(X)$ and $\beta_2(X') = \beta_2(X) + 2$ otherwise.*

Proof. Since X' is formed by removing $S^1 \times B^3$ and replacing it with $B^2 \times S^2$, $\chi(X') = \chi(X) + 2$. If α is of infinite order, $\beta_1(X') = \beta_1(X) - 1$, and similarly $\beta_3(X') = \beta_3(X) - 1$ by duality; so $\beta_2(X') = \beta_2(X)$. On the other hand, if α is of finite order, β_1 is unchanged by surgery, by duality β_3 is unchanged, and so the change in the Euler characteristic must come from an increase in β_2 by 2. □

Proof of Theorem 6. Form the connected sum $X \# T^4$ of X with the 4-torus T^4. This increases the second Betti number by six. Next, perform surgery on four curves to identify the generators of $\pi_1(T^4)$ with the elements w_i. This does not change the second Betti number because the curves being surgered are of infinite order in $H_1(X \# T^4)$. Since the generators of $\pi_1(T^4)$ commute, the effect of this is that now the four elements w_i commute. Thus the manifold that results from the surgeries has the stated properties. □

The main algebraic result concerning 4–reduction, and the key to our geometric constructions via Theorem 6, is the following.

Theorem 8. *For $m > 2$ and $n > 2$, the free product $\mathbf{Z}^m * \mathbf{Z}^n$ can be 4–reduced to $\mathbf{Z}^{m+n}$ using $\frac{mn}{6}$ 4–tuples if mn is divisible by 6.*

Proof. If the free product $\mathbf{Z}^m * \mathbf{Z}^n$ can be 4–reduced to $\mathbf{Z}^{m+n}$ using $\frac{mn}{6}$ 4–tuples, we will say that the pair (m, n) is realizable. Let $\mathcal{R}$ denote the set of realizable pairs with $m > 2, n > 2$.

First we show that $(3, 4)$, $(3, 6)$, and $(5, 6)$ are in $\mathcal{R}$.

- Consider first the pair $(3, 4)$. Denote the generators of $\mathbf{Z}^3$ by $\{x_1, x_2, x_3\}$, and let $\mathbf{Z}^4$ be generated by $\{y_1, y_2, y_3, y_4\}$. We now observe that the two 4–tuples $[x_1, y_1, x_2y_2, x_3y_3]$ and $[x_2, x_1y_3, x_3y_2, y_4]$ carry out the desired 4–reduction. For the convenience of the reader we provide the details next, but in subsequent examples similar calculations will be omitted.

  We must show that the subgroup U generated by these commutator 4–relations contain all 12 commutators $[x_i, y_j]$. It is helpful to recall that the set of elements in a group that commute with a fixed element forms a subgroup.

  From the first 4–relation, $[x_1, y_1, x_2y_2, x_3y_3]$, we see, using the commutators $[x_1, y_1]$, $[x_1, x_2y_2]$, and $[x_1, x_3y_3]$, that the commutators $[x_1, y_1]$, $[x_1, y_2]$, and $[x_1, y_3]$ are in U. The commutators $[y_1, x_2y_2]$ and $[y_1, x_3y_3]$ give that the commutators $[x_2, y_1]$ and $[x_3, y_1]$ are in U. The last commutator, $[x_2y_2, x_3y_3]$, will be returned to momentarily.

  From the second 4–relation, $[x_2, x_1y_3, x_3y_2, y_4]$, we have first the commutator $[x_2, y_3] \in U$. Then, from the previous relation ($[x_2y_2, x_3y_3]$) it follows that $[x_3, y_2] \in U$. Next, that the commutators $[x_2, y_2]$ and $[x_2, y_4]$ are in U follows immediately. From the commutator $[x_1y_3, x_3y_2]$ we see that $[x_3, y_3] \in U$ (since we already had that $[x_1, y_2]$ is in U). From $[x_1y_3, y_4]$ we have $[x_1, y_4] \in U$. The commutator $[x_3y_2, y_4]$ gives the last needed commutator, $[x_3, y_4]$.

- In the case of the pair $(3,6)$, using similar notation, the following three 4–tuples $[x_1, y_1, x_2y_2, x_3y_3], [x_2, x_1y_3, x_3y_4, x_2y_5], [y_6, x_1y_2, x_2x_3, x_3y_4]$ reduce to $\mathbf{Z}^3 * \mathbf{Z}^6$.
- Finally, for $(5,6)$, the following five 4–tuples suffice: $[x_1, y_1, x_2y_2, x_3y_3]$, $[x_2, y_3, x_3y_4, x_4y_5]$, $[x_5, y_6, x_4y_3, x_1y_2y_5]$, $[x_3, y_5, x_5y_6, x_1y_4]$, and $[x_1y_1, x_2y_4, x_4y_6, x_5y_2]$.

For the general case of (m,n), assume first that m is divisible by 6. Using the realization of $(3,4)$ we can realize $(6,4)$ and have already realized $(6,3)$. (Separate the six generators into two groups of three, and make each set commute with the other four using the construction used for $(3,4)$.) Combining these, we can realize $(6k,3)$ and $(6k,4)$ for any k. Now, combining these we can realize $(6k, 3a+4b)$ for any a and b. But all integers greater than 2, other than 5, can be written as $3a+4b$ for some a and b.

In the case that neither m nor n is divisible by 6, we can assume 3 divides m and we want to realize $(3k,n)$. Notice that n must be even. Since we can realize $(3,4)$ and $(3,6)$, we can realize $(3k,4)$ and $(3k,6)$ for all k. Thus we can realize $(3k, 4a+6b)$ for all a and b, but $4a+6b$ realizes all even integers greater than 3. $\square$

5. Basic realizing examples

We begin with the exceptional cases of $n=3$ and $n=5$ and then move on to a set of basic examples for which $h(n) = C(n,2)+\epsilon_n$. In the next section we note that these basic examples can be used to construct the necessary examples for the proof of Theorem 1. Recall that F_n denotes the closed, orientable surface of genus $n/2$.

- $n=3$: Start with the 4–torus, T^4, with $\beta_2(T^4)=6$. Surgery on a single curve representing a generator of $\pi_1(T^4)$ results in a manifold M with $\pi_1(M)=\mathbf{Z}^3$ and $H_2(M)=\mathbf{Z}^6$.
- $n=5$: Begin with $X = F_2 \times F_4$ with $\beta_2(X)=10$ and π_1 generated by $\{x_1,x_2\}$ and $\{y_1,y_2,y_3,y_4\}$. Perform a surgery to identify y_3 and y_4, so that the group is generated by $\{x_1,x_2,y_1,y_2,y_3\}$. Notice that y_1 and y_2 commute, as follows from the original surface commutator relationship $[y_1,y_2][y_3,y_4]=1$. This surgery, since it is along an element of infinite order in H_1, does not change $H_2(X)$. Hence, it only remains to arrange that the pairs of elements $\{y_1,y_3\}$ and $\{y_2,y_3\}$ commute. Performing surgery on a (rationally) null homologous curve raises β_2 by two, so performing surgeries to kill these two commutators raises the rank of $H_2(X)$ by 4, and the resulting 4–manifold M has $H_2(M)=\mathbf{Z}^{14}$ as desired.

We will say that the integer n *is realizable* if there is a closed oriented 4–manifold M_n with $\pi_1(M_n)=\mathbf{Z}^n$ and $\beta_2(M_n)=C(n,2)+\epsilon_n$. Let $\mathcal{S}$ be the set of realizable integers. We now show that $0,1,2,4,6,7,8,9,11,12 \subset \mathcal{S}$ by describing the construction of realizing 4–manifolds M_n for each of these n.

- $n=0$: $M_0=S^4$.
- $n=1$: $M_1=S^1\times S^3$.
- $n=2$: $M_2=F_2\times S^2$. Notice that $C(2,2)=1$, so that $\beta_2(M_2)=2=C(2,2)+\epsilon_2$.
- $n=4$: $M_4=T^4$.

- $n = 6$: Build M_6 as follows. Let $X = F_2 \times F_4$ with π_1 generated by the 6 elements $\{x_1, x_2\}$, $\{y_1, y_2, y_3, y_4\}$. Note that $\beta_2(X) = 10$. Apply Theorem 6 to perform the 4–reduction $[y_1, y_2, y_3, y_4]$ and arrive at the 4–manifold M_6 with $\pi_1(M_6) = \mathbf{Z}^6$ and $\beta_2(M_6) = 16 = C(6, 2) + \epsilon_6$ as desired.
- $n = 7$: Begin with $X = F_2 \times F_4 \# T^4$, so $\beta_2(X) = 16$. Let the generators of π_1 be $\{x_1, x_2\}$, $\{y_1, y_2, y_3, y_4\}$, and $\{z_1, z_2, z_3, z_4\}$. Perform surgeries giving $y_1 = z_2$, $y_2 = z_3$, $y_4 = z_4$. Now use Theorem 6 to perform the 4–tuple reduction $[z_1, x_1y_2, x_2y_1, y_3]$. (In checking that this abelianizes the group, use the fact that $[y_1, y_2] = 1$ if and only if $[y_3, y_4] = 1$.) The resulting 4–manifold M_7 has $\pi_1(M_7) = \mathbf{Z}^7$ and $\beta_2(M_7) = 22 = C(7, 2) + \epsilon_7$.
- $n = 8$: Take $X = F_4 \times F_4 \# F_2 \times F_4$, with $\beta_2(X) = 28$ and π_1 generated by $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$, $\{z_1, z_2\}$ and $\{w_1, w_2, w_3, w_4\}$, respectively. Now perform surgeries to introduce the following relations:

  $z_1 = x_1y_1$,
  $z_2 = x_2y_2$,
  $w_1 = x_3y_2$,
  $w_2 = x_4$,
  $w_3 = x_1y_3$,
  $w_4 = x_2y_4$.

  Since $[w_1, w_2][w_3, w_4] = 1$ we have that $[x_3y_2, x_4][x_1y_3, x_2y_4] = 1$. Since the x_i commute with the y_i, this implies that $[x_3, x_4][x_1, x_2][y_3, y_4] = 1$. From the surface relation for the x_i, $[x_1, x_2][x_3, x_4] = 1$, it then follows that $[y_3, y_4] = 1$. From this and the surface relation for the y_i, it follows that $[y_1, y_2] = 1$. Now, the fact that $[z_1, z_2] = 1$ gives that $[x_1, x_2] = 1$, which implies that $[x_3, x_4] = 1$ too. The remaining needed four commutator relations between the x_i and the four commutator relations between the y_i give a total of 8 needed commutator relations. These are obtained by first considering the relations $[z_1, w_i]$ and then the relations $[z_2, w_i]$. The resulting M_8 has $\pi_1(M_8) = \mathbf{Z}^8$ and $\beta_2(M_8) = 28 = C(8, 2) + \epsilon_8$.
- $n = 9$: Start with $X = T^4 \# T^4 \# T^4$, with $\beta_2(X) = 18$ and π_1 generated by $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$, and $\{z_1, z_2, z_3, z_4\}$. Perform three surgeries to give the identifications: $z_2 = x_3y_3$, $z_3 = x_4$, and $z_4 = y_4$. Notice that since the 4–tuple relation $[z_1, z_2, z_3, z_4]$ held in the original group, we now have the 4–tuple relation $[z_1, x_3y_3, x_4, y_4]$.

  Use Theorem 6 to add the following three more 4–tuple relations, raising β_2 to 36:

  $[z_1, y_1, x_3, x_2y_2]$,
  $[z_1x_3, x_1, y_2, x_2y_3]$,
  $[x_2y_4, x_4y_3, x_2y_1, x_1y_2y_4]$.

  The resulting manifold M_9 has $\pi_1(M_9) = \mathbf{Z}^9$ and $\beta_2(M_9) = 36 = C(9, 2) + \epsilon_9$.
- $n = 11$: Start with $X = F_4 \times F_6 \# T^4$, with $\beta_2(X) = 32$ and π_1 generated by $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4, y_5, y_6\}$, and $\{z_1, z_2, z_3, z_4\}$. Perform surgery to get the following identifications:

  $y_4 = z_2$,
  $y_5 = z_3$,
  $y_6 = z_4$.

This leaves a generating set with eleven elements:

$$\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, y_5, y_6, z_1\}.$$

(Notice that $[x_1, x_2][x_3, x_4] = 1$ and $[y_1, y_2][y_3, y_4] = 1$, since y_5 and y_6 now commute.) Apply Theorem 6 four times to perform the following 4–reductions:

$[x_1, x_2, x_3, z_1]$,
$[y_1, y_2, y_3, z_1]$,
$[x_4y_1, y_5, y_3y_6, y_2z_1]$,
$[x_1y_4, x_4y_6, x_2y_2z_1, y_1y_3]$.

The resulting manifold M_{11} has $\pi_1(M_{11}) = \mathbf{Z}^{11}$ and $\beta_2(M_{11}) = 32 + 24 = 56 = C(11, 2) + \epsilon_{11}$.

- $n = 12$: Start with $X = F_4 \times F_4 \# T^4$ with $\beta_2(X) = 24$ and π_1 generated by $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$, and $\{z_1, z_2, z_3, z_4\}$ as before. Now apply Theorem 6 to add seven 4–tuple relations:

  $[x_1, x_2, x_3, z_1]$,
  $[y_1, y_2, y_3, z_1]$,
  $[x_1, x_4, z_2, y_1z_3]$,
  $[y_1, y_4, z_3, x_3z_4]$,
  $[x_2y_2, x_4, y_4z_1, z_1z_4]$,
  $[x_2z_3, z_4, y_3, x_3z_2]$,
  $[x_3z_3, y_4z_2, x_1y_2, z_4x_2z_2]$.

  The resulting M_{12} has $\beta_2(M_{12}) = 24 + 42 = 66 = C(12, 2) + \epsilon_{12}$ and $\pi_1(M_{12}) = \mathbf{Z}^{12}$ as desired.

6. Constructing more examples: The proof of Theorem 1

Theorem 9. *If $m \in \mathcal{S}$, $n \in \mathcal{S}$, $(m, n) \in \mathcal{R}$, and if one of m, $m - 1$, n, or $n - 1$ is congruent to 0 modulo 4, then $m + n \in \mathcal{S}$.*

Proof. The stated mod 4 condition together with the fact that $mn \equiv 0 \pmod 6$ assures that $C(n+m, 2) + \epsilon_{n+m} = (C(n, 2) + \epsilon_n) + (C(m, 2) + \epsilon_m) + mn$. Thus, one can build the desired M_{m+n} by performing mn surgeries on $M_m \# M_n \# \frac{mn}{6} T^4$; that is, by performing $\frac{mn}{6}$ 4–reductions as in Theorem 6. □

We have that $\{0, 1, 2, 4, 6, 7, 8, 9, 11, 12\} \subset \mathcal{S}$. Furthermore, all pairs (n, m) with $mn \equiv 0 \mod 6$ and $m \geq 3$ and $n \geq 3$ are in $\mathcal{R}$.

Using Theorem 9 and the pair $(4, 6) \in \mathcal{R}$ gives $10 \in \mathcal{S}$. Similarly, using the pair $(4, 9) \in \mathcal{R}$ gives $13 \in \mathcal{S}$. The pair $(6, 8) \in \mathcal{R}$ shows that $14 \in \mathcal{S}$. The pair $(6, 9) \in \mathcal{R}$ shows that $15 \in \mathcal{S}$. The pair $(4, 12) \in \mathcal{R}$ shows that $16 \in \mathcal{S}$. The pair $(8, 9) \in \mathcal{R}$ shows that $17 \in \mathcal{S}$.

Next the pairs $(12, n)$ for $n = 6, \ldots, 17$ show that $\{18, \ldots, 29\} \subset \mathcal{S}$. Then the pairs $(12, n)$ for $n = 18, \ldots, 29$ show that $\{30, \ldots, 41\} \subset \mathcal{S}$. Repeating inductively in this way shows that $n \in \mathcal{S}$ for all $n \geq 6$, as desired.

7. Finitely generated abelian groups

We next use the manifolds constructed in the previous sections as building blocks to prove Theorem 2. Let

$$G = \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k \oplus \mathbf{Z}^n$$

where $d_i | d_{i+1}$ and $d_i > 1$. Fix a prime p that divides d_1. Then

$$\mathrm{rk}_{\mathbf{Z}/p} H_1(G; \mathbf{Z}/p) = k + n \text{ and } \mathrm{rk}_{\mathbf{Z}/p} H_2(G; \mathbf{Z}/p) = C(k+n, 2) + k.$$

If X is a closed, oriented 4–manifold with $\pi_1(X) \cong G$, we have $H_1(X; \mathbf{Z}/p) \cong H_1(G; \mathbf{Z}/p)$ and $H_2(X; \mathbf{Z}/p)$ surjects to $H_2(G; \mathbf{Z}/p)$. This gives the lower bound on $q(G)$:

$$q(G) \geq 2 - 2(k+n) + C(k+n, 2) + k = 1 - n + C(n+k-1, 2). \tag{7.1}$$

To construct upper bounds, consider the following constructions of closed 4–manifolds X with $\pi_1(X) \cong G$. Let $L(d)$ denote a 3-dimensional lens space with $\pi_1(L(d)) = \mathbf{Z}/d$, and let B_n denote a closed 4–manifold with $\pi_1(B_n) = \mathbf{Z}^n$ and $\chi(B_n) = q(\mathbf{Z}^n)$.

- Let X be the manifold obtained by starting with B_{k+n} and doing surgeries on the first k generators of $\pi_1(B_{k+n}) = \mathbf{Z}^{k+n}$ in such a way as to kill d_1 times the first generator, d_2 times the second, and so forth. Then $\pi_1(X) = \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k \oplus \mathbf{Z}^n$ and $\mathrm{rk}_{\mathbf{Q}}(H_2(X; \mathbf{Q})) = \mathrm{rk}_{\mathbf{Q}}(H_2(B_{n+k}; \mathbf{Q}))$. Therefore
$$\chi(X) = 2 - 2n + \mathrm{rk}_{\mathbf{Q}}(H_2(B_{n+k}; \mathbf{Q})).$$
Thus, if $n + k \neq 3, 5$, simplifying yields
$$\chi(X) = 1 - n + C(n+k-1, 2) + k + \epsilon_{n+k},$$
and hence
$$0 \leq q(G) - \big(1 - n + C(n+k-1, 2)\big) \leq k + \epsilon_{n+k}. \tag{7.2}$$
- Suppose that $n \geq 1$. Start with
$$Y = \big((L(d_1) \# L(d_2) \# \cdots \# L(d_k)) \times S^1\big) \# B_{k+n-1}.$$
Then perform k surgeries that identify the k generators of the connected sum of lens spaces with the first k generators of $\pi_1(B_{k+n-1})$. These surgeries do not change the rank of $H_2(Y; \mathbf{Q})$. Finally perform $n-1$ surgeries along circles representing the commutator of the S^1 factor and the last $n-1$ generators of $\pi_1(B_{k+n-1})$. Each of these surgeries increases the rank of the second rational homology by 2 since the commutators are nullhomologous. This produces a 4–manifold X with $\pi_1(X) \cong G$ and $\mathrm{rk}_{\mathbf{Q}} H_2(X; \mathbf{Q}) = 2(n-1) + \mathrm{rk}_{\mathbf{Q}} H_2(B_{k+n-1}; \mathbf{Q})$. Hence
$$\chi(X) = \mathrm{rk}_{\mathbf{Q}} H_2(B_{k+n-1}; \mathbf{Q}).$$
Thus, if $n + k - 1 \neq 3, 5$, $\chi(X) = C(n+k-1, 2) + \epsilon_{n+k-1}$.
Referring to Equation (7.1), this gives the upper bound
$$0 \leq q(G) - \big(1 - n + C(n+k-1, 2)\big) \leq n - 1 + \epsilon_{n+k-1} \tag{7.3}$$
(for n, k satisfying $n + k - 1 \neq 3, 5$ and $n \geq 1$).
- Consider now the case $n = 0$. Start with the 4–manifold obtained from $\big((L(d_1) \# \cdots \# L(d_{k-1})) \times S^1\big) \# B_{k-1}$ by performing surgery to identify the generators of $\pi_1(B_{k-1})$ with the generators for the lens spaces. This yields a closed 4–manifold Y with $\pi_1(Y) \cong \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_{k-1} \oplus \mathbf{Z}$. Surgering d_k times the last generator gives a closed 4–manifold X with $\pi_1(X) \cong \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k$ and $\mathrm{rk}_{\mathbf{Q}}(H_2(X; \mathbf{Q})) = \mathrm{rk}_{\mathbf{Q}}(H_2(B_{k-1}; \mathbf{Q}))$. Therefore,
$$\chi(X) = 2 + \mathrm{rk}_{\mathbf{Q}}(H_2(B_{k-1}; \mathbf{Q})).$$

Thus when $k-1 \neq 3, 5$, $\chi(X) = 2 + C(k-1,2) + \epsilon_{k-1}$, and so for $G = \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k$,

$$0 \leq q(G) - \big(1 + C(k-1,2)\big) \leq 1 + \epsilon_{k-1}. \tag{7.4}$$

- In two cases the rational homology gives better lower bounds than the $\mathbf{Z}/p$ homology. For $G = \mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_k$, $\mathrm{rk}_{\mathbf{Q}}(H_1(G;\mathbf{Q})) = 0$ and $\mathrm{rk}_{\mathbf{Q}}(H_2(G;\mathbf{Q})) = 0$ and hence $q(G) \geq 2$. Combined with Equation (7.4), this shows that $q(\mathbf{Z}/d) = 2$ and $q(\mathbf{Z}/d_1 \oplus \mathbf{Z}/d_2) = 2$.

The estimates (7.1), (7.3), (7.2), and (7.4) and the discussion of the previous paragraph combine to give a proof of Theorem 2.

8. Remarks and questions

A variant of $q(G)$ is obtained by defining $p(G)$ to be the smallest value of $\chi(X) - \sigma(X)$ for all closed oriented 4–manifolds X with $\pi_1(X) = G$. Here $\sigma(X)$ denotes the signature of X. Notice that all the examples we constructed for G abelian have signature zero. We conjecture that $p(\mathbf{Z}^n) = 2 - 2n + C(n, 2)$. This guess is motivated by a slight amount of redundancy which occurs in the constructions given above when $\epsilon_n = 1$. For example, in the case of $n = 6$, the construction we gave starts with $F_2 \times F_4$ and uses one 4–reduction to abelianize the fundamental group of F_4. In particular, the surface relation $[y_1, y_2][y_3, y_4] = 1$ shows that one of the 6 relations coming from the 4–reduction is unnecessary. This extra bit of flexibility may perhaps be used to twist the geometric construction slightly to introduce some signature.

An interesting question is whether the invariant q depends on the category of the manifold or the choice of geometric structure. For example, one might consider the infimum of $\chi(X)$ over smooth 4–manifolds or topological 4–manifolds or even 4–dimensional Poincaré complexes with $\pi_1(X) = G$. The manifolds we constructed for $G = \mathbf{Z}^n$ are smooth, and the lower bounds are homotopy invariants, so that for $G = \mathbf{Z}^n$ the value of q is independent of the category.

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