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INFINITE SYSTEMS OF LINEAR EQUATIONS FOR REAL ANALYTIC FUNCTIONS

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Abstract. We study the problem when an infinite system of linear functional equations

$$\mu_n(f) = b_n \quad \text{for } n \in \mathbb{N}$$

has a real analytic solution f on $\omega \subseteq \mathbb{R}^d$ for every right-hand side $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ and give a complete characterization of such sequences of analytic functionals (μ_n). We also show that every open set $\omega \subseteq \mathbb{R}^d$ has a complex neighbourhood $\Omega \subseteq \mathbb{C}^d$ such that the positive answer is equivalent to the positive answer for the analogous question with solutions holomorphic on Ω.

Introduction

One of the first classical examples of an infinite system of linear equations is related to the so-called *moment problem* solved by Hausdorff. It is the question of finding a Borel measure μ on $[0,1]$ such that a given sequence of reals $(b_n)_{n\in\mathbb{N}}$ is the sequence of moments of μ (i.e., $\int_0^1 t^n d\mu(t) = b_n$ for $n \in \mathbb{N}$). More generally, we can look for a functional $f \in X'$ such that for the given sequence of scalars (b_n) and of vectors (x_n) belonging to a locally convex (Banach) space X the following holds:

$$f(x_n) = b_n \qquad \text{for } n \in \mathbb{N}. \tag{1}$$

First solutions for spaces $X = C[0,1]$ or $L_p[0,1]$ are due to F. Riesz (1909) and the functional analytic approach to the problem is contained in the famous book of Banach [1, Ch. IV §7, §8]; see also [23, pp. 106-107].

Later on, Eidelheit, a colleague of Banach, characterized in all Fréchet spaces what are now called Eidelheit sequences [9] (see [17, Th. 26.27], [12, II.38.6]). Let X be a locally convex space. We call a sequence $(\mu_n)_n \in \mathbb{N}$ of continuous linear functionals on X an *Eidelheit sequence on X* if for every sequence of scalars $(b_n)_n \in \mathbb{N}$ there is $f \in X$ satisfying

$$\mu_n(f) = b_n \quad \text{for } n \in \mathbb{N}. \tag{2}$$

The notion of Eidelheit sequences has been extensively studied in Fréchet spaces later on; see [9], [18], [20], [21] and [22], compare also [17, §26].

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We will prove a completely analogous result for the highly non-metrizable space of real analytic functions $\mathscr{A}(\omega)$ on an arbitrary open set $\omega \subseteq \mathbb{R}^d$ (Theorem 2.2). We show that for every open set $\omega \subseteq \mathbb{R}^d$ there is a domain of holomorphy $\Omega \subseteq \mathbb{C}^d$, $\Omega \cap \mathbb{R}^d = \omega$, such that every Eidelheit sequence on $\mathscr{A}(\omega)$ is automatically Eidelheit on the space $H(\Omega)$ of holomorphic functions on Ω (Theorem 2.2). The main tool in the proof is given by the so-called *distinguished sets*, which might be of independent interest (see Lemma 1.1 and Lemma 1.3).

Let $(f_{1,n})_n \in \mathbb{N}$, $(f_{2,n})_n \in \mathbb{N}$ be two Eidelheit sequences on $\mathscr{A}(\omega_1)$ and $\mathscr{A}(\omega_2)$, respectively. We consider the question if there is a continuous linear map (= *operator*) $T : \mathscr{A}(\omega_1) \to \mathscr{A}(\omega_2)$ such that $f_{1,n} = f_{2,n} \circ T$ for every $n \in \mathbb{N}$ (compare an analogous problem on Fréchet spaces due to Mityagin [18] and its solution in [21]). It turns out that for every $(f_{1,n})_{n\in\mathbb{N}}$ there is $(f_{2,n})_n \in \mathbb{N}$ without such factorization (Proposition 3.1). On the other hand, we show a positive result for sequences related to the interpolation problem for analytic functions (Theorem 3.2).

Let us recall that the space $\mathscr{A}(\omega)$, $\omega \subseteq \mathbb{R}^d$ an open subset, of real analytic functions $f : \omega \longrightarrow \mathbb{C}$ is equipped with the topology of the projective limit $\operatorname{proj}_N H(K_N)$, where (K_N) is an exhaustion of ω by a sequence of compact sets

$$K_1 \subset\subset K_2 \subset\subset \ldots \subset\subset K_N \subset\subset \ldots \subset \omega, \quad \bigcup_{N\in\mathbb{N}} K_N = \omega,$$

and $H(K)$ denotes the space of germs of analytic functions over K with its natural LB-space topology. It is known (see [16, Prop. 1.7, 1.2]) that this topology is equal to the inductive limit topology $\operatorname{ind} H(U)$, where U runs over all open neighborhoods of ω in $\mathbb{C}^d$ and $H(U)$ denotes the Fréchet space of holomorphic functions on U with the compact open topology. Thus $\mathscr{A}(\omega)$ is a complete, separable, ultrabornological, reflexive, webbed nuclear space with the approximation property (but non-metrizable and without a basis [7]) with the dual being a complete LF-space $\mathscr{A}(\omega)'_\beta = \operatorname{ind} H(K_N)'_\beta$. The space $\mathscr{A}(\omega)$ is the projective limit of a sequence of LB-spaces. Spaces of this type are called *PLB-spaces*. For more information on the space of real analytic functions, see [2], [4], [5], [7], [8] and [16].

For smooth functions on $\omega \subseteq \mathbb{R}^d$ we use the standard multi-index notation. So if $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, then

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} f, \quad \delta_z(f) := f(z), \quad \delta_z^\alpha(f) := (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} f(z),$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_d$.

The elements $\mu \in H(\mathbb{C}^d)'$ are called analytic functionals. A compact subset $K \subset \mathbb{C}^d$ is called a *carrier* for μ if for every neighborhood $U \subset\subset \mathbb{C}^d$ of K there is a continuity estimate

$$|\mu(f)| \leq C \sup_{z\in U} |f(z)|.$$

We may identify $\mathscr{A}(\mathbb{R}^d)'$ with the analytic functionals that are carried by some $K \subset \mathbb{R}^d$. In this case there is a smallest real carrier which is called the support of μ and denoted by $\operatorname{supp} \mu$ (see [19, p. 44 ff.]). For open $\omega \subset \mathbb{R}^d$ we may identify $\mathscr{A}(\omega)'$ with the analytic functionals μ with $\operatorname{supp} \mu \subset \omega$.

The other notation is standard. We refer for functional analysis to [17] and for complex analysis to [13]. For analytic functionals see [19] or [15].

1. Distinguished open sets

We will need some auxiliary notions. We call a nonnegative C^∞-function $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ *distinguished* whenever it vanishes at infinity and

$$\sup_{x\in\mathbb{R}^d} \|H_\psi(x)\| < 1,$$

where H_ψ denotes the Hessian of ψ and $\|\cdot\|$ is the norm in the space of quadratic forms on the d-dimensional euclidean space. For every distinguished function ψ we define

$$\Omega^\psi := \{z = x + iy \in \mathbb{C}^d : |y|^2 < \psi(x)\}$$

and call it a *distinguished open set* with *basis* $\omega := \Omega^\psi \cap \mathbb{R}^d$. These sets and the construction below are variations of the proof of the Cartan-Grauert Theorem as given in [3, Prop. 1] and [10, Prop. 6 and Prop. 7].

Lemma 1.1. (a) *Every distinguished open set Ω^ψ is a domain of holomorphy and for every $\alpha > 0$, $\Omega^\psi_\alpha := \{z : |y|^2 < \psi(x) - \alpha\} \subset\subset \Omega^\psi$. Moreover, polynomials are dense in $H(\Omega^\psi)$ equipped with the compact-open topology.*

(b) *For every pair of open sets $U \subset W \subset \mathbb{C}^d$, U distinguished, there is a distinguished set Ω^ψ with the basis $W \cap \mathbb{R}^d$ such that $U \subseteq \Omega^\psi \subseteq W$. If $U \subset\subset W$, then we can choose ψ such that $U \subset\subset \Omega^\psi$.*

(c) *Every open set $\omega \subset \mathbb{R}^d$ has a basis of open neighbourhoods in $\mathbb{C}^d$ consisting of distinguished sets.*

(d) *Every compact set in $\mathbb{C}^d$ is contained in some bounded distinguished set.*

Parts (a) and (c) imply a corollary going back to Cartan (see [11, Cor. II. 3.15]):

Corollary 1.2. *Every open set ω in $\mathbb{R}^d$ has a basis of neighbourhoods in $\mathbb{C}^d$ consisting of domains of holomorphy.*

Proof of Lemma 1.1. (a): Let $f(z) := |y|^2 - \psi(x)$ for $z = x + iy \in \mathbb{C}^d$. Since $\|H_\psi(x)\| < 1$, the Levi form of f is positive definite. Thus, f is plurisubharmonic on $\mathbb{C}^d$. Clearly $\Omega^\psi = \{z : f(z) < 0\}$. Since ψ vanishes at infinity, $\Omega^\psi_\alpha \subset\subset \Omega^\psi$. By [13, Cor. 5.4.3], every function holomorphic on a neighbourhood of $\overline{\Omega^\psi_\alpha}$ can be approximated uniformly on this set by entire functions.

(b): Let $U = \Omega^{\psi_0}$ and

$$s(x) := \inf\{|w| : x + i(y + w) \notin W \text{ for some } y, |y|^2 \le \psi_0(x)\}.$$

Observe that

(i) since ψ_0 is continuous and W is open, s is lower semi-continuous;
(ii) $V := \{x \in W \cap \mathbb{R}^d : s(x) > 0\}$ is open and contains $(W \setminus U) \cap \mathbb{R}^d$;
(iii) if $U \subset\subset W$, then $V = W \cap \mathbb{R}^d$.

We choose a covering of V by sets

$$U_{r_j}(x_j) = \{x \in \mathbb{R}^d : |x - x_j| < r_j\} \subset\subset W \cap \mathbb{R}^d.$$

We set

$$\varphi(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \ge 1, \end{cases} \qquad \psi_1(x) = \sum_{j=1}^\infty \varepsilon_j \varphi\left(\frac{1}{r_j}(x - x_j)\right).$$

By (i) and a suitable choice of (ε_j), we get that the series is convergent and

(iv) $\psi_1(x) < s^2(x)$;
(v) $\sup_{x\in\mathbb{R}^d} \|H_{\psi_1}(x)\| < 1 - \sup_{x\in\mathbb{R}^d} \|H_{\psi_0}(x)\|$.

By (v), $\psi := \psi_0 + \psi_1$ is a distinguished function. It is immediate that $U \subseteq \Omega^\psi$ and, by (iv), we get $\Omega^\psi \subseteq W$. By (ii), the basis of ψ is equal to $W \cap \mathbb{R}^d$. Finally, if $U \subset\subset W$, then , by (iii), $\psi_1|_{\overline{U\cap\mathbb{R}^d}} > \delta > 0$. Thus $U \subset \Omega^\psi_\delta \subset\subset \Omega$.

(c): Take in (b) $U = \emptyset$ and $\omega = W \cap \mathbb{R}^d$.

(d): It suffices to observe that if φ is defined as in (b) and $\varphi_a := a\varphi(\frac{x}{a})$, then $H_{\varphi_a} = a^{-1}H_\varphi$. Therefore, for a big enough, φ_a is a distinguished function but $\varphi_a \longrightarrow +\infty$ uniformly on compact sets as $a \longrightarrow \infty$. □

The proof of the following lemma follows the line of, e.g., the proof of [19, Théorème 111].

Lemma 1.3. *Let ω be open and bounded in $\mathbb{R}^d$, Ω^ψ a distinguished set with basis ω. Then for every $\mu \in H(\Omega^\psi)' \cap \mathscr{A}(\mathbb{R}^d)'$ we have* $\mathrm{supp}\mu \subset \overline{\omega}$.

Proof. Let $\Omega \supset \overline{\omega}$ be open and bounded in $\mathbb{C}^d$. Let Ω_1 be an open neighbourhood of $\mathbb{R}^d$ in $\mathbb{C}^d$ such that $\Omega^\psi \cap \Omega_1 \subseteq \Omega$. By Lemma 1.1, there is a distinguished set Ω^{ψ_1} with basis $\mathbb{R}^d$ such that $\Omega^\psi \subseteq \Omega^{\psi_1} \subseteq \Omega^\psi \cup \Omega_1$. Since Ω^{ψ_1} is a domain of holomorphy with covering $\Omega_1 \cap \Omega^{\psi_1}$ and $\Omega^\psi \cap \Omega^{\psi_1} = \Omega^\psi$, the solution of the first Cousin problem yields the following exact sequence:

$$0 \longrightarrow H(\Omega^{\psi_1}) \xrightarrow{\ j\ } H(\Omega_1 \cap \Omega^{\psi_1}) \oplus H(\Omega^\psi) \xrightarrow{\ \delta\ } H(\Omega_1 \cap \Omega^\psi) \longrightarrow 0$$

where $j(f) = (f, f)$ and $\delta(f, g) = f - g$.

On $H(\Omega_1 \cap \Omega^{\psi_1}) \oplus H(\Omega^\psi)$ we define $u(f, g) = \mu f - \mu g$. Since $u \circ j = 0$ on $H(\mathbb{C}^d)$ and, by density of $H(\mathbb{C}^d)$ in $H(\Omega^{\psi_1})$ (see Lemma 1.1 (a)), we have $u \circ j = 0$ on $H(\Omega^{\psi_1})$ and u gives rise to an element $\mu \in H(\Omega_1 \cap \Omega^\psi)'$ that clearly extends the given $\mu \in \mathscr{A}(\mathbb{R}^d)'$.

Since $\Omega^\psi \cap \Omega_1 \subseteq \Omega$ this proves the result. □

2. Eidelheit sequences

It turns out that a condition very similar to Eidelheit's characterization of Eidelheit sequences on Fréchet spaces is also necessary for any PLB-space.

Lemma 2.1. *If $(f_n)_{n\in\mathbb{N}}$ is an Eidelheit sequence on a PLB-space $X = \mathrm{proj}_{N\in\mathbb{N}} X_N$, then for every $N \in \mathbb{N}$ we have*

$$\dim(\mathrm{span}\{f_n : n \in \mathbb{N}\} \cap X_N') < \infty. \tag{3}$$

Proof. Let us assume that for some N the condition (3) does not hold. Since X_N is an LB-space, there is a bounded set $B \subset (X_N)'_\beta$ such that $\mathrm{span}\{f_n : n \in \mathbb{N}\} \cap B$ has infinite dimension. Clearly, for a continuous surjection $T\colon X \longrightarrow \mathbb{C}^\mathbb{N}$, $T(g) := (f_n(g))_{n\in\mathbb{N}}$,

$$\begin{aligned} (T')^{-1}(B) &\subseteq (T')^{-1}(B^{\circ\circ}) = \{g \in (\mathbb{C}^\mathbb{N})' \colon T'g \in B^{\circ\circ}\} \\ &= \{g \in (\mathbb{C}^\mathbb{N})' \colon |g(Tx)| \le 1 \text{ for } x \in B^\circ\} = (T(B^\circ))^\circ. \end{aligned}$$

Since T is open (as a surjective continuous operator from the webbed space onto a Fréchet space), $T(B^\circ)$ is a 0-neighbourhood and $(T(B^\circ))^\circ$ is bounded in $\varphi = (\mathbb{C}^\mathbb{N})'$.

Therefore $(T(B^\circ))^\circ$ and $(T')^{-1}(B)$ are finite dimensional. Since T' is injective we get

$$\dim(\mathrm{Im}T' \cap B) = \dim(\mathrm{span}\{f_n : n \in \mathbb{N}\} \cap B) < \infty, \quad \text{a contradiction.}$$

□

Remark. The condition (2) does not characterize Eidelheit sequences on an arbitrary PLB-space. In [6, Ex. 2.8] the authors constructed an example of a PLB-space X with nuclear steps such that its ultrabornological associated topology makes it a nuclear LB-space. Clearly, no sequence on X is Eidelheit (since it must be an Eidelheit sequence on its ultrabornological associated space, and therefore, on an LB-space). On the other hand, by the construction, X cannot be represented as a projective limit of LB-spaces with surjective linking maps, and therefore, there exists a sequence $(f_n)_n \in \mathbb{N} \subseteq X'$ satisfying (2).

Surprisingly, for the space $\mathscr{A}(\omega)$ the above condition is also sufficient.

Theorem 2.2. *Let $\omega \subseteq \mathbb{R}^d$ be an arbitrary domain. Then there exists a domain of holomorphy Ω in $\mathbb{C}^d$ ($\Omega = \mathbb{C}^d$ if $\omega = \mathbb{R}^d$), $\Omega \cap \mathbb{R}^d = \omega$, such that for any sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathscr{A}(\omega)'$ of linearly independent analytic functionals the following assertions are equivalent:*

(a) *$(\mu_n)_{n\in\mathbb{N}}$ is an Eidelheit sequence on $\mathscr{A}(\omega)$;*
(b) *$(\mu_n)_{n\in\mathbb{N}}$ is an Eidelheit sequence on $H(\Omega)$;*
(c) *for every compact set $K \subset\subset \omega$ we have $\dim(\mathrm{span}\{\mu_n : n \in \mathbb{N}\} \cap \{\mu \in \mathscr{A}(\mathbb{R}^d)' : \mathrm{supp}\mu \subset K\}) < \infty$.*

Proof. Clearly (b)$\Longrightarrow$(a)$\Longrightarrow$(c) follows from Lemma 2.1. Observe that by [19, Lemma 112], μ extends to $\mu \in H(K)'$ if and only if $\mathrm{supp}\mu \subseteq K$.

(c)$\Longrightarrow$(b): Let (ω_n) be an exhaustion of ω, $\omega_n \subset\subset \omega_{n+1}$ for $n \in \mathbb{N}$. By Lemma 1.1 (b), there exists an increasing family of distinguished sets Ω^{ψ_n} with bases ω_n, $\Omega^{\psi_n} \subset\subset \Omega^{\psi_{n+1}}$ for $n \in \mathbb{N}$. In the case of $\omega = \mathbb{R}^d$ we can take Ω^{ψ_n} containing the ball of radius n in $\mathbb{C}^d$ (use Lemma 1.1 (d)). We define $\Omega := \bigcup_{n\in\mathbb{N}} \Omega^{\psi_n}$, which is a domain of holomorphy by Lemma 1.1 (a) and [13, 3.3.7]. By Lemma 1.3, the assumption implies that for every compact set $L \subset\subset \Omega$ we have

$$\dim(\mathrm{span}\{\mu_n : n \in \mathbb{N}\} \cap \{\mu \in H(\Omega)' : \mathrm{supp}\mu \subset L\}) < \infty.$$

Therefore the classical Eidelheit theorem (see [9], [17, Th. 26.27]) yields the result. □

3. Factorization of Eidelheit sequences

Proposition 3.1. *For every Eidelheit sequence $(f_n)_n \in \mathbb{N}$ on $\mathscr{A}(\omega)$ there is an Eidelheit sequence $(g_n)_n \in \mathbb{N}$ on $\mathscr{A}(\omega)$ such that there is no operator $T : \mathscr{A}(\omega) \to \mathscr{A}(\omega)$ so that $g_n \circ T = f_n$ for all $n \in \mathbb{N}$.*

Proof. By Theorem 2.2, there is a domain of holomorphy $\Omega \subseteq \mathbb{C}^d$ such that $\Omega \cap \mathbb{R}^d = \omega$ and $(f_n)_n \in \mathbb{N}$ is an Eidelheit sequence on $H(\Omega)$ as well. By [8] (compare [4]), there is a Fréchet quotient F of $\mathscr{A}(\omega)$ that is not a quojection. Since $H(\Omega)$ is quasinormable, by [21, Th. 2.4], there is an Eidelheit sequence $(h_n)_{n\in\mathbb{N}}$ on F such that for no operator $S : H(\Omega) \to F$ the factorization $h_n \circ S = f_n$ holds.

Let $q : \mathscr{A}(\omega) \to F$ be the quotient map. Observe that $g_n := h_n \circ q$ is the Eidelheit sequence we are looking for. Indeed, if $g_n \circ T = f_n$, then $h_n \circ q \circ T = f_n$, which contradicts the choice of $(h_n)_{n\in\mathbb{N}}$. □

We will consider now Eidelheit sequences of the form $(\delta^\alpha_{z_n})_{n\in\mathbb{N},|\alpha|\le k_n}$ on $\mathscr{A}(\omega)$, $\omega\subseteq\mathbb{R}^d$, where $(k_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of natural numbers and (z_n) is a discrete sequence. We call such sequences *interpolation sequences*.

Theorem 3.2. *Let $(f_n^{(1)})\subseteq\mathscr{A}(\omega_1)'$ and $(f_n^{(2)})\subseteq\mathscr{A}(\omega_2)'$ be two arbitrary interpolation sequences. There is always an operator $T\colon\mathscr{A}(\omega_1)\longrightarrow\mathscr{A}(\omega_2)$ such that T' maps $(f_n^{(2)})_{n\in\mathbb{N}}$ one-to-one onto $(f_n^{(1)})_n\in\mathbb{N}$.*

The proof of Theorem 3.2 follows immediately from Corollary 3.4, and Lemmas 3.5 and 3.6 below.

Lemma 3.3. *Let $\omega_1\subseteq\mathbb{R}^d$ and $\omega_2\subseteq\mathbb{R}^d$ be open domains, and let $(z_n)\subseteq\omega_1$ and $(y_n)\subseteq\omega_2$ be discrete sequences. Let $(k_n)_n\in\mathbb{N}$ be a sequence of natural numbers. Then there is a real analytic map $\varphi\colon\omega_1\longrightarrow\omega_2$ such that for every $n\in\mathbb{N}$, $\varphi(z_n)=y_n$, the Jacobian matrix $\mathscr{J}\varphi$ of φ at the points z_n is just the unit matrix and all the partial derivatives of φ at z_n of rank strictly larger than one and less than or equal to k_n are vanishing.*

Corollary 3.4. *Under the notation of the previous lemma, the composition operator $C_\varphi\colon\mathscr{A}(\omega_2)\longrightarrow\mathscr{A}(\omega_1)$, $C_\varphi(f):=f\circ\varphi$, satisfies*

$$C'_\varphi(\delta^\alpha_{z_n})=\delta^\alpha_{y_n}\quad \text{for } |\alpha|\le k_n,\ n\in\mathbb{N}.$$

Proof of Lemma 3.3. By [5, Lemma 4.2], there is a surjective real analytic map $\psi\colon\mathbb{R}^d\longrightarrow\omega_2$ of constant rank d. It is easily seen that for every $z\in\mathbb{R}^d$ the map C'_ψ maps $\operatorname{span}\{\delta^\alpha_z:|\alpha|\le k\}\subseteq\mathscr{A}(\mathbb{R}^d)'$ onto $\operatorname{span}\{\delta^\alpha_{\psi(z)}:|\alpha|\le k\}\subseteq\mathscr{A}(\omega_2)'$. By the classical interpolation problem on domains of holomorphy and Cor. 1.2, we find $\eta:\omega_1\to\mathbb{R}^d$ such that $\varphi:=\psi\circ\eta$ is the map we are looking for. □

Lemma 3.5. *Let (y_n), (x_n) be sequences in $\mathbb{R}^d$ tending to infinity. Let $(\alpha(n))$ and $(\beta(n))$ be sequences of multi-indices such that if $y_n=y_k$, then $\alpha(n)\ne\alpha(k)$. Then there is an operator $T\colon\mathscr{A}(\mathbb{R}^d)\longrightarrow\mathscr{A}(\mathbb{R}^d)$ such that*

$$\left(\frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}}Tf\right)(y_n)=\left(\frac{\partial^{\beta(n)}}{\partial x^{\beta(n)}}f\right)(x_n)\qquad \text{for every } n\in\mathbb{N}.$$

Proof. Let (k_n) be an increasing sequence of natural numbers such that $k_n\ge\max\{|\alpha(l)|:y_l=y_n\}$. Let us choose $\varphi_n:\mathbb{C}^d\to\mathbb{C}^d$, $\varphi_n(\mathbb{R}^d)\subseteq\mathbb{R}^d$, holomorphic such that

$$\varphi_n(y_k)=\begin{cases} x_n & \text{for } y_k=y_n,\\ 0 & \text{for } y_k\ne y_n,\end{cases}\qquad \text{and}\qquad \left(\frac{\partial^\beta}{\partial x^\beta}\varphi_n\right)(y_k)=0\quad\text{for } 1\le|\beta|\le k_n.$$

By the solution of the classical interpolation problem for entire functions, such φ_n exist and we may assume that $\varphi_n\to0$ uniformly on compact sets. Analogously, we define functions $g_n:\mathbb{C}^d\to\mathbb{C}$ such that $\left(\frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}}g_n\right)(y_n)=1$ and

$$\left(\frac{\partial^\beta}{\partial x^\beta}g_n\right)(y_k)=0\quad\text{for } |\beta|\le k_n \text{ and either } y_k\ne y_n \text{ or } y_k=y_n,\beta\ne\alpha(n).$$

Since the map

$$R:H(\mathbb{C}^d)\to\mathbb{C}^\mathbb{N},\qquad R(f):=\left(\left(\frac{\partial^\beta}{\partial x^\beta}f\right)(y_n)\right)_{n\in\mathbb{N},|\beta|\le k_n},$$

is surjective and open, we may choose g_n such that for every compact set $K \subseteq \mathbb{C}^d$,

$$\sup_{z\in K} \sum_{n\in\mathbb{N}} |g_n(z)| n^{k_n+1} < +\infty.$$

We can define the map

$$T : \mathscr{A}(\mathbb{R}^d) \to \mathscr{A}(\mathbb{R}^d), \qquad (Tf)(z) := \sum_{n\in\mathbb{N}} g_n(z) \cdot \left(\frac{\partial^{\beta(n)}}{\partial x^{\beta(n)}} f \right) (\varphi_n(z)).$$

Clearly, the series converges for every $f \in \mathscr{A}(\mathbb{R}^d)$ and, by the Closed Graph Theorem, the map is continuous. Moreover, for each n,

$$\begin{aligned} \left(\frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} Tf \right)(y_n) &= \sum_{y_k=y_n} \frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} \left[g_k \cdot \left(\frac{\partial^{\beta(k)}}{\partial x^{\beta(k)}} f \right) \circ \varphi_k \right](y_k) \\ &= \sum_{y_k=y_n} \left(\frac{\partial^{\beta(k)}}{\partial x^{\beta(k)}} f \right)(x_n) \cdot \left(\frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} g_k \right)(y_k) \\ &= \frac{\partial^{\beta(n)}}{\partial x^{\beta(n)}} f(x_n). \end{aligned}$$

This completes the proof. □

Lemma 3.6. *Let $\mathbb{R}$ be embedded in a standard way into $\mathbb{R}^d$, and let (z_n) be a discrete sequence in $\mathbb{R}$. Then there are operators $T\colon \mathscr{A}(\mathbb{R}) \longrightarrow \mathscr{A}(\mathbb{R}^d)$ and $S\colon \mathscr{A}(\mathbb{R}^d) \longrightarrow \mathscr{A}(\mathbb{R})$ such that $T'(\delta_{z_n}) = (\delta_{z_n})$ and $S'(\delta_{z_n}) = (\delta_{z_n})$ for $n \in \mathbb{N}$.*

Proof. One takes $T(f)(x_1, \ldots, x_d) := f(x_1)$, $S(f)(x) := f(x, 0, \ldots, 0)$. □

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