PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 9, Pages 2609–2615
S 0002-9939(02)06356-6
Article electronically published on February 4, 2002

FINITE UNIONS OF INTERPOLATION SEQUENCES

PETER DUREN AND ALEXANDER P. SCHUSTER

(Communicated by Juha M. Heinonen)

ABSTRACT. A unified and relatively simple proof is given for some well-known results involving finite unions of uniformly separated sequences.

In this note we consider finite unions of uniformly separated sequences, which appear in several well-known theorems involving both Hardy and Bergman spaces. The techniques of proof, some of which are quite involved, differ greatly from result to result. The purpose of this note is to provide a relatively simple and self-contained proof that unifies all of these results.

A sequence $\{z_k\}$ of points in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is called a *Blaschke sequence* if

$$\sum_{k=1}^{\infty}(1 - |z_k|) < \infty.$$

It is *uniformly separated* if

$$\prod_{j\neq k}\left|\frac{z_j - z_k}{1 - \overline{z_j}z_k}\right| \geq \delta\,, \qquad k = 1, 2, \ldots,$$

for some constant $\delta > 0$ that is independent of k.

The importance of uniformly separated sequences is evident in their role as interpolation sequences for the Hardy space H^p, the set of functions f analytic in the disk satisfying

$$\|f\|_{H^p} = \sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right\}^{1/p} < \infty.$$

A sequence $\{z_k\}$ in the disk is said to be a *universal interpolation sequence* if for each complex sequence $\{w_k\} \in \ell^\infty$ there exists a bounded analytic function f with $f(z_k) = w_k$ for $k = 1, 2, \ldots$. Carleson [1] proved that $\{z_k\}$ is a universal interpolation sequence if and only if it is uniformly separated. Shapiro and Shields [13] then generalized the theorem to arbitrary H^p spaces ($1 \leq p < \infty$) by showing that the operator

$$T_p(f) = \left\{(1 - |z_k|^2)^{1/p} f(z_k)\right\}$$

Received by the editors August 14, 2000 and, in revised form, March 26, 2001.
2000 *Mathematics Subject Classification.* Primary 30H05, 46E15.

maps H^p onto ℓ^p if and only if $\{z_k\}$ is uniformly separated, a result that Kabaĭla [8] extended to $0 < p < 1$. One aspect of the theorem is the assertion that

$$\sum_{k=1}^{\infty}(1-|z_k|^2)|f(z_k)|^p < \infty \qquad \text{for every } f \in H^p$$

if $\{z_k\}$ is uniformly separated.

A finite measure μ on the disk is called a *Carleson measure* if $H^p \subset L^p(\mathbb{D}, \mu)$. These measures are easily seen to be independent of p and were described geometrically in a theorem of Carleson [2]. (See also Duren [3].) The convergence of the above sum for every $f \in H^p$ is equivalent to saying that

$$\mu = \sum_{k=1}^{\infty}(1-|z_k|^2)\delta_{z_k}$$

is a Carleson measure. McDonald and Sundberg [10] completed this result to the following theorem.

Theorem A. *A sequence $\{z_k\}$ of points in $\mathbb{D}$ generates a Carleson measure $\mu = \sum(1-|z_k|^2)\delta_{z_k}$ if and only if $\{z_k\}$ is a finite union of uniformly separated sequences.*

We turn now to Bergman spaces, where uniformly separated sequences also play an important, albeit different, role. Recall that a function f analytic in $\mathbb{D}$ is in the Bergman space A^p if

$$\|f\|_p = \left\{\frac{1}{\pi}\int_{\mathbb{D}}|f(z)|^p dm(z)\right\}^{1/p} < \infty,$$

where m denotes Lebesgue area measure.

Given a Blaschke sequence $\{z_k\}$, we can form the Blaschke product

$$B(z) = \prod_k b_{z_k}(z), \qquad \text{where} \qquad b_{z_k}(z) = \frac{|z_k|}{z_k}\frac{z_k - z}{1-\overline{z_k}z}.$$

While the quotient f/B is in H^p whenever f is an H^p function vanishing on $\{z_k\}$, this is not always the case in A^p. The Blaschke product B is called a *universal divisor for* A^p if $f/B \in A^p$ whenever a function $f \in A^p$ vanishes on $\{z_k\}$. By the closed graph theorem it is equivalent to require the existence of a constant C_p such that $\|f/B\|_p \le C_p\|f\|_p$ for all $f \in A^p$ vanishing on $\{z_k\}$.

The complete description of universal divisors, due to Horowitz [7], is as follows.

Theorem B. *For any Blaschke product B, the following statements are equivalent.*
(*i*) *B is a universal divisor for some space A^p , $0 < p < \infty$.*
(*ii*) *B is a universal divisor for every space A^p .*
(*iii*) *The zero-set of B is a finite union of uniformly separated sequences.*

We shall need an additional result, also due to Horowitz [6]. The zero-set $\{z_k\}$ of a function $f \in A^p$ need not be a Blaschke sequence, but it has the property $\sum_k(1-|z_k|)^2 < \infty$, which implies the convergence of the *Horowitz product*

$$H(z) = \prod_{k=1}^{\infty} b_{z_k}(z)\big(2 - b_{z_k}(z)\big).$$

In fact, the product

$$H^*(z) = \prod_{k=1}^{\infty} |\varphi_{z_k}(z)|(2 - |\varphi_{z_k}(z)|),$$

where $\varphi_\zeta(z) = (\zeta - z)/(1 - \overline{\zeta} z)$, also converges. Note that $H^*(z) \leq |H(z)|$.

Theorem C. *For $0 < p < \infty$, let $f \in A^p$, and let H be the Horowitz product formed with its zero-set. Then $f/H \in A^p$ and $\|f/H\|_p \leq C_p \|f\|_p$, where C_p is a constant depending only on p. Moreover, $f/H^* \in L^p$ and $\|f/H^*\|_p \leq C_p \|f\|_p$.*

We now combine Theorems A and B with two auxiliary conditions to form a single theorem that unifies all of these results. This arrangement allows a self-contained proof, invoking only the basic theorem of Horowitz (Theorem C above). One important advantage of this approach is that it avoids all appeal to the geometric characterization of Carleson measures, a relatively deep result. Moreover, it proves Theorem A without the complicated geometric argument in [10], and it proves Theorem B without the appeal in [7] to a rather obscure property of level-curves of Blaschke products.

Main Theorem. *For a Blaschke sequence $\{z_k\}$ of points in $\mathbb{D}$, the following five statements are equivalent.*

(*i*) *$\{z_k\}$ is a finite union of uniformly separated sequences.*

(*ii*) *$\sum_{k=1}^{\infty} (1 - |z_k|^2) \delta_{z_k}$ is a Carleson measure.*

(*iii*) *$\sup_{z \in \mathbb{D}} \sum_{k=1}^{\infty} (1 - |\varphi_{z_k}(z)|) < \infty$.*

(*iv*) *The associated Blaschke product B is a universal divisor of A^p; that is, $f/B \in A^p$ for every function $f \in A^p$ that vanishes on $\{z_k\}$.*

(*v*) *The operator of multiplication by the associated Blaschke product B is bounded below on A^p; that is, $\|Bf\|_p \geq c\,\|f\|_p$ for some constant $c > 0$ and every function $f \in A^p$.*

The last two statements are ambiguous because they do not specify the values of p for which the property is to hold. In fact, it will be clear from the proof that if either property holds for *some* p ($0 < p < \infty$), then it must hold for *all* p.

Proof of the Main Theorem. (*i*) $\implies$ (*ii*). Here we refer the reader to the self-contained and relatively simple proof due to Neville [11].

(*ii*) $\implies$ (*iii*). Now we assume that

$$\sum_{k=1}^{\infty} (1 - |z_k|^2) |f(z_k)|^p \leq C\, \|f\|_{H^p}^p$$

for all functions $f \in H^p$. Choosing

$$f(z) = f_\zeta(z) = \left(\frac{1 - |\zeta|^2}{1 - \overline{\zeta} z} \right)^{2/p}, \qquad \zeta \in \mathbb{D},$$

with $\|f_\zeta\|_{H^p}^p = 1 - |\zeta|^2$, we infer from the identity

$$1 - |\varphi_{z_k}(z)|^2 = \frac{(1 - |z|^2)(1 - |z_k|^2)}{|1 - \overline{z_k} z|^2}, \tag{1}$$

that

$$\sum_{k=1}^{\infty}(1-|\varphi_\zeta(z_k)|^2) = \sum_{k=1}^{\infty}\frac{(1-|\zeta|^2)(1-|z_k|^2)}{|1-\overline{\zeta}z_k|^2}$$
$$= \frac{1}{1-|\zeta|^2}\sum_{k=1}^{\infty}(1-|z_k|^2)|f_\zeta(z_k)|^p \le \frac{C}{1-|\zeta|^2}\|f_\zeta\|_{H^p}^p = C$$

for all $\zeta \in \mathbb{D}$, which gives (iii).

$(iii) \implies (iv)$. Let B be the Blaschke product of $\{z_k\}$, and let f be a function in A^p that vanishes on $\{z_k\}$. Assuming that (iii) holds, we want to show that $f/B \in A^p$. But

$$\frac{|f(z)|}{|B(z)|} = \frac{|f(z)|}{|H^*(z)|}\prod_{k=1}^{\infty}(2-|\varphi_z(z_k)|)\,,$$

so by Theorem C it will be sufficient to show that the infinite product is uniformly bounded for $z \in \mathbb{D}$. But this is a consequence of property (iii), since

$$\log\left(\prod_{k=1}^{\infty}(2-|\varphi_z(z_k)|)\right) \le \sum_{k=1}^{\infty}(1-|\varphi_z(z_k)|)\,.$$

Thus (iv) holds if (iii) does.

$(iv) \implies (v)$. A simple argument using the closed graph theorem shows that multiplication by a function $\psi \in H^\infty$ is bounded below on A^p if and only if the set ψA^p is closed. On the other hand, it is true for any Blaschke product B that BA^p is contained in the subspace N^p of all functions in A^p that vanish on the zero-set of B. Property (iv) says that $N^p \subset BA^p$, so that $BA^p = N^p$ and BA^p is therefore closed. Thus a sequence $\{z_k\}$ has property (v) whenever it has property (iv). (We remark that one could also prove directly that $(v) \implies (iv)$. Note first that $[B] \subset BA^p \subset N^p$ when BA^p is closed, where $[B]$ denotes the subspace of A^p generated by all polynomial multiples of B. However, it was shown by H.S. Shapiro [12] (see [4] for a proof) that $[B] = N^p$ for any Blaschke product B. Thus $BA^p = N^p$ if BA^p is closed, which shows that $(v) \implies (iv)$.)

$(v) \implies (iii)$. This part of the argument follows Horowitz [7]. Under the assumption that (iii) does *not* hold, or equivalently that

$$\sup_{z\in\mathbb{D}}\sum_{k=1}^{\infty}(1-|\varphi_{z_k}(z)|^2) = \infty\,,$$

we shall produce a family of functions $f_\zeta \in A^p$ with $\|f_\zeta\|_p = 1$ for all $\zeta \in \mathbb{D}$, such that

$$\inf_{\zeta\in\mathbb{D}}\|Bf_\zeta\|_p = 0\,.$$

Define $f_\zeta = (\varphi_\zeta')^{2/p}$ and observe that

$$\|f_\zeta\|_p^p = \frac{1}{\pi}\int_{\mathbb{D}}|\varphi_\zeta'(z)|^2\,dm(z) = \frac{1}{\pi}\int_{\mathbb{D}}dm(w) = 1\,,$$

where $w = \varphi_\zeta(z)$. Also note that

$$\|Bf_\zeta\|_p = \|B\circ\varphi_\zeta\|_p\,,$$

since $\varphi_\zeta^{-1} = \varphi_\zeta$. But the composition $B \circ \varphi_\zeta$ is, up to rotation, a Blaschke product with zeros at the points $\zeta_k = \varphi_\zeta(z_k)$. Thus

$$\begin{aligned}\log|B(\varphi_\zeta(z))| &= \frac{1}{2}\sum_{k=1}^{\infty}\log|\varphi_{\zeta_k}(z)|^2 \le \frac{1}{2}\sum_{k=1}^{\infty}\left\{|\varphi_{\zeta_k}(z)|^2 - 1\right\}\\ &= -\frac{1}{2}(1-|z|^2)\sum_{k=1}^{\infty}\frac{1-|\zeta_k|^2}{|1-\overline{\zeta_k}z|^2} \le -\frac{1}{8}(1-r^2)\sum_{k=1}^{\infty}(1-|\zeta_k|^2)\end{aligned}$$

for $|z| \le r < 1$. Consequently, given any $\varepsilon > 0$ we may choose $M > 0$ so large that $\|B \circ \varphi_\zeta\|_p < \varepsilon$ when

$$\sum_{k=1}^{\infty}(1-|\zeta_k|^2) = \sum_{k=1}^{\infty}(1-|\varphi_{z_k}(\zeta)|^2) > M\,.$$

Specifically, we have only to choose $r < 1$ so that

$$\int_{r<|z|<1}|B(\varphi_\zeta(z))|^p\,dm \le \pi(1-r^2) < \frac{\varepsilon^p}{2}$$

for all $\zeta \in \mathbb{D}$, and then to choose M large enough to ensure that

$$\int_{|z|\le r}|B(\varphi_\zeta(z))|^p\,dm < \frac{\varepsilon^p}{2}$$

for any ζ with $\sum(1-|\zeta_k|^2) > M$. Adding the two integrals, we see that

$$\|B\circ\varphi_\zeta\|_p^p \le \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p$$

for some $\zeta \in \mathbb{D}$. Thus $\|Bf_\zeta\|_p < \varepsilon$ while $\|f_\zeta\|_p = 1$. This shows that (v) does not hold if (iii) does not.

$(iii) \implies (i)$. This final stage of the proof will consist of two main steps. The first step is to show that a sequence $\{z_k\}$ with property (iii) is a finite union of uniformly discrete sequences. (A sequence $\{z_k\}$ is *uniformly discrete* if there is a positive constant δ such that $|\varphi_{z_j}(z_k)| \ge \delta$ for all j and k with $j \ne k$.) The second step is to show that any uniformly discrete sequence with property (iii) is in fact uniformly separated. To carry out the first step, we begin by observing that if for some $\delta > 0$ each pseudohyperbolic disk

$$\Delta(\zeta,\delta) = \{z \in \mathbb{D} : |\varphi_\zeta(z)| < \delta\}$$

contains at most N points of the sequence $\{z_k\}$, then $\{z_k\}$ is the union of at most N uniformly discrete sequences. Thus if $\{z_k\}$ is *not* a finite union of uniformly discrete sequences, there must exist a sequence $\{\zeta_m\}$ of points in $\mathbb{D}$ for which $N_m \to \infty$ as $m \to \infty$, where N_m is the number of points z_k in the disk $\Delta(\zeta_m, \frac{1}{2})$. Consequently,

$$\sum_{k=1}^{\infty}(1-|\varphi_{z_k}(\zeta_m)|^2) \ge \sum_{z_k\in\Delta(\zeta_m,\frac{1}{2})}(1-|\varphi_{z_k}(\zeta_m)|^2) \ge \frac{3}{4}N_m \to \infty$$

as $m \to \infty$, contradicting the property (iii). Thus (iii) implies that $\{z_k\}$ is a finite union of uniformly discrete sequences.

Finally, suppose that (iii) holds and $\{z_k\}$ is uniformly discrete, so that

$$\sum_{k=1}^{\infty}(1-|\varphi_{z_k}(z)|^2) \le M < \infty$$

and $|\varphi_{z_k}(z_j)| \geq \delta > 0$ for $j \neq k$. Then in view of the inequality $-\log x \leq \frac{1}{x}(1-x)$, we see that

$$\sum_{k \neq j} -\log\left|\varphi_{z_k}(z_j)\right|^2 \leq \frac{1}{\delta^2} \sum_{k \neq j} \left(1 - \left|\varphi_{z_k}(z_j)\right|^2\right) \leq \frac{M}{\delta^2}\,.$$

Exponentiating, we conclude that

$$\prod_{k \neq j} \left|\varphi_{z_k}(z_j)\right| \geq e^{-M/2\delta^2} > 0\,, \qquad j = 1, 2, \dots\,,$$

which says that $\{z_k\}$ is uniformly separated. Combining the two main steps of the proof, we conclude that any sequence $\{z_k\}$ with the property (iii) is a finite union of uniformly separated sequences. This completes the proof of the Main Theorem. □

For an alternate proof that properties (ii) and (iii) are equivalent, one can invoke the theory of Carleson measures. By appeal to Carleson's geometric description, it is not difficult to show (*cf.* Garnett [5], p. 239) that a positive measure μ is a Carleson measure if and only if

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \overline{\zeta} z|^2}\, d\mu(\zeta) < \infty\,.$$

Applying this criterion to the discrete measure

$$\mu = \sum_{k=1}^{\infty} (1 - |z_k|^2)\delta_{z_k}$$

and using the identity (1), one sees immediately that (ii) and (iii) are equivalent.

McDonald and Sundberg [10] gave a stronger form of the implication $(v) \implies (i)$, showing that multiplication by an arbitrary inner function ψ is bounded below on A^p if and only if ψ is a Blaschke product whose zero-set is a finite union of uniformly separated sequences. Luecking [9] carried the problem further, showing that multiplication by a function $\psi \in H^\infty$ is bounded below on A^p if and only if

$$m\big(\{z \in \mathbb{D} \cap \Delta \;:\; |\psi(z)| > \varepsilon\}\big) > \delta\, m(\mathbb{D} \cap \Delta)$$

for some positive constants δ, ε and for all disks Δ centered on the unit circle.

References

1. L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921-930. MR **22:**8129
2. L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. **76** (1962), 547-559. MR **25:**5186
3. P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970; reprinted with supplement by Dover Publications, New York, 2000. MR **42:**3552
4. P. Duren, D. Khavinson, and H. S. Shapiro, *Extremal functions in invariant subspaces of Bergman spaces*, Illinois J. Math. **40** (1996), 202-210. MR **97h:**30069
5. J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981. MR **83g:**30037
6. C. Horowitz, *Zeros of functions in the Bergman spaces*, Duke Math. J. **41** (1974), 693-710. MR **50:**10215
7. C. Horowitz, *Factorization theorems for functions in the Bergman spaces*, Duke Math. J. **44** (1977), 201-213. MR **55:**681
8. V. Kabaĭla, Interpolation sequences for the H_p classes in the case $p < 1$, Litovsk. Mat. Sb. **3** (1963), no. 1, 141-147 (in Russian). MR **32:**217
9. D. Luecking, *Inequalities in Bergman spaces*, Illinois J. Math. **25** (1981), 1-11. MR **82e:**30072
10. G. McDonald and C. Sundberg, *Toeplitz operators on the disc*, Indiana Univ. Math. J. **28** (1979), 595-611. MR **80h:**47034

11. C. W. Neville, *A short proof of an inequality of Carleson's*, Proc. Amer. Math. Soc. **65** (1977), 131-132. MR **56:**3303
12. H. S. Shapiro, *Comparative approximation in two topologies*, in *Approximation Theory* (Banach Center Publications, Vol. 4, PWN – Polish Scientific Publishers, Warsaw 1979), pp. 225-232. MR **80j:**41064
13. H. S. Shapiro and A. L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. **83** (1961), 513-532. MR **24:**A3280

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109
E-mail address: `duren@umich.edu`

Department of Mathematics, San Francisco State University, San Francisco, California 94131
E-mail address: `schuster@sfsu.edu`