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TWO-WEIGHT CACCIOPPOLI INEQUALITIES FOR SOLUTIONS OF NONHOMOGENEOUS A-HARMONIC EQUATIONS ON RIEMANNIAN MANIFOLDS

SHUSEN DING

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Abstract. In this paper, we first prove the local two-weight Caccioppoli inequalities for solutions to the nonhomogeneous A-harmonic equation of the form $d^\star A(x, d\omega) = B(x, d\omega)$. Then, as applications of the local results, we prove the global two-weight Caccioppoli-type inequalities for these solutions on Riemannian manifolds.

1. Introduction and notation

The A-harmonic equation for differential forms has received much investigation in recent years. The A-harmonic equation is an important generalization of the p-harmonic equation in $\mathbf{R}^n$, $p > 1$, and the p-harmonic equation is a natural extension of the usual Laplace equation. Caccioppoli-type inequalities have been widely studied and frequently used in analysis and related fields, including partial differential equations and the theory of elasticity. Roughly speaking, these inequalities provide upper bounds for the L^s-norm of ∇u if u is a function (0-form) or du if u is a form in terms of the L^s-norm of the differential form u. Many mathematicians have made their contributions to this topic; see [6], [7], [9], [13], [14]. Different versions of the Caccioppoli-type inequality have been established in [1], [3], [12], [15] for solutions of the homogeneous A-harmonic equation $d^\star A(x, d\omega) = 0$ in a domain in $\mathbf{R}^n$. In this paper, we shall prove the Caccioppoli-type inequality for solutions of the nonhomogeneous A-harmonic equation $d^\star A(x, d\omega) = B(x, d\omega)$ on Riemannian manifolds on $\mathbf{R}^n$. Our main results are presented in Theorem 2.9 and Theorem 3.1, respectively. Each of them contains two weights and two parameters that make the inequalities more powerful and useful.

As usual, let $e_1, e_2, \cdots, e_n$ denote the standard orthogonal basis of $\mathbf{R}^n$. Suppose that $\Lambda^l = \Lambda^l(\mathbf{R}^n)$ is the linear space of l-vectors, generated by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, $l = 0, 1, \cdots, n$. The Grassmann algebra $\Lambda = \bigoplus_{l=0}^n \Lambda^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \Lambda$ and $\beta = \sum \beta^I e_I \in \Lambda$, the inner product in Λ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, i_2, \cdots, i_l)$ and all integers $l = 0, 1, \cdots, n$. We define the

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Hodge star operator $\star$: $\Lambda \to \Lambda$ by $\star\omega = \mathrm{sign}(\pi)\alpha_{i_1,i_2,\cdots,i_k}(x_1,x_2,\cdots,x_n)dx_{j_1} \wedge \cdots \wedge dx_{n-k}$, where $\omega = \alpha_{i_1,i_2,\cdots,i_k}(x_1,x_2,\cdots,x_n)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, $i_1 < i_2 < \cdots < i_k$, is a differential k-form, $\pi = (i_1,\cdots,i_k,j_1,\cdots,j_{n-k})$ is a permutation of $(1,\cdots,n)$ and $\mathrm{sign}(\pi)$ is the signature of the permutation. The norm of $\alpha \in \Lambda$ is given by the formula $|\alpha|^2 = \langle \alpha,\alpha\rangle = \star(\alpha \wedge \star\alpha) \in \Lambda^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on Λ with $\star: \Lambda^l \to \Lambda^{n-l}$ and $\star\star\,(-1)^{l(n-l)}: \Lambda^l \to \Lambda^l$.

Throughout this paper, we always assume that M is a compact Riemannian manifold on $\mathbf{R}^n$; see [10] for basic properties of a compact Riemannian manifold. Let $\Lambda^l M$ be the l-th exterior power of the cotangent bundle. We use $D'(M,\Lambda^l)$ to denote the space of all differential l-forms and $L^p(\Lambda^l M)$ to denote the l-forms $\omega(x) = \sum_I \omega_I(x)dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ on M satisfying $\int_M |\omega_I|^p < \infty$ for all ordered l-tuples I. Thus $L^p(\Lambda^l M)$ is a Banach space with norm $||\omega||_{p,M} = \left(\int_M |\omega(x)|^p dx\right)^{1/p} = \left(\int_M (\sum_I |\omega_I(x)|^2)^{p/2} dx\right)^{1/p}$. We denote the exterior derivative by $d: D'(M,\Lambda^l) \to D'(M,\Lambda^{l+1})$ for $l = 0,1,\cdots,n$. The Hodge codifferential operator $d^\star: D'(M,\Lambda^{l+1}) \to D'(M,\Lambda^l)$ is given by $d^\star = (-1)^{nl+1} \star d\star$ on $D'(M,\Lambda^{l+1})$, $l = 0,1,\cdots,n$. We use B to denote a ball, and σB, $\sigma > 0$, is the ball with the same center as B and with $diam(\sigma B) = \sigma diam(B)$. For a measurable set $E \subset \mathbf{R}^n$, we write $|E|$ for the n-dimensional Lebesgue measure of E. We call w a weight if $w \in L^1_{loc}(\mathbf{R}^n)$ and $w > 0$ a.e. For $0 < p < \infty$, we write $f \in L^p(\Lambda^l E, w^\alpha)$ if the weighted L^p-norm of f over E satisfies $||f||_{p,E,w^\alpha} = \left(\int_E |f(x)|^p w(x)^\alpha dx\right)^{1/p} < \infty$, where α is a real number.

The nonhomogeneous A-harmonic equation for differential forms is a nonlinear elliptic equation of the form

$$d^\star A(x,d\omega) = B(x,d\omega), \tag{1.1}$$

where $A: M \times \Lambda^l(\mathbf{R}^n) \to \Lambda^l(\mathbf{R}^n)$ and $B: M \times \Lambda^l(\mathbf{R}^n) \to \Lambda^{l-1}(\mathbf{R}^n)$ satisfy the conditions

$$|A(x,\xi)| \le a|\xi|^{p-1}, \quad A(x,\xi)\cdot\xi \ge |\xi|^p \quad \text{and} \quad |B(x,\xi)| \le |\xi|^{p-1} \tag{1.2}$$

for almost every $x \in M$ and all $\xi \in \Lambda^l(\mathbf{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.1). A solution to (1.1) is an element of the Sobolev space $W^{1,p}_{loc}(M,\Lambda^{l-1})$ such that

$$\int_M A(x,d\omega)\cdot d\varphi + B(x,d\omega)\cdot\varphi = 0 \tag{1.3}$$

for all $\varphi \in W^{1,p}_{loc}(M,\Lambda^{l-1})$ with compact support. The solutions of the A-harmonic equation are called A-harmonic tensors. We should notice that if the operator $B = 0$ in (1.1), then equation (1.1) is called the homogeneous A-harmonic equation, or the A-harmonic equation.

2. The local Caccioppoli-type inequalities

We first prove the following local Caccioppoli-type inequality for solutions to the nonhomogeneous A-harmonic equation. This will play an important role in this paper.

Theorem 2.1. *Let $u \in D'(M,\Lambda^l)$ be a solution to the nonhomogeneous A-harmonic equation* (1.1) *on a manifold M, and let $\sigma > 1$ be a constant. Then there exists a*

constant C, independent of u, such that

$$\|du\|_{p,B} \le C diam(B)^{-1}\|u-c\|_{p,\sigma B} \tag{2.2}$$

for all balls or cubes B with $\sigma B \subset M$ and all closed forms c. Here $1<p<\infty$.

Proof. Let $\sigma > 1$ be a real number, $\eta \in C_0^\infty(\sigma B)$, $0 \le \eta \le 1$ with $\eta \equiv 1$ in B, and $|\nabla \eta| \le \frac{C_1}{diam(B)}$. Choosing the test form $\varphi = -u\eta^p$ for (1.1), we have $d\varphi = -\eta^p du - pu\eta^{p-1}d\eta$.

From (1.3), we obtain

$$\int_{\sigma B} A(x,du)\cdot(\eta^p du) = -\int_{\sigma B} A(x,du)\cdot(pu\eta^{p-1}d\eta) - \int_{\sigma B} B(x,du)\cdot(u\eta^p). \tag{2.3}$$

Applying (1.2) yields

$$\int_{\sigma B} |A(x,du)\cdot(\eta^p du)| = \int_{\sigma B} |\eta^p A(x,du)\cdot du| \ge \int_{\sigma B} |\eta|^p |du|^p. \tag{2.4}$$

Note the fact that $|\nabla\eta| = |d\eta|$ and $diam(B) \le diam(M) < \infty$. Using (2.4), (2.3), (1.2) and the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\sigma B} |\eta|^p|du|^p \le \int_{\sigma B} |A(x,du)\cdot(\eta^p du)| \\
&\le \int_{\sigma B} |A(x,du)|\cdot(p|u||\eta|^{p-1}|d\eta|) + \int_{\sigma B} |B(x,du)|\cdot(|u||\eta|^p) \\
&\le \int_{\sigma B} a|du|^{p-1}\cdot(p|u||\eta|^{p-1}|\nabla\eta|) + \int_{\sigma B} b|du|^{p-1}\cdot(|u||\eta|^p) \\
&\le \frac{C_2}{diam(B)}\int_{\sigma B}|du|^{p-1}|\eta|^{p-1}|u| + C_3\int_{\sigma B}|du|^{p-1}|u||\eta|^p \\
&\le \frac{C_2}{diam(B)}\|u\|_{p,\sigma B}\Big(\int_{\sigma B}\big((|du|\eta)^{p-1}\big)^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} + C_3\|\eta u\|_{p,\sigma B}\Big(\int_{\sigma B}(\eta|du|)^p\Big)^{\frac{p-1}{p}} \\
&\le \Big(\frac{C_2}{diam(B)}\|u\|_{p,\sigma B} + C_3\|\eta u\|_{p,\sigma B}\Big)\Big(\int_{\sigma B}(\eta|du|)^p\Big)^{\frac{p-1}{p}} \\
&\le \frac{C_2 + C_3 diam(B)}{diam(B)}\|u\|_{p,\sigma B}\Big(\int_{\sigma B}(\eta du)^p\Big)^{\frac{p-1}{p}} \\
&\le \frac{C_4}{diam(B)}\|u\|_{p,\sigma B}\Big(\int_{\sigma B}|\eta|^p|du|^p\Big)^{\frac{p-1}{p}},
\end{aligned}$$

which is equivalent to

$$\|\eta du\|_{p,\sigma B} \le \frac{C_4}{diam(B)}\|u\|_{p,\sigma B}.$$

Thus, we have

$$\|du\|_{p,B} = \|\eta du\|_{p,B} \le \|\eta du\|_{p,\sigma B} \le \frac{C_4}{diam(B)}\|u\|_{p,\sigma B}. \tag{2.5}$$

It is clear that $u-c$ is also a solution of (1.1) if u is a solution of (1.1) and c is any closed form. Therefore, from (2.5), we conclude that

$$\|du\|_{p,B} \le C diam(B)^{-1}\|u-c\|_{p,\sigma B}.$$

We have completed the proof of Theorem 2.1. □

Definition 2.6. We say that a pair of weights $(w_1(x), w_2(x))$ satisfies the $A_{r,\lambda}(E)$-condition in a set $E \subset \mathbf{R}^n$, and write $(w_1(x), w_2(x)) \in A_{r,\lambda}(E)$ for some $\lambda \geq 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$ if

$$\sup_{B \subset E} \left(\frac{1}{|B|} \int_B (w_1)^\lambda dx \right)^{1/\lambda r} \left(\frac{1}{|B|} \int_B \left(\frac{1}{w_2} \right)^{\lambda r'/r} dx \right)^{1/\lambda r'} < \infty.$$

The class of $A_{r,\lambda}(E)$-weights (or the two-weight) appears in [11]. It is easy to see that the $A_{r,\lambda}(E)$-weight is an extension of the usual A_r-weight [8] and $A_r(\lambda)$-weight [4]. See [2] and [5] for more applications of the two-weight. We need the following generalized Hölder inequality.

Lemma 2.7. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbf{R}^n$, then*

$$\| fg \|_{s,E} \leq \| f \|_{\alpha,E} \cdot \| g \|_{\beta,E}$$

for any $E \subset \mathbf{R}^n$.

The following weak reverse Hölder inequality appears in [12].

Lemma 2.8. *Let u be a solution of* (1.1) *in M, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \leq C|B|^{(t-s)/st} \|u\|_{t,\sigma B}$$

for all balls or cubes B with $\sigma B \subset M$.

Now, we are ready to prove the following two-weight Caccioppoli-type inequality.

Theorem 2.9. *Let $u \in D'(M, \Lambda^l)$, $l = 0, 1, \cdots, n$, be a solution of the nonhomogeneous A-harmonic equation* (1.1) *on a manifold $M \subset \mathbf{R}^n$, and let $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation* (1.1) *and that $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \geq 1$ and $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\left(\int_B |du|^s w_1^\alpha dx \right)^{1/s} \leq \frac{C}{diam(B)} \left(\int_{\rho B} |u - c|^s w_2^\alpha dx \right)^{1/s} \tag{2.10}$$

or

$$\|du\|_{s,B,w_1^\alpha} \leq C diam(B)^{-1} \|u - c\|_{s,\rho B,w_2^\alpha} \tag{2.10$'$}$$

for all balls B with $\rho B \subset M$ and all closed forms c. Here α is any constant with $0 < \alpha < \lambda$.

Proof. Choose $t = \lambda s/(\lambda - \alpha)$. Since $1/s = 1/t + (t-s)/(st)$, using Lemma 2.7 and Theorem 2.1, we obtain

$$\begin{aligned} \left(\int_B |du|^s w_1^\alpha dx \right)^{1/s} &= \left(\int_B \left(|du| w_1^{\alpha/s} \right)^s dx \right)^{1/s} \\ &\leq \left(\int_B |du|^t dx \right)^{1/t} \left(\int_B \left(w_1^{\alpha/s} \right)^{st/(t-s)} dx \right)^{(t-s)/(st)} \\ &\leq \|du\|_{t,B} \cdot \left(\int_B w_1^\lambda dx \right)^{\alpha/(\lambda s)} \\ &= C_1 diam(B)^{-1} \|u - c\|_{t,\sigma B} \left(\int_B w_1^\lambda dx \right)^{\alpha/(\lambda s)} \end{aligned} \tag{2.11}$$

for all balls B with $\sigma B \subset M$ and all closed forms c. Note that c is a closed form and u is a solution of (1.1). Then $u - c$ is still a solution of (1.1). Taking $m = \lambda s/(\lambda + \alpha(r-1))$, then $m < s < t$. Using Lemma 2.8, we have

$$\|u - c\|_{t,\sigma B} \leq C_2 |B|^{(m-t)/mt} \|u - c\|_{m,\rho B}, \tag{2.12}$$

where $\rho > \sigma > 1$. Substituting (2.12) in (2.11), we have

$$\left(\int_B |du|^s w_1^\alpha dx\right)^{1/s} \leq C_3 diam(B)^{-1} |B|^{(m-t)/mt} \|u - c\|_{m,\rho B} \left(\int_B w^\lambda dx\right)^{\alpha/(\lambda s)}. \tag{2.13}$$

Using Lemma 2.7 with $1/m = 1/s + (s-m)/(sm)$ again yields

$$\begin{aligned} \|u - c\|_{m,\rho B} &= \left(\int_{\rho B} |u - c|^m dx\right)^{1/m} \\ &= \left(\int_{\rho B} \left(|u - c| w_2^{\alpha/s} w_2^{-\alpha/s}\right)^m dx\right)^{1/m} \\ &\leq \left(\int_{\rho B} |u - c|^s w_2^\alpha dx\right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/(\lambda s)} \end{aligned} \tag{2.14}$$

for all balls B with $\rho B \subset M$ and all closed forms c. Substituting (2.14) into (2.13), we find that

$$\begin{aligned} &\left(\int_B |du|^s w_1^\alpha dx\right)^{1/s} \\ &\leq C_3 diam(B)^{-1} |B|^{(m-t)/(mt)} \|w_1^\lambda\|_{\lambda,B}^{\alpha/s} \|1/w_2\|_{\lambda/(r-1),\rho B}^{\alpha/s} \left(\int_{\rho B} |u - c|^s w_2^\alpha dx\right)^{1/s}. \end{aligned} \tag{2.15}$$

Using the condition $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$, we obtain

$$\begin{aligned} \|w_1^\lambda\|_{\lambda,B}^{\alpha/s} \|1/w_2\|_{\lambda/(r-1),\rho B}^{\alpha/s} &\leq \left(\left(\int_{\rho B} w_1^\lambda dx\right) \left(\int_{\rho B} (1/w_2)^{\lambda/(r-1)} dx\right)^{r-1}\right)^{\alpha/(\lambda s)} \\ &= \left(|\rho B|^r \left(\frac{1}{|\rho B|} \int_{\rho B} w_1^\lambda dx\right) \left(\frac{1}{|\rho B|} \int_{\rho B} \left(\frac{1}{w_2^\lambda}\right)^{1/(r-1)} dx\right)^{r-1}\right)^{\alpha/(\lambda s)} \\ &\leq C_4 |B|^{\alpha r/(\lambda s)}. \end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16), we conclude that

$$\left(\int_B |du|^s w_1^\alpha dx\right)^{1/s} \leq \frac{C_5}{diam(B)} \left(\int_{\rho B} |u - c|^s w_2^\alpha dx\right)^{1/s} \tag{2.17}$$

for all balls B with $\rho B \subset M$ and all closed forms c. This ends the proof of Theorem 2.9. □

Note that Theorem 2.9 contains two weights, $w_1(x)$ and $w_2(x)$, and two parameters, λ and α. These features make the Caccioppoli-type inequalities more flexible and more useful. The existing versions of the Caccioppoli-type inequality in [1],

[3], and [12] are special cases of Theorem 2.9 with suitable choices of weights and parameters. For example, if we choose $\alpha = 1$ in Theorem 2.9, we have the following corollary.

Corollary 2.18. *Let $u \in D'(M, \Lambda^l)$, $l = 0, 1, \cdots, n$, be a solution of the nonhomogeneous A-harmonic equation* (1.1) *on a manifold $M \subset \mathbf{R}^n$, and let $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation* (1.1) *and that $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \geq 1$ and $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\|du\|_{s,B,w_1} \leq C diam(B)^{-1} \|u - c\|_{s,\rho B,w_2} \tag{2.19}$$

for all balls B with $\rho B \subset M$ and all closed forms c.

If we put $w_1(x) = w_2(x) = w(x)$ in (2.19), we have

$$\|du\|_{p,B,w} \leq C diam(B)^{-1} \|u - c\|_{p,\rho B,w}, \tag{2.20}$$

where the weight $w(x)$ satisfies

$$\sup_{B \subset E} \left(\frac{1}{|B|} \int_B (w)^\lambda dx \right)^{1/\lambda r} \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{\lambda r'/r} dx \right)^{1/\lambda r'} < \infty, \tag{2.21}$$

which is a generalization of the usual A_r-weights [8].

It is easy to see that if we choose $w_1(x) = w_2(x)$ and $\lambda = 1$ in Definition 2.6, we have

$$\sup_{B \subset E} \left(\frac{1}{|B|} \int_B w \, dx \right)^{1/r} \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{r'/r} dx \right)^{1/r'} < \infty,$$

that is,

$$\sup_{B \subset E} \left(\left(\frac{1}{|B|} \int_B w \, dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} \right)^{1/r} < \infty,$$

since $r'/r = 1/(r-1)$. Thus, we see that the $A_{r,\lambda}(M)$-weight reduces to the usual $A_r(M)$-weight if $w_1(x) = w_2(x)$ and $\lambda = 1$. Hence, setting $w_1(x) = w_2(x)$ and $\lambda = 1$ in Theorem 2.9, we obtain the following local $A_r(M)$-weighted Caccioppoli-type estimate.

Theorem 2.22. *Let $u \in D'(M, \Lambda^l)$, $l = 0, 1, \cdots, n$, be a solution of the nonhomogeneous A-harmonic equation* (1.1) *on a manifold $M \subset \mathbf{R}^n$, and let $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation* (1.1) *and that $w(x) \in A_r(M)$ for some $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\|du\|_{s,B,w^\alpha} \leq C diam(B)^{-1} \|u - c\|_{s,\rho B,w^\alpha} \tag{2.23}$$

for all balls B with $\rho B \subset M$ and all closed forms c. Here α is any constant with $0 < \alpha < 1$.

Remark. We can obtain many interesting versions of the Caccioppoli-type inequality from different choices of weights and parameters α and λ. Considering the length of the paper, we can only list a few of them here.

3. The global two-weight Caccioppoli-type inequality

As applications of the local results, we prove the global $A_{r,\lambda}(M)$-weighted Caccioppoli-type inequality on manifolds.

Theorem 3.1. *Let $u \in D'(M, \Lambda^l)$, $l = 0, 1, \cdots, n$, be a solution of the nonhomogeneous A-harmonic equation* (1.1) *on a compact Riemannian manifold $M \subset \mathbf{R}^n$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation* (1.1) *and that $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \geq 1$ and $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\left(\int_M |du|^s w_1^\alpha dx\right)^{1/s} \leq \frac{C}{diam(M)} \left(\int_M |u-c|^s w_2^\alpha dx\right)^{1/s} \tag{3.2}$$

for all closed forms c. Here α is any constant with $0 < \alpha < \lambda$.

Proof. Let $\mathcal{U} = \{B_i\}_{i \in I}$ be an open cover of M. Since the manifold M is compact, then M has a finite open cover $\mathcal{V} = \{B_1, B_2, \cdots, B_m\} \subset \mathcal{U}$. Assume that $d_i = diam(B_i) > 0, i = 1, 2, \cdots, m$, and let $d = \min\{d_1, d_2, \cdots, d_m\}$. Since M is compact, hence it is bounded. Thus, there exists a constant C_1 such that

$$\frac{1}{d} \leq \frac{C_1}{diam(M)}. \tag{3.3}$$

Using (3.3) and Theorem 2.9, we have

$$\begin{aligned}
\left(\int_M |du|^s w_1^\alpha dx\right)^{1/s} &\leq \sum_{B \in \mathcal{V}} \left(\int_B |du|^s w_1^\alpha dx\right)^{1/s} \\
&\leq \sum_{B \in \mathcal{V}} \frac{C_2}{diam(B)} \left(\int_{\rho B} |u-c|^s w_2^\alpha dx\right)^{1/s} \\
&\leq \sum_{B \in \mathcal{V}} \frac{C_2}{d} \left(\int_{\rho B} |u-c|^s w_2^\alpha dx\right)^{1/s} \\
&\leq \sum_{B \in \mathcal{V}} \frac{C_3}{diam(M)} \left(\int_M |u-c|^s w_2^\alpha dx\right)^{1/s} \\
&\leq \frac{C_4}{diam(M)} \left(\int_M |u-c|^s w_2^\alpha dx\right)^{1/s}.
\end{aligned}$$

Hence, (3.2) follows. The proof of Theorem 3.1 is complete. □

Letting $\alpha = 1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.4. *Let $u \in D'(M, \Lambda^l)$, $l = 0, 1, \cdots, n$, be a solution of the nonhomogeneous A-harmonic equation* (1.1) *on a manifold $M \subset \mathbf{R}^n$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation* (1.1) *and that $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \geq 1$ and $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\|du\|_{s,M,w_1} \leq C diam(M)^{-1} \|u-c\|_{s,M,w_2} \tag{3.5}$$

for all closed forms c.

Select $w_1 = w_2 = w$ in Theorem 3.1 and $\lambda = 1$. We have the following $A_r(M)$-weighted Caccioppoli-type inequality.

Corollary 3.6. *Let $u \in D'(M, \Lambda^l)$, $l = 0, 1, \cdots, n$, be a solution of the nonhomogeneous A-harmonic equation* (1.1) *on a manifold $M \subset \mathbf{R}^n$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation* (1.1) *and that $(w_1(x), w_2(x)) \in A_r(M)$ for some r, $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\|du\|_{s,M,w^\alpha} \leq C diam(M)^{-1} \|u - c\|_{s,M,w^\alpha} \tag{3.7}$$

for all closed forms c. Here α is any constant with $0 < \alpha < 1$.

We shall obtain some desired versions of the Caccioppoli-type inequalities which have rich symmetric properties by different choices of weights and parameters in Theorem 3.1. For example, choosing $\alpha = 1/s$ and $\alpha = 1/r$ in Theorem 3.1, respectively, we have the following $A_{r,\lambda}(M)$-weighted Caccioppoli-type inequalities:

$$\left(\int_M |du|^s w_1^{1/s} dx\right)^{1/s} \leq \frac{C}{diam(M)} \left(\int_M |u - c|^s w_2^{1/s} dx\right)^{1/s}, \tag{3.8}$$

$$\left(\int_M |du|^s w_1^{1/r} dx\right)^{1/s} \leq \frac{C}{diam(M)} \left(\int_M |u - c|^s w_2^{1/r} dx\right)^{1/s}. \tag{3.9}$$

Similarly, let $\lambda > 1$ and set $\alpha = 1/\lambda$. Then $\alpha = 1/\lambda < \lambda$. From Theorem 3.1, we find that

$$\left(\int_M |du|^s w_1^{1/\lambda} dx\right)^{1/s} \leq \frac{C}{diam(M)} \left(\int_M |u - c|^s w_2^{1/\lambda} dx\right)^{1/s} \tag{3.10}$$

for $\lambda > 1$. Finally, setting $\alpha = 1 - t$, $0 < t < 1$, in Theorem 3.1, we obtain the following result.

Corollary 3.11. *Let $u \in D'(M, \Lambda^l)$, $l = 0, 1, \cdots, n$, be a solution of the nonhomogeneous A-harmonic equation* (1.1) *on a manifold $M \subset \mathbf{R}^n$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation* (1.1) *and that $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \geq 1$ and $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\left(\int_M |du|^s w_1^{-t} d\mu\right)^{1/s} \leq \frac{C}{diam(M)} \left(\int_M |u - c|^s w_2^{-t} d\nu\right)^{1/s}$$

for all closed forms c and some real number t with $0 < t < 1$, where the measures μ and ν are defined by $d\mu = w_1(x)dx$ and $d\nu = w_2(x)dx$, respectively.

We have the following $A_r(M)$-weighted Caccioppoli-type inequality for solutions of the nonhomogeneous A-harmonic equation if we choose $w_1 = w_2 = w$ and $\lambda = 1$ in Corollary 3.11:

$$\left(\int_M |du|^s w^{-t} d\mu\right)^{1/s} \leq \frac{C}{diam(M)} \left(\int_M |u - c|^s w^{-t} d\mu\right)^{1/s} \tag{3.12}$$

for all closed forms c and some real number t with $0 < t < 1$, where the measure μ is defined by $d\mu = w(x)dx$ and $w(x) \in A_r(M)$.

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Department of Mathematics, Seattle University, Seattle, Washington 98122
E-mail address: sding@seattleu.edu