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ON SUPPORT POINTS OF UNIVALENT FUNCTIONS AND A DISPROOF OF A CONJECTURE OF BOMBIERI

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Abstract. We consider the linear functional $\mathrm{Re}(a_3 + \lambda a_2)$ for $\lambda \in i\mathbb{R}$ on the set of normalized univalent functions in the unit disk and use the result to disprove a conjecture of Bombieri.

1. Introduction

Let $\mathcal{S}$ be the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ analytic and univalent in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. We consider for a fixed constant $\lambda \in \mathbb{C}$ the linear functional

$$L_\lambda(f) = L_\lambda(a_2, a_3) := \mathrm{Re}\,(a_3 + \lambda a_2)\,, \qquad f \in \mathcal{S}. \tag{1.1}$$

Every function $F \in \mathcal{S}$ maximizing L_λ over $\mathcal{S}$ is called a *support point* for L_λ.

The coefficient functional L_λ has been studied by Brown [Bro81] who obtained a complete picture of the support points for L_λ for all $\lambda \in \mathbb{C}$ apart from the case $\lambda = \pm i|\lambda|$, $6 \le |\lambda| < 8$. In a completely different manner the functional L_λ was investigated by Tammi and Kortram in [KT80] and by Tammi in [Tam82]. However, the reasoning in [KT80] and [Tam82] in the most difficult case $\lambda \in i\mathbb{R}$, $|\lambda| < 8$, is not complete as it was pointed out for instance by Leung, [Leu85], p. 9. The problem is to show that a certain system of non-linear equations (*cf.* (54) in [Tam82], p. 87) has a unique solution; see also the remarks in [Haa83], p. 65.

It is the purpose of this note to fill in this gap and to give a rigorous proof of the following result stated by Tammi [Tam82], p. 90.

Theorem 1.1. *Let $\lambda \in i\mathbb{R} \setminus \{0\}$ and let $F \in \mathcal{S}$ be a support point for the functional* (1.1) *over $\mathcal{S}$:*

(a) *If $|\lambda| \ge 4e/(e-1) = 6.3279\ldots$ and $\lambda = i|\lambda|$ (resp. $\lambda = -i|\lambda|$), then $F(z) = iK(-iz)$ (resp. $F(z) = -iK(iz)$) where $K(z) = z/(1-z)^2$ is the Koebe function.*

(b) *If $0 < |\lambda| < 4e/(e-1)$, then F is not a rotation of the Koebe function.*

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The proof of Theorem 1.1 will be given in Section 2.

The information on the linear functional L_λ provided by Theorem 1.1 can be used to solve a *linear fractional* extremal problem on $\mathcal{S}$ and leads to a precise statement about the coeffcient body $V_3 := \{(a_2, a_3)^T : f \in \mathcal{S}\}$ near the "Koebe-point":

Corollary 1.2. *We have*

$$\min_{(a_2,a_3)\in V_3} \frac{2-\operatorname{Re} a_2}{3-\operatorname{Re} a_3} = \liminf_{a_2\to 2} \frac{2-\operatorname{Re} a_2}{3-\operatorname{Re} a_3} = \frac{1}{4}\frac{e-1}{e} = 0.15803\ldots, \tag{1.2}$$

where the $\liminf$ *are taken over all functions of* $\mathcal{S}$.

Proof. Using the rotation invariance of $\mathcal{S}$ it is easy to see that

$$\min_{f\in\mathcal{S}} \operatorname{Re}(a_3 - \lambda a_2) = -\max_{f\in\mathcal{S}} \operatorname{Re}(a_3 + i\lambda a_2), \qquad \lambda \in \mathbb{C}. \tag{1.3}$$

Therefore, Theorem 1.1 (a) implies

$$\frac{3-\operatorname{Re} a_3}{2-\operatorname{Re} a_2} \le 4\frac{e}{e-1}$$

for all $f \in \mathcal{S}$, $f \neq K$, i.e.,

$$\min_{(a_2,a_3)\in V_3} \frac{2-\operatorname{Re} a_2}{3-\operatorname{Re} a_3} \ge \frac{1}{4}\frac{e-1}{e}. \tag{1.4}$$

Let $0 < \lambda_n < 4e/(e-1)$ such that $\lim_{n\to\infty} \lambda_n = 4e/(e-1)$. In view of (1.3) and Theorem 1.1 (b) there are functions $F_n(z) = z + a_2^{(n)} z^2 + a_3^{(n)} z^3 + \cdots \in \mathcal{S} \setminus \{K\}$ such that $F_n \to K$ locally uniformly in $\mathbb{D}$ and

$$\min_{f\in\mathcal{S}} \operatorname{Re}(a_3 - \lambda_n a_2) = \operatorname{Re}(a_3^{(n)} - \lambda_n a_2^{(n)}) < 3 - 2\lambda_n.$$

In particular, $3 - \operatorname{Re} a_3^{(n)} \ge \lambda_n (2 - \operatorname{Re} a_2^{(n)})$, i.e.,

$$\liminf_{a_2\to 2} \frac{2-\operatorname{Re} a_2}{3-\operatorname{Re} a_3} \le \lim_{n\to\infty} \frac{2-\operatorname{Re} a_2^{(n)}}{3-\operatorname{Re} a_3^{(n)}} \le \lim_{n\to\infty} \frac{1}{\lambda_n} = \frac{1}{4}\frac{e-1}{e}. \tag{1.5}$$

The assertion follows now from inequalities (1.4) and (1.5). □

In particular, Corollary 1.2 disproves a conjecture of Bombieri [Bom67], p. 51 (see also [BH85], [BH87]), which asserts that

$$\liminf_{a_m\to m} \frac{n-\operatorname{Re} a_n}{m-\operatorname{Re} a_m} = \min_{\theta\in\mathbb{R}} \frac{n - \frac{\sin(n\theta)}{\sin\theta}}{m - \frac{\sin(m\theta)}{\sin\theta}} \qquad \text{for all } m, n \ge 2, \tag{1.6}$$

in the case $n = 2$ and $m = 3$, because for $n = 2$ and $m = 3$ the right-hand side of (1.6) equals $1/4$ which is strictly larger than the bound $(e-1)/(4e)$ in Corollary 1.2.

2. Proof of Theorem 1.1

The standard method of boundary variation (*cf.* [SSp50], [Dur83]) shows that every support point $F(z) = z + A_2 z^2 + A_3 z^3 + \cdots$ of the functional L_λ, $\lambda \in \mathbb{C}$ fixed, is a solution of the Schiffer differential equation

$$\left[\frac{zF'(z)}{F(z)}\right]^2 \frac{1 + AF(z)}{F(z)^2} = \frac{1}{z^2} + \frac{A}{z} + B_0 + \overline{A}z + z^2, \tag{2.1}$$

where

$$B_0 = 2A_3 + \lambda A_2 > 0, \qquad A = 2A_2 + \lambda. \tag{2.2}$$

In the sequel we will work only with conditions (2.1) and (2.2). It is therefore convenient to introduce the following terminology; *cf.* [Pfl88].

Definition 2.1. A function $f(z) = z + a_2 z^2 + \cdots \in \mathcal{S}$ is called *A-admissible* for $A \in \mathbb{C}$ if f admits a piecewise analytic extension to $\overline{\mathbb{D}}$ such that

$$z \mapsto \left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1 + Af(z)}{f(z)^2}$$

is positive on $|z| = 1$ except possibly for one or two points on $|z| = 1$. If, in addition, $A = 2a_2 + \lambda$, then f is called a critical point for the functional L_λ.

Therefore, every support point $F(z) = z + A_2 z^2 + \cdots$ for L_λ is A-admissible for $A = 2A_2 + \lambda$, i.e., every support point for L_λ is a critical point for L_λ.

Lemma 2.2. *If $F(z) = z + A_2 z^2 + \cdots \in \mathcal{S}$ is a support point for L_λ, then $A = 2A_2 + \lambda \in \mathbb{C} \setminus (-4, 4)$.*

This follows from the fact that A-admissible functions for $A \in (-4, 4)$ are two-slit mappings whereas support points in $\mathcal{S}$ are necessarily one-slit maps; *cf.* [SSp50] for details.

In the next lemma we shall consider A-admissible functions for $A \notin (-4, 4)$. We parametrize A in terms of $\rho \in (0, 1]$ and $\phi \in (-\pi, \pi]$ by

$$A = A(\rho, \phi) := \left(\rho + \frac{1}{\rho}\right) e^{i\phi} + 2e^{-i\phi}. \tag{2.3}$$

(This is the parametrization used by Schaeffer and Spencer [SSp50], Chapter XIII.)

Lemma 2.3. *If $A \in \mathbb{C} \setminus (-4, 4)$, then there exists a uniquely determined A-admissible function $f_A(z) = z + a_2(A) z^2 + \cdots \in \mathcal{S}$. Moreover $\mathbb{C} \setminus f_A(\mathbb{D})$ is an analytic Jordan arc extending to ∞ and*

$$2a_2(A) = 4e^{-i\phi} - A \log\left(1 + \rho^2 + 2\rho e^{-2i\phi}\right) + A \log(1 - \rho^2) + \overline{A} \log \frac{1 + \rho}{1 - \rho}. \tag{2.4}$$

Equation (2.4) is exactly formula (13.5.8) in [SSp50] for the part of the coefficient body V_3 which corresponds to one-slit mappings.

We now characterize the critical points of the functional L_λ which are A-admissible for $A \notin (-4, 4)$.

Lemma 2.4. *If f is a critical point of the functional L_λ for $\lambda \in i\mathbb{R}$ which is A-admissible for $A = A(\rho, \phi) \in \mathbb{C} \setminus (-4, 4)$, then $h(\rho, \phi) = 0$ where*

$$\begin{aligned} h(\rho, \phi) \quad := \quad & (\rho - 1)^2 \cos\phi - 2(\rho + 1)^2 \log(1 + \rho) \cos\phi \\ & + (\rho + 1)^2 \cos\phi \, \mathrm{Re}(\log(1 + \rho^2 + 2\rho e^{-2i\phi})) \\ & + (\rho - 1)^2 \sin\phi \, \mathrm{Im}(\log(1 + \rho^2 + 2\rho e^{-2i\phi})). \end{aligned} \tag{2.5}$$

Moreover, $\lambda = ip(\rho,\phi)$*, where*

$$
\begin{aligned}
p(\rho,\phi) &= \operatorname{Im} A(\rho,\phi) - 2\operatorname{Im} a_2(A(\rho,\phi)) \\
&= \frac{(1+\rho)^2}{\rho}\cos\phi\,\operatorname{Im}(\log(1+2\,\rho e^{-2\,i\,\phi}+\rho^2)) \\
&\quad + \frac{(1+\rho)^2 - 2\,(\rho-1)^2\,\log(1-\rho)}{\rho}\sin\phi \\
&\quad + \frac{(\rho-1)^2}{\rho}\operatorname{Re}(\log(1+2\,\rho e^{-2\,i\,\phi}+\rho^2))\,\sin\phi.
\end{aligned} \tag{2.6}
$$

Proof. We know from (2.2) that $A = 2a_2(A) + \lambda$ where $a_2(A)$ is given by (2.4). Taking the real part leads to $h(\rho,\phi) = 0$; taking the imaginary part gives (2.6). □

We now consider the equation $h(\rho,\phi) = 0$. Since $h(\rho,\pi\pm\phi) = -h(\rho,\phi)$ we may restrict ourselves to the case $0 < \rho \le 1$ and $0 \le \phi \le \pi/2$. For $\phi = 0$ we have $h(\rho,\phi) = 0$ if and only if $\rho = 1$. In this case $\lambda = 0$ and there are exactly two support points, namely $K(z)$ and $-K(-z)$.

Lemma 2.5. *If* $h(\rho,\phi) = 0$ *with* $0 < \rho \le 1$, $0 < \phi \le \pi/2$, *then either*:

(a) $\phi = \pi/2$ *and* $p(\rho,\phi) = (1+\rho)^2/\rho \in [4,\infty)$, *or*

(b) $\phi < \pi/2$. *In this case* $(\sqrt{e}-1)/(\sqrt{e}+1) \doteq 0.244919 < \rho \le 1$ *and* $\phi = \phi(\rho)$ *is a continuously differentiable and strictly decreasing function of* $(0,1]$ *onto* $[0,\pi/2)$. *Moreover the function* $\rho \mapsto p(\rho,\phi(\rho))$ *is continuously differentiable and strictly decreasing on* $(0,1]$ *and takes on its values in* $[0, 4e/(e-1))$.

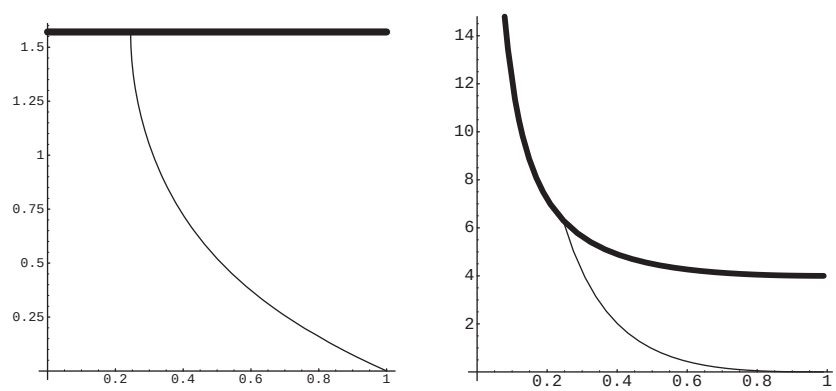

FIGURE 1. The locus of the zeros of $h(\rho,\phi)$ consisting of two curves in the ρ-ϕ-plane (on the left) and the values of p as a function of ρ along these curves (on the right). The thick parts correspond to the Koebe function $z/(1-iz)^2$.

Proof. (a) is obvious. To prove (b) we introduce the functions

$$
\begin{aligned}
g(v,x) &:= v + \frac{1}{2}\,[L(v,x) + 2\log v] + \frac{v\,s(x)}{4(1+x)}T(v,x), \\
q(v,x) &:= \frac{2}{1-v}s(x) - \frac{1}{1-v}T(v,x) + \frac{v}{1-v}s(x)L(v,x),
\end{aligned}
$$

defined for $(v,x) \in X := (0,1) \times (-1,1]$, where we used the shorthand notations

$$\begin{aligned} s(x) &:= \sqrt{2-2x}, \\ T(v,x) &:= 2\sqrt{2+2x}\arctan\frac{\sqrt{1-x^2}}{\frac{1+v}{1-v}+x}, \\ L(v,x) &:= \log\frac{\frac{1+v^2}{1-v^2}+x}{\frac{1+v^2}{1-v^2}-1}. \end{aligned}$$

The following estimate for $T(v,x)$ on X will be useful later on:

$$(1-v)(1+x)s(x) < T(v,x) < \frac{2(1+x)s(x)}{\frac{1+v}{1-v}+x}. \tag{2.7}$$

The first inequality in (2.7) may be obtained by comparing the partial derivatives with respect to v for fixed x, the second one readily follows from $\arctan y < y$ for $y > 0$.

By the transformation

$$v = v(\rho) := \left(\frac{1-\rho}{1+\rho}\right)^2, \qquad x = x(\phi) := \cos 2\phi, \tag{2.8}$$

we define a bijective map $(\rho,\phi) \mapsto (v(\rho), x(\phi))$ of $(0,1) \times (0,\pi/2]$ onto X. A straightforward calculation leads to the relations

$$\begin{aligned} h(\rho,\phi) &= (1+r)^2\cos\phi\; g(v(\rho),x(\phi)), \\ p(\rho,\phi) &= q(v(\rho),x(\phi)), \end{aligned} \tag{2.9}$$

between h and p and the new functions g and q. We claim that the locus of the zeros of $g(v,x)$ is a curve $\gamma : t \mapsto (t, x(t))$, $t \in (0, 1/e]$, with

$$\lim_{t\to 0}\gamma(t) = (0,1), \qquad \lim_{t\to e^{-1}-}\gamma(t) = (e^{-1},-1),$$

where $x'(t) < 0$ is continuous. The existence of such a curve γ is guaranteed by the implicit function theorem since the partial derivatives

$$\begin{aligned} g_v(v,x) &= 1 + \frac{s(x)}{4(1+x)}T(v,x), \\ g_x(v,x) &= \frac{-1+v+x-vx+\frac{v}{2(1+x)}s(x)T(v,x)}{2(x^2-1)}, \end{aligned} \tag{2.10}$$

of g appear to be positive on $(0,1)\times(-1,1)$ by (2.7) and

$$\lim_{v\to 0+} g(v,x) = \frac{1}{2}\log\frac{1+x}{2} < 0, \qquad \lim_{v\to 1-} g(v,x) = 1,$$

for fixed $x \in (-1,1)$. A computation of the limits

$$\lim_{x\to 1-} g(v,x) = v, \qquad \lim_{x\to -1+} g(v,x) = 1 + \log v,$$

for fixed v proves the statement about the endpoints of γ.

We shall prove now that $q(v,x)$ is increasing on γ. To see this consider

$$\frac{d}{dt}q(t,x(t)) = q_v(t,x(t)) - q_x(t,x(t))\frac{g_v(t,x(t))}{g_x(t,x(t))}$$

where

$$
\begin{aligned}
q_v(v,x) &= \frac{-T(v,x)+s(x)[2+L(v,x)]}{(1-v)^2},\\
q_x(v,x) &= \frac{-T(v,x)-\frac{2v(1+x)}{s(x)}[2+L(v,x)]}{2(1-v)(1+x)}.
\end{aligned}
\tag{2.11}
$$

A straightforward computation involving (2.7) and $L(v,x)\geq 0$ shows $q_v(v,x)>0$ and $q_x(v,x)<0$ on $(0,1)\times(-1,1)$. Hence $\frac{d}{dt}q(t,x(t))>0$.

Translating our result via (2.8) and (2.9) back to the functions $h(\rho,\phi)$ and $p(\rho,\phi)$ we obtain the assertion. □

We deduce from Lemma 2.5 that for $\lambda=ip$ and $p\geq 4e/(e-1)$ the unique support point for L_λ is the Koebe function $iK(-iz)$. If $p\leq -4e/(e-1)$, then, by symmetry, the unique support point for L_λ is $-iK(iz)$. This proves part (a) of Theorem 1.1.

To prove part (b), i.e., to show that for $0<p<4e/(e-1)$, no rotation of the Koebe function is a support point for L_{ip}, we establish the following:

Lemma 2.6. *For $0<p<4e/(e-1)$ let F_0 be the uniquely determined $A(\rho,\phi(\rho))$-admissible function such that $p=p(\rho,\phi(\rho))$ for some $\rho\in((\sqrt{e}-1)/(\sqrt{e}+1),1)$. Then $L_{ip}(F_0)>L_{ip}(iK(-iz))$.*

Proof. We adopt the notation from the proof of Lemma 2.5. Using (2.2) we get $2L_{ip}(F_0)=B_0-p\operatorname{Im}A_2$ with $B_0=2(\rho+1/\rho+\cos 2\phi)$; *cf.* [SSp50], and, in view of (2.4) and the transformation (2.8),

$$
\operatorname{Im}A_2=\frac{T(v,x)-vL(v,x)s(x)}{2(1-v)}-s(x).
$$

Thus we have to show that

$$
L_{ip}(F_0)-L_{ip}(iK(-iz))=2\frac{1+v}{1-v}+x-\frac{q(v,x)}{2}\operatorname{Im}A_2-2q(v,x)+3 \tag{2.12}
$$

is non-negative for all $(v,x)\in X$ with $g(v,x)=0$. In fact, we will prove this for all $(v,x)\in X$. Let us denote the expression (2.12) by $R(v,x)$. Then we have

$$
\begin{aligned}
R(v,x) = \frac{1}{4(1-v)^2}\big[&v^2s(x)^2L(v,x)^2-2vs(x)T(v,x)L(v,x)+T(v,x)^2\\
&-2s(x)v[4-4v-2s(x)+vs(x)]L(v,x)\\
&+2[4-4v-2s(x)+vs(x)]T(v,x)\\
&+4(1-v)[5-v+x-4s(x)+s(x)^2-vx]\big].
\end{aligned}
$$

A straightforward calculation leads to

$$
4(1-v)^2R(v,x)=[T(v,x)-vs(x)L(v,x)]^2+2[c_1(v,x)a(v,x)+c_0(v,x)], \tag{2.13}
$$

where

$$
\begin{aligned}
a(v,x) &:= T(v,x)-vs(x)L(v,x),\\
c_1(v,x) &:= 4(1-v)-(2-v)s(x),\\
c_0(v,x) &:= (1-v)[2-s(x)][2(3-v)-(1+v)s(x)].
\end{aligned}
$$

Obviously, $c_0(v,x) > 0$ for all $(v,x) \in X$. Furthermore, $L(v,x) \geq 0$ in X, since

$$L_v(v,x) = -2\frac{1+x}{v(1+x)+v^3(1-x)} \leq 0, \qquad \lim_{v\to 1} L(v,x) = 0.$$

Thus, the relations

$$a_v(v,x) = -s(x)L(v,x), \qquad \lim_{v\to 1} a(v,x) = 0,$$

show also $a(v,x) \geq 0$. In view of equation (2.13) it remains to prove that $c_1(v,x)a(v,x) + c_0(v,x) \geq 0$ for all $(v,x) \in X$ that satisfy $c_1(v,x) < 0$, i.e. $v > 2[2-s(x)]/[4-s(x)]$. We denote the set of all such points by Δ and show below that the partial derivative

$$\begin{aligned}\left(a + \frac{c_0}{c_1}\right)_v (v,x) &= -s(x)L(v,x) - \frac{4-s(x)^2}{4-s(x)} \\ &\quad + \frac{s(x)[2-s(x)][16-10s(x)+3s(x)^2]}{[4-s(x)][4-4v-2s(x)+vs(x)]^2}\end{aligned} \tag{2.14}$$

is positive for all $(v,x) \in \Delta$. Then the obvious limit relation

$$\lim_{v\to 1}\left[a(v,x) + \frac{c_0(v,x)}{c_1(v,x)}\right] = 0$$

shows $a(v,x) + c_0(v,x)/c_1(v,x)$ is negative in Δ. Thus $c_1(v,x)a(v,x) + c_0(v,x)$ is positive for all $(v,x) \in \Delta$.

To complete the proof we have to show that (2.14) is positive for all $(v,x) \in \Delta$. First, we use the estimates

$$L(v,x) = \log\left(1 + \frac{1+x}{\frac{1+v^2}{1-v^2}-1}\right) \leq \frac{1+x}{\frac{1+v^2}{1-v^2}-1} = -\frac{4-s(x)^2}{4}\left(1 - \frac{1}{v^2}\right)$$

and $16 - 10s + 3s^2 \geq 2(2+s)$ to obtain

$$\begin{aligned}\left(a + \frac{c_0}{c_1}\right)_v (v,x) &\geq -\frac{[2-s(x)]^3[2+s(x)]}{4[4-s(x)]} + s(x)[4-s(x)^2] \\ &\quad \times \left(\frac{2}{[4-s(x)][4-4v-2s(x)+vs(x)]^2} - \frac{1}{4v^2}\right).\end{aligned}$$

Let us denote this lower bound by $P(v,x)$. The partial derivative of P with respect to v turns out to be

$$P_v(v,x) = \frac{[4-s(x)^2]s(x)[s(x)-2](v-2)}{2v^3[4-4v-2s(x)+vs(x)]^3}Q(v,s(x)) \tag{2.15}$$

where

$$Q(v,s) := v^2s^2 - 10v^2s - 4vs^2 + 28v^2 + 28vs + 4s^2 - 40v - 16s + 16.$$

In (2.15), the denominator is negative, since $(v,x) \in \Delta$, and the numerator is obviously non-negative. Since the critical values of Q are $Q(0,2) = 0$ and $Q(2,6) = 48$, a limit argument shows that $Q(v,s) \geq 0$ for all $(v,s) \in \mathbb{R}^2$. Thus P is monotonously decreasing as a function of v. Finally, using

$$P(1,x) = \frac{[2-s(x)]^2[2+s(x)]}{s(x)[4-s(x)]} > 0$$

we conclude that $P(v,x) > 0$ for all $(v,x) \in \Delta$. $\square$

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