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ON THE ALGEBRA OF FUNCTIONS $\mathcal{C}^k$-EXTENDABLE FOR EACH k FINITE

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ABSTRACT. For each positive integer l we construct a $\mathcal{C}^l$-function of one real variable, the graph Γ of which has the following property: there exists a real function on Γ which is $\mathcal{C}^k$-extendable to $\mathbb{R}^2$, for each k finite, but it is not $\mathcal{C}^\infty$-extendable.

INTRODUCTION

Let X be a locally closed subset of $\mathbb{R}^n$, i.e. closed in an open subset G of $\mathbb{R}^n$. Consider the following $\mathbb{R}$-algebras of functions:

$$\mathcal{C}^k(X) = \{\, f : X \longrightarrow \mathbb{R} \mid \exists\, \tilde{f} : G \longrightarrow \mathbb{R} \text{ of class } \mathcal{C}^k : \tilde{f}|X = f\},$$

where $k \in \mathbb{N} \cup \{\infty\}$, and the $\mathbb{R}$-algebra of functions which can be called *almost $\mathcal{C}^\infty$-functions* on X:

$$\mathcal{C}^{(\infty)}(X) = \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(X) = \varprojlim_{k\in\mathbb{N}} \mathcal{C}^k(X).$$

Obviously, we have

$$\mathcal{C}^\infty(X) \subset \mathcal{C}^{(\infty)}(X) \subset \mathcal{C}^k(X), \quad k \in \mathbb{N}.$$

A fundamental question concerning singularities of the set X is the following: *When does $\mathcal{C}^{(\infty)}(X) = \mathcal{C}^\infty(X)$?*

The answer is affirmative in the following cases:

1) Clearly, when X is open, it means when $X = G$. More generally, if X is a closed C^∞-submanifold of G.

2) When $X = \overline{\operatorname{int} X} \cap G$, because then $\mathcal{C}^k(X)$ is naturally isomorphic to the algebra $\mathcal{E}^k(X)$ of $\mathcal{C}^k$-Whitney fields on X ($k \in \mathbb{N}\cup\{\infty\}$) (cf. [W]), and consequently,

$$\mathcal{C}^{(\infty)}(X) = \varprojlim_{k\in\mathbb{N}} \mathcal{C}^k(X) = \varprojlim_{k\in\mathbb{N}} \mathcal{E}^k(X) = \mathcal{E}^\infty(X) = \mathcal{C}^\infty(X).$$

More generally, when $X \subset M$, M is a closed $\mathcal{C}^\infty$-submanifold of G and X is the closure of its interior in M.

3) When $n = 1$ (cf. [M]).

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4) When X is a closed *semianalytic* subset of G. Not all *subanalytic* subsets have this property, and this property distinguishes an important class of subanalytic sets (cf. [BMP]).

In [P] the author gave an example of a subset of $\mathbb{R}^2$ on which there are almost $\mathcal{C}^\infty$-functions that are not $\mathcal{C}^\infty$. Simplifying and clarifying the construction from [P], here we will prove the following.

Theorem. *For each positive integer l there exists a function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ of class $\mathcal{C}^l$ such that $\mathcal{C}^{(\infty)}(\tilde{\varphi}) \neq \mathcal{C}^\infty(\tilde{\varphi})$, where $\tilde{\varphi} \subset \mathbb{R} \times \mathbb{R}$ stands for the graph of the function φ.*

Proof of the Theorem

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ and $(a_\nu)_\nu \subset \mathbb{R}$ be such that

(I) $a_1 > a_2 > ... > a_\nu > ...$, $a_\nu \to 0$ $(\nu \to \infty)$;

(II) $\varphi|\mathbb{R} \setminus (\{a_\nu : \nu \in \mathbb{N}^*\} \cup \{0\}) : \mathbb{R} \setminus (\{a_\nu : \nu \in \mathbb{N}^*\} \cup \{0\}) \longrightarrow \mathbb{R}$ is $\mathcal{C}^\infty$ ($\mathbb{N}^*$ (resp. $\mathbb{N}$) will denote the set of positive (resp. non-negative) integers);

(III) $\varphi|(a_{\nu+1}, a_{\nu-1})$ is $\mathcal{C}^\nu$ but not $\mathcal{C}^{\nu+1}$ $(a_0 := +\infty)$;

(IV) $\forall\, k \in \mathbb{N}$: $\lim_{x\to 0}\varphi^{(k)}(x)$ exists in $\mathbb{R}$ and $\lim_{x \to 0}\varphi(x) = \varphi(0)$.

Lemma. *Fix ν. If $f, g : U \longrightarrow \mathbb{R}$ are $\mathcal{C}^{\nu+1}$-functions in a neighbourhood U of $(a_\nu, \varphi(a_\nu))$ in $\mathbb{R}^2$ such that $f = g$ in $U \cap \tilde{\varphi}$, then*

$$\frac{\partial f}{\partial y}(a_\nu, \varphi(a_\nu)) = \frac{\partial g}{\partial y}(a_\nu, \varphi(a_\nu)).$$

Proof of the Lemma. Put $\omega(x) := f(x, \varphi(x)) = g(x, \varphi(x))$, for x near a_ν. Then

$$\omega^{(k)}(x) = P_k(\{\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x, \varphi(x))\}_{i+j \leqslant k}, \varphi'(x), ..., \varphi^{(k-1)}(x)) + \varphi^{(k)}(x)\frac{\partial f}{\partial y}(x, \varphi(x)),$$

for x near a_ν, $x \neq a_\nu$ and any $k \in \mathbb{N}$, where P_k is a polynomial depending only on k.

In particular,

$$\omega^{(\nu+1)}(x) = $$
$$P_{\nu+1}(\{\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x, \varphi(x))\}_{i+j \leqslant \nu+1}, \varphi'(x), ..., \varphi^{(\nu)}(x)) \quad + \quad \varphi^{(\nu+1)}(x)\frac{\partial f}{\partial y}(x, \varphi(x)).$$

$\forall\, k = 0, \dots, \nu\, \exists\, \alpha_k \in \mathbb{R} : \lim_{x \to a_\nu} \varphi^{(k)}(x) = \alpha_k$ and $\lim_{x \to a_\nu} \varphi^{(\nu+1)}(x)$ does not exist in $\mathbb{R}$.

Two cases:

(1) There are two sequences $(b_n)_n$, $(c_n)_n \subset \mathbb{R}$ converging to a_ν such that

$$\lim_{n\to\infty} \varphi^{(\nu+1)}(b_n) = \beta, \qquad \lim_{n\to\infty} \varphi^{(\nu+1)}(c_n) = \gamma, \beta \neq \gamma.$$

(2) There is a sequence $(b_n)_n \subset \mathbb{R}$ converging to a_ν such that

$$\lim_{n\to\infty} \varphi^{(\nu+1)}(b_n) = \pm\infty.$$

In case (1),

$$\lim_{n\to\infty} \omega^{(\nu+1)}(b_n) = P_{\nu+1}(\{\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(a_\nu, \varphi(a_\nu))\}_{i+j \leqslant \nu+1}, \alpha_1, ..., \alpha_\nu) + \beta\frac{\partial f}{\partial y}(a_\nu, \varphi(a_\nu)),$$
$$\lim_{n\to\infty} \omega^{(\nu+1)}(c_n) = P_{\nu+1}(\{\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(a_\nu, \varphi(a_\nu))\}_{i+j \leqslant \nu+1}, \alpha_1, ..., \alpha_\nu) + \gamma\frac{\partial f}{\partial y}(a_\nu, \varphi(a_\nu)).$$

Consequently,

$$\frac{\lim_{n\to\infty}[\omega^{(\nu+1)}(b_n)-\omega^{(\nu+1)}(c_n)]}{\beta-\gamma}=\frac{\partial f}{\partial y}(a_\nu,\varphi(a_\nu))=\frac{\partial g}{\partial y}(a_\nu,\varphi(a_\nu)).$$

In case (2),

$$\omega^{(\nu+1)}(b_n)=\text{sequence with a finite limit}+\varphi^{(\nu+1)}(b_n)\frac{\partial f}{\partial y}(b_n,\varphi(b_n)).$$

Since $\varphi^{(\nu+1)}(b_n)\to\pm\infty$ we have

$$\lim_{n\to\infty}\frac{\omega^{(\nu+1)}(b_n)}{\varphi^{(\nu+1)}(b_n)}=\frac{\partial f}{\partial y}(a_\nu,\varphi(a_\nu))=\frac{\partial g}{\partial y}(a_\nu,\varphi(a_\nu)).$$

□

To finish the proof of the theorem first take a $\mathcal{C}^\infty$-function $\lambda:\mathbb{R}\longrightarrow\mathbb{R}$ such that $\lambda^{(k)}(0)=\lim_{x\to 0}\varphi^{(k)}(0)$ for each $k\in\mathbb{N}$ (by Borel's theorem), and then define

$$f(x,y):=\frac{y-\lambda(x)}{x},\quad \text{for } (x,y)\in\varphi\setminus\{(0,\varphi(0))\},\quad \text{and } f(0,\varphi(0)):=0.$$

Fix any $k\in\mathbb{N}$. For $(x,y)\neq(0,\varphi(0))$, $f(x,y)=\psi(x)$, where

$$\psi(x):=\frac{\varphi(x)-\lambda(x)}{x},\quad \text{for } x\in\mathbb{R}\setminus\{0\},\quad \text{and } \psi(0):=0.$$

ψ is $\mathcal{C}^k$ on the set $(-\infty,a_{k-1})\setminus\{0\}$, due to the properties (II)-(IV). On the other hand, by l'Hôpital's rule,

$$\forall p,q\in\mathbb{N}:\lim_{x\to 0}\frac{\varphi^{(p)}(x)-\lambda^{(p)}(x)}{x^q}=0.$$

This implies in an easy way that $\lim_{x\to 0}\psi^{(p)}(x)=0$, for all $p\in\mathbb{N}$.

Consequently, ψ is a $\mathcal{C}^k$-function on $(-\infty,a_{k-1})$, which can be treated as a $\mathcal{C}^k$-function on $(-\infty,a_{k-1})\times\mathbb{R}$ not depending on y. On the other hand, $\dfrac{y-\lambda(x)}{x}$ is a $\mathcal{C}^\infty$-function on $(a_k,+\infty)\times\mathbb{R}$, so it suffices now to glue smoothly these two functions along the strip $(a_k,a_{k-1})\times\mathbb{R}$.

To check that f cannot be extended to a $\mathcal{C}^\infty$-function $F:\mathbb{R}^2\longrightarrow\mathbb{R}$, suppose that such an extension F exists. Then from the Lemma

$$\frac{\partial F}{\partial y}(a_\nu,\varphi(a_\nu))=\frac{\partial(\frac{y-\lambda(x)}{x})}{\partial y}(a_\nu,\varphi(a_\nu))=\frac{1}{a_\nu}\to+\infty,$$

but, of course,

$$\frac{\partial F}{\partial y}(a_\nu,\varphi(a_\nu))\to\frac{\partial F}{\partial y}(0,\varphi(0)),$$

a contradiction.

Remark. It follows from [G] (the author is indebted to Rémi Soufflet for this reference) that the function φ in our theorem can be chosen in such a way that the germ of φ at 0 belongs to a Hardy field of germs of real functions at 0.

References

[BMP] E. Bierstone, P. D. Milman and W. Pawłucki, *Composite differentiable functions*, Duke Math. J. **83** (1996), 607–620. MR 1390657 (97k:32011)

[G] D. Gokhman, *Functions in a Hardy field not ultimately* $\mathcal{C}^\infty$, Complex Variables Theory Appl. **(1) 32** (1997), 1–6. MR 1448476 (98e:26024)

[M] J. Merrien, *Prolongateurs de fonctions différentiables d'une variable réelle*, J. Math. Pures Appl. **(9) 45** (1966), 291–309. MR 0207937 (34:7750)

[P] W. Pawłucki, *Examples of functions* $\mathcal{C}^k$*-extendable for each* k *finite, but not* $\mathcal{C}^\infty$*-extendable. Singularities Symposium - Łojasiewicz 70*, Banach Center Publ. Polish Acad. Sci., Warsaw **44** (1998), 183–187. MR 1677379 (99m:32008)

[W] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Am. Math. Soc. **36** (1934), 63–89. MR 1501735

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