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UNIVERSALLY MEAGER SETS

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Abstract. We study category counterparts of the notion of a universal measure zero set of reals.

We say that a set $A \subseteq \mathbb{R}$ is universally meager if every Borel isomorphic image of A is meager in $\mathbb{R}$. We give various equivalent definitions emphasizing analogies with the universally null sets of reals.

In particular, two problems emerging from an earlier work of Grzegorek are solved.

1. Introduction

Since special subsets of the real line are the subject of this note, a good general reference is Miller's article [7].

A subset A of the reals or, more generally, of a perfect (i.e. with no isolated points) Polish (i.e. separable, metrizable) topological space X is a *universal measure zero* set if it has (outer) measure zero with respect to every Borel measure on X. By a Borel measure we mean a countably additive, continuous (i.e. points have measure zero) finite measure defined on the σ-algebra $\mathbf{B}(X)$ of Borel subsets of X. It is well known and easy to prove that $A \subseteq X$ is a universal measure zero set if and only if either of the following conditions holds:

1. every Borel isomorphic image of A in $\mathbb{R}$ has (outer) Lebesgue measure zero,
2. A does not contain any Borel one-to-one image of a set of positive outer Lebesgue measure.

A subset A of the reals or, more generally, of a perfect Polish space X is *perfectly meager* if for all perfect subsets P of X, the set $A \cap P$ is meager relative to the topology of P. Quoting Miller [7] we may note that this notion is "somewhat analogous" to the one of a universal measure zero set. However, the category counterparts of the conditions above do not characterize perfectly meager sets since, assuming the continuum hypothesis (CH), there exists a set of reals which is perfectly meager but has a non-meager image under a Borel isomorphism (see [2]). This led to the study of the following two apparently different classes of small sets which was undertaken by Grzegorek in [1], [2], [3].

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Using Grzegorek's definitions, we will call a set $A \subseteq X$:

1. *absolutely of the first category* if every Borel isomorphic image of A in X is meager (see [1]),
2. an $\overline{\mathrm{AFC}}$ set if it does not contain any Borel one-to-one image of a non-meager subset of X (see [2]).

Clearly, every $\overline{\mathrm{AFC}}$ set is absolutely of the first category. The natural question, whether the converse is true, seems to have been left open.

In this note we give a simple proof that the two definitions above describe just the same class. We characterize it in topological and Boolean-algebraic terms collecting further evidence that its members are "more analogous" to universal measure zero sets and should therefore be called *universally meager* sets. In particular, the product of two universally meager sets is universally meager. This was earlier proved by Grzegorek [2] for $\overline{\mathrm{AFC}}$ sets using CH. On the other hand Recław [10] showed that assuming CH there exist two perfectly meager sets whose product is not perfectly meager.

2. Properties of universally meager sets

The equivalent statements constituting the theorem below are motivated by their respective counterparts concerning universal measure zero sets. In particular, a subset A of a perfect Polish space X is a universal measure zero set if $\mu(A) = 0$ for every (countably additive, continuous, finite) measure defined on the σ-algebra $\mathbf{B}(A)$ of (relative) Borel subsets of A. Equivalently, there is no σ-ideal $\mathcal{I}$ in $\mathbf{B}(A)$ such that $\mathbf{B}(A)/\mathcal{I} \cong \mathbf{R}$, the factor algebra of $\mathbf{B}(\mathbb{R})$ modulo the σ-ideal of Lebesgue null sets. By a σ-ideal in $\mathbf{B}(A)$ we mean here a proper subfamily of $\mathbf{B}(A)$ containing all singletons which is closed under taking subsets and countable unions.

Let $\mathrm{MGR}(A)$ denote the collection of all meager subsets of A. If τ is a topology on A, then $\mathbf{B}(A,\tau)$ and $\mathrm{MGR}(A,\tau)$ denote the respective families of Borel and meager subsets of A in the topology τ. The factor algebra of $\mathbf{B}(\mathbb{R})$ modulo the σ-ideal $\mathrm{MGR}(\mathbb{R})$ will be denoted by $\mathbf{C}$; it is the unique, up to an isomorphism, complete, atomless Boolean algebra with a countable dense subset.

Theorem 2.1. *For a subset A of a perfect Polish space X, the following are equivalent:*

(i) *A is an $\overline{AFC}$ set.*

(ii) *A is absolutely of the first category.*

(iii) *For every σ-ideal I in $\mathbf{B}(X)$ such that $\mathbf{B}(X)/I \cong \mathbf{C}$ there is a Borel set $B \in I$ such that $A \subseteq B$.*

(iv) *A is meager in every Polish topology τ on X such that X has no isolated points and $\mathbf{B}(X,\tau) = \mathbf{B}(X)$.*

(v) *A is meager in every second countable Hausdorff topology τ on X such that X has no isolated points and all Borel sets (in the original Polish topology) have Baire Property in the topology τ.*

(vi) *There is no σ-ideal $\mathcal{I}$ in $\mathbf{B}(A)$ such that $\mathbf{B}(A)/\mathcal{I} \cong \mathbf{C}$.*

(vii) *A is meager in every second countable Hausdorff topology τ on A such that A has no isolated points and all Borel subsets of A (in the topology inherited from the original Polish topology on X) have Baire Property in the topology τ.*

(viii) *A is meager in every separable metrizable topology τ on A such that A has no isolated points and $\mathbf{B}(A) = \mathbf{B}(A,\tau)$.*

Proof. (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v): It suffices to notice that:

- Due to Sikorski's theorem (see [5], 15.10), if I is a σ-ideal in $\mathbf{B}(X)$ such that $\mathbf{B}(X)/I \cong \mathbf{C}$, then there is a Borel automorphism $\Phi : X \to X$ such that
$$\forall A \in \mathbf{B}(X)\ A \in \mathrm{MGR}(X) \iff \Phi[A] \in I.$$
- If τ is a second countable Hausdorff topology on X such that X has no isolated points, all Borel sets (in the original Polish topology) have Baire Property in the topology τ and $A \notin \mathrm{MGR}(A, \tau)$, then
$$\mathbf{B}(X)/(\mathrm{MGR}(X, \tau) \cap \mathbf{B}(X)) \cong \mathbf{C}.$$

(iii) $\Rightarrow$ (vi): Clear, since if $\mathcal{I}$ is a σ-ideal in $\mathbf{B}(A)$, then
$$I = \{B \in \mathbf{B}(X) :\ B \cap A \in \mathcal{I}\}$$
is a σ-ideal in $\mathbf{B}(X)$ with $\mathbf{B}(A)/\mathcal{I} \cong \mathbf{B}(X)/I$.

(vi) $\Rightarrow$ (iv): Suppose that $A \notin \mathrm{MGR}(X, \tau)$ for a certain Polish topology τ on X such that X has no isolated points and $\mathbf{B}(X, \tau) = \mathbf{B}(X)$.

Let $\mathcal{I} = \mathrm{MGR}(X, \tau) \cap \mathbf{B}(A)$. But then $\mathcal{I}$ is a σ-ideal in $\mathbf{B}(A)$ with $\mathbf{B}(A)/\mathcal{I} \cong \mathbf{C}$ (see [12], Lemma 2.2).

(vi) $\Rightarrow$ (vii): It is easy to see that if τ is a second countable Hausdorff topology on A such that A has no isolated points, all sets from $\mathbf{B}(A)$ have Baire Property in the topology τ and $A \notin \mathrm{MGR}(A, \tau)$, then
$$\mathbf{B}(A)/(\mathrm{MGR}(A, \tau) \cap \mathbf{B}(A)) \cong \mathbf{C}.$$

(vii) $\Rightarrow$ (viii): Obvious.

(viii) $\Rightarrow$ (vi): Suppose that $\mathcal{I}$ is a σ-ideal in $\mathbf{B}(A)$ such that $\mathbf{B}(A)/\mathcal{I} \cong \mathbf{C}$. We will find a separable metrizable topology τ on A such that A has no isolated points, $\mathbf{B}(A) = \mathbf{B}(A, \tau)$ and $A \notin \mathrm{MGR}(A, \tau)$.

Define a σ-ideal J in $\mathbf{B}(X)$ by
$$B \in J \iff B \cap A \in \mathcal{I} \text{ for } B \in \mathbf{B}(X).$$
Since
$$\mathbf{B}(X)/J \cong \mathbf{B}(A)/\mathcal{I} \cong \mathbf{C},$$
there is a Polish topology τ_1 on X for which we have $\mathbf{B}(X) = \mathbf{B}(X, \tau_1)$ and $J = \mathrm{MGR}(X, \tau_1) \cap \mathbf{B}(X)$ (see [5], 15.10).

Let $\tau = \tau_1 | A$ be the topology on A inherited from the topology τ_1. Clearly, τ is second countable and $\mathbf{B}(A) = \mathbf{B}(A, \tau)$.

Also, the topological space (A, τ) is perfect. Otherwise, there is a $U \in \tau_1$, $U \neq \emptyset$ with $|A \cap U| = 1$. But then $U \setminus A \notin \mathrm{MGR}(X, \tau_1)$ as a non-empty open subset of a Polish space contradicting the fact that $U \setminus A \in J$ and $J = \mathrm{MGR}(X, \tau_1) \cap \mathbf{B}(X)$.

Finally, $A \notin \mathrm{MGR}(A, \tau)$. Otherwise, $A \in \mathrm{MGR}(X, \tau_1)$, so there is $B \in \mathbf{B}(X)$ with $A \subseteq B$ and $B \in \mathrm{MGR}(X, \tau_1)$. Then $B \in J$ which means $A \in \mathcal{I}$ – a contradiction. (It could even be proved that $\mathrm{MGR}(A, \tau) = \mathcal{I}$.)

(iii) $\Rightarrow$ (i): Suppose A is not an $\overline{\mathrm{AFC}}$ set. Let $f : Z \to A$ be a Borel one-to-one function with $Z \notin \mathrm{MGR}(X)$. We will find a σ-ideal I in $\mathbf{B}(X)$ with $\mathbf{B}(X)/I \cong \mathbf{C}$ and such that no Borel set $B \in I$ covers A.

Define a σ-ideal I in $\mathbf{B}(X)$ by letting
$$B \in I \iff f^{-1}[B] \in \mathrm{MGR}(X) \text{ for } B \in \mathbf{B}(X).$$

Note that f induces a complete embedding of the Boolean algebra $\mathbf{B}(X)/I$ into the algebra $\mathbf{B}(Z)/(\mathrm{MGR}(X)\cap\mathbf{B}(Z))$. Moreover, $\mathbf{B}(Z)/(\mathrm{MGR}(X)\cap\mathbf{B}(Z))\cong\mathbf{C}$ (see [12], Lemma 2.2). It follows that the algebra $\mathbf{B}(X)/I$ is also isomorphic to $\mathbf{C}$ as a complete subalgebra of the latter which has no atoms. To verify the last statement, suppose that there is $C\in\mathbf{B}(X)\setminus I$ such that for every $B\in\mathbf{B}(X)$ either $B\cap C\in I$ or $C\setminus B\in I$. Then, since I contains all singletons, every point x of C has an open neighbourhood V_x with $V_x\cap C\in I$. This immediately implies that $C\in I$ – a contradiction.

Finally, since $f^{-1}[A]=Z\notin\mathrm{MGR}(X)$, A cannot be covered by a Borel set from I. □

From now on a subset A of a perfect Polish space X will be called *universally meager* if it satisfies any of the equivalent conditions above.

Clearly, every universally meager set is perfectly meager but (at least consistently) not vice versa (see [2]). In particular, the two classes differ as far as the closure under products is concerned. Assuming CH there exist two perfectly meager sets whose product is not perfectly meager (cf. [10]). On the other hand the following result (and its proof) closely resembles Marczewski's theorem (and its proof – see [7], Theorem 8.1) that the product of two universal measure zero sets has universal measure zero. It improves Grzegorek's theorem that, assuming CH, the class of $\overline{\mathrm{AFC}}$ sets is closed under taking products.

Theorem 2.2. *The product of two universally meager sets is universally meager.*

Proof. Let D and E be universally meager subsets of perfect Polish spaces X and Y, respectively. Suppose, towards a contradiction, that $A=D\times E$ is not universally meager. By Theorem 2.1(vi) this means that there is a σ-ideal $\mathcal{J}$ in $\mathbf{B}(A)$ such that

$$\mathbf{B}(A)/\mathcal{J}\cong\mathbf{C}.$$

Define a σ-ideal $\mathcal{I}$ in $\mathbf{B}(E)$ by

$$B\in\mathcal{I}\ \text{ if }\ D\times B\in\mathcal{J}\ \text{ for }\ B\in\mathbf{B}(E).$$

Since D is universally meager, so is $D\times\{y\}$ for $y\in Y$; it follows that $D\times\{y\}\in\mathcal{J}$ and so $\{y\}\in\mathcal{I}$. This implies that the quotient algebra $\mathbf{B}(E)/\mathcal{I}$ is atomless. Moreover, using the formula

$$[B]_{\mathcal{I}}\mapsto[D\times B]_{\mathcal{J}}$$

we can completely embed it into the algebra $\mathbf{B}(A)/\mathcal{J}$. But, since the latter is isomorphic to $\mathbf{C}$, so is the algebra $\mathbf{B}(E)/\mathcal{I}$, contrary to the fact that E is universally meager. □

Despite the differences between universally meager and perfectly meager sets indicated above, in all standard examples uncountable perfectly meager sets turn out to be universally meager as well. These examples include, in particular, a selector of the constituents of a non-trivial coanalytic set (see [7], Theorem 5.3), a Hausdorff (ω_1,ω_1^*)-gap and a tower, i.e., any set well-ordered by a Borel relation (see [9]).

Recall that a set $A\subseteq 2^\omega$ has the *Hurewicz property* if every continuous image of A in ω^ω is bounded, i.e., contained in a K_σ subset of ω^ω. In Theorem 5.5 of Just-Miller-Scheepers-Szeptycki [4] it was shown that if A has the Hurewicz property and contains no perfect subset, then A is perfectly meager. This can be strengthened as follows.

Proposition 2.3. *If a set $A \subseteq 2^\omega$ has the Hurewicz property and contains no perfect subset, then A is universally meager.*

Proof. By a result of Nowik (see [8], the proof of Theorem 2), a sufficient condition for A to be universally meager is that for every countable family $\{Q_n : n \in \mathbb{N}\}$ of perfect subsets of 2^ω there exists an F_σ set F such that $A \subseteq F$ and $Q_n \nsubseteq F$ for any $n \in \omega$.

So let $Q_n,\ n \in \mathbb{N}$, be perfect subsets of 2^ω. Since A contains no perfect subsets, for each $n \in \mathbb{N}$ there is an open set U_n such that $A \subseteq U_n$ but $Q_n \nsubseteq U_n$.

Let $G = \bigcap_{n\in\mathbb{N}} U_n$. Now, since A has the Hurewicz property, Theorem 5.7 of Just-Miller-Scheepers-Szeptycki [4] tells us that there is a F_σ-set F such that $A \subseteq F \subseteq G$. Clearly, $Q_n \nsubseteq F$ for any $n \in \mathbb{N}$. □

Let us finally indicate that the problem, whether one can prove in ZFC alone that the classes of perfectly meager and universally meager sets are different, seems to be open.*

The same question is apparently open for the classes of universal measure zero sets and universally meager sets, though both inclusions between the two are known to be relatively consistent. By the results of Miller [6], the models are the Cohen real model, in which every universally meager set has size less than continuum and has therefore universal measure zero, and the random real model, in which the situation is symmetric.

Note that as a simple consequence of the equivalence "(ii) ⇔ (vi)" from Theorem 2.1 and the well known dual fact concerning universal measure zero we have the following:

Proposition 2.4. *The following are equivalent:*

(i) *The classes of universal measure zero sets and universally meager sets are equal.*

(ii) *If a measurable space $(X, \mathcal{A})$ is countably generated and separates points, then there exists a σ-ideal $\mathcal{I}$ in $\mathcal{A}$ such that $\mathcal{A}/\mathcal{I} \cong \mathbf{C}$ if and only if there exists a σ-ideal $\mathcal{J}$ in $\mathcal{A}$ such that $\mathcal{A}/\mathcal{J} \cong \mathbf{R}$.*

References

1. E. Grzegorek *Solution to a problem of Banach on σ-fields without continuous measures*, Bull. Acad. Pol. Sci. **28** (1980), 7–10. MR **82h:**04005
2. E. Grzegorek *Always of the first category sets*, Rend. Circ. Mat. Palermo, II. Ser. Suppl. **6** (1984), 139–147. CMP 17:10
3. E. Grzegorek *Always of the first category sets. II*, Rend. Circ. Mat. Palermo, II. Ser. Suppl. **10** (1985), 43–48. MR **88j:**54054
4. W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, *The combinatorics of open covers (II)*, Topology Appl. **73** (1996), 241–266. MR **98g:**03115a
5. A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Math. 156, Springer-Verlag 1995. MR **96e:**03057
6. A.W. Miller, *Mapping a set of reals onto the reals*, J. Symb. Logic **48** (1983), 575–584. MR **84k:**03125
7. A.W. Miller, *Special subsets of the real line* in *Handbook of set–theoretic topology*, North-Holland 1984, 201–233. MR **86i:**54037

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* *Added in proof.* Bartoszyński has recently proved that the two classes are consistently the same.

8. A. Nowik, T. Weiss, *Not every Q-set is perfectly meager in the transitive sense*, to appear in Proc. Amer. Math. Soc. CMP 99:06
9. S. Plewik, *Towers are universally measure zero and always of first category*, Proc. Amer. Math. Soc. **119(3)** (1993), 865–868. MR **93m:**04003
10. I. Recław, *Products of perfectly meager sets*, Proc. Amer. Math. Soc. **112(4)** (1991), 1029–1031. MR **91j:**28001
11. I. Recław and P. Zakrzewski, *Strong Fubini properties of ideals*, Fund. Math. **159** (1999), 135–152. CMP 99:08
12. P. Zakrzewski, *Extending Baire property by countably many sets*, to appear in Proc. Amer. Math. Soc. CMP 99:14

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