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TOWERS OF BOREL FUNCTIONS

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Abstract. We give mathematical reformulations of the cardinals $\mathfrak{p}$ and $\mathfrak{t}$ in terms of families of Borel functions. As an application we show that $\mathfrak{t}$ is invariant under the addition of a single Cohen real.

1. Introduction

Definitions and notation. Let $\mathbb{N}$ be the set of all nonnegative integers. We define a relation "almost set inclusion" on $\mathscr{P}(\mathbb{N})$ by

$$A \subseteq^* B \quad \text{iff} \quad |A \setminus B| < \aleph_0,$$

where $A, B \subseteq \mathbb{N}$. And $A \supseteq^* B$ iff $B \subseteq^* A$. For a family $\mathcal{F} \subseteq \mathscr{P}(\mathbb{N})$, $A \subseteq \mathbb{N}$ is a *pseudo-intersection* of $\mathcal{F}$ iff $A \subseteq^* F$ for every $F \in \mathcal{F}$.

A family of (infinite) subsets of $\mathbb{N}$ is called a *filter base* iff every nonempty finite subfamily has an infinite intersection. Let $\mathfrak{p}$ be the cardinality of the smallest filter base with no infinite pseudo-intersection. Let $\mathscr{P}_\infty(\mathbb{N})$ denote the set of all infinite subsets of $\mathbb{N}$. A *tower* is a subfamily of $\mathscr{P}_\infty(\mathbb{N})$ that is well-ordered by $\supseteq^*$. Let $\mathfrak{t}$ be the cardinality of the smallest tower with no infinite pseudo-intersection.

Now we define the corresponding notions in the realm of functions from $\mathbb{R}$ into $\mathscr{P}(\mathbb{N})$. For a property of the reals $P(x)$, we say that $P(x)$ for *almost all* x in $\mathbb{R}$ iff there is a comeager $X \subseteq \mathbb{R}$ such that $P(x)$ holds for all $x \in X$. We define a relation on the set of all functions from $\mathbb{R}$ to $\mathscr{P}(\mathbb{N})$ by

$$f \subseteq^* g \quad \text{iff} \quad f(x) \subseteq^* g(x) \text{ for almost all } x \in \mathbb{R},$$

where $f, g : \mathbb{R} \to \mathscr{P}(\mathbb{N})$. For a family $\mathcal{F}$ of functions from $\mathbb{R}$ to $\mathscr{P}(\mathbb{N})$, $f : \mathbb{R} \to \mathscr{P}(\mathbb{N})$ is a *pseudo-intersection* of $\mathcal{F}$ iff $f \subseteq^* g$ for every $g \in \mathcal{F}$.

Definition 1.1. A family $\mathcal{F}$ of functions from $\mathbb{R}$ to $\mathscr{P}(\mathbb{N})$ is *filtered* iff for every nonempty finite subfamily $\mathcal{A} \subseteq \mathcal{F}$,

$$\bigcap_{f \in \mathcal{A}} f(x) \text{ is infinite for almost all } x \in \mathbb{R}.$$

Definition 1.2. A family of functions $\mathcal{T}$ from $\mathbb{R}$ to $\mathscr{P}(\mathbb{N})$ is a *tower* iff

1. $\mathcal{T}$ is well-ordered by $\supseteq^*$, and
2. for every $f \in \mathcal{T}$, $f(x)$ is infinite for almost all $x \in \mathbb{R}$.

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Denote the Cantor set, i.e. $\{0,1\}^{\mathbb{N}}$ with the product topology, by $\mathscr{C}$. We give $\mathscr{P}(\mathbb{N})$ a topology by identifying $\mathscr{P}(\mathbb{N})$ with $\mathscr{C}$.

Definition 1.3. Let $\mathfrak{p}_1$ be the cardinality of the smallest filtered family $\mathcal{F}$ of Borel functions from $\mathbb{R}$ to $\mathscr{P}(\mathbb{N})$ such that there is no Borel function $f : \mathbb{R} \to \mathscr{P}_\infty(\mathbb{N})$ which is a pseudo-intersection of $\mathcal{F}$.

Our first claim is that $\mathfrak{p} = \mathfrak{p}_1$.

Definition 1.4. Let $\mathfrak{t}_1$ be the cardinality of the smallest tower $\mathcal{T}$ of Borel functions from $\mathbb{R}$ to $\mathscr{P}(\mathbb{N})$ such that there is no Borel function $f : \mathbb{R} \to \mathscr{P}_\infty(\mathbb{N})$ which is a pseudo-intersection of $\mathcal{T}$.

The other main result of this paper is that $\mathfrak{t} = \mathfrak{t}_1$. No specialized knowledge of set theory is required in order to understand the proofs.

I wish to thank Stevo Todorčević for his handwritten notes: "$\mathscr{C}_\omega \Vdash \dot{\mathfrak{p}} \leq \check{\mathfrak{p}}$" from the summer of 1997.

2. The proof

To start, we need the following basic facts about $\mathfrak{p}$ and $\mathfrak{t}$. Let $\mathbb{N}^{\mathbb{N}}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. We define the relation $\leq^*$ on $\mathbb{N}^{\mathbb{N}}$ by

$$f \leq^* g \quad \text{iff} \quad \text{there is an } n \in \mathbb{N} \text{ such that } f(k) \leq g(k) \text{ for all } k \geq n,$$

where $f, g \in \mathbb{N}^{\mathbb{N}}$. Let $\mathfrak{b}$ be the size of the smallest subfamily of $\mathbb{N}^{\mathbb{N}}$ that is unbounded in $\langle \mathbb{N}^{\mathbb{N}}, \leq^* \rangle$.

Theorem 2.1. $\omega_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b}$.

Proof. See [2].

We also use the following trivial fact.

Fact 2.2. $\mathfrak{t}$ *is regular.*

Proof. See [2].

Let $\mathcal{I}$ be an ideal. Define

$$\operatorname{add}(\mathcal{I}) = \min\bigl\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\bigr\}.$$

For a topological space X, we let $\mathcal{M}$ denote the σ-ideal of meager subsets of X. We shall need the following lower bound for $\operatorname{add}(\mathcal{M})$.

Theorem 2.3 (Piotrowski–Szymański)**.** *For any Polish space,* $\operatorname{add}(\mathcal{M}) \geq \mathfrak{t}$.

Proof. See [4].

Theorem 2.4. $\mathfrak{p} = \mathfrak{p}_1$.

Theorem 2.5. $\mathfrak{t} = \mathfrak{t}_1$.

We prove Theorems 2.4 and 2.5 simultaneously.

Proofs. $\mathfrak{t}_1 \leq \mathfrak{t}$: Let $\{A_\xi\}_{\xi<\mathfrak{t}} \subseteq \mathscr{P}_\infty(\mathbb{N})$ be a tower with no infinite pseudo-intersection, such that the enumeration respects the well-ordering of the tower, i.e. $\xi \leq \eta \to A_\eta \subseteq^* A_\xi$. For each $\xi < \mathfrak{t}$, define $f_\xi : \mathbb{R} \to \mathscr{P}_\infty(\mathbb{N})$ by $f_\xi(x) = A_\xi$ for all $x \in \mathbb{R}$. Then obviously $\{f_\xi\}_{\xi<\mathfrak{t}}$ is a tower of Borel functions. We need to show that there is no Borel function $f : \mathbb{R} \to \mathscr{P}_\infty(\mathbb{N})$ which is a pseudo-intersection of $\{f_\xi\}_{\xi<\mathfrak{t}}$.

Suppose to the contrary that f is such a function. Find a comeager $G \subseteq \mathbb{R}$ such that $f \restriction G$ is continuous. For each $\xi < \mathfrak{t}$ and $n \in \mathbb{N}$, define

$$F_{\xi n} = \{x \in G : f(x) \setminus n \subseteq A_\xi\}.$$

Then each $F_{\xi n}$ is a relatively closed subset of G. Since $\bigcup_{n=0}^{\infty} F_{\xi n}$ is a relatively comeager—and in particular, nonmeager—subset of G, there is a nonempty rational interval I_ξ and an $n_\xi \in \mathbb{N}$ such that

$$G \cap I_\xi \subseteq F_{\xi n_\xi}.$$

Since $\mathfrak{t}$ is an uncountable regular cardinal (Theorem 2.1 and Fact 2.2), there are $\overline{n} \in \mathbb{N}$ and a rational interval $\bar{I}$ such that

$$\Gamma = \{\xi < \mathfrak{t} : n_\xi = \overline{n} \text{ and } I_\xi = \bar{I}\}$$

is cofinal in $\mathfrak{t}$. Pick $x \in G \cap \bar{I}$. Then $f(x) \setminus \overline{n} \subseteq \bigcap_{\xi \in \Gamma} A_\xi$, from which it follows that $f(x)$ is an infinite pseudo-intersection of $\{A_\xi\}_{\xi<\mathfrak{t}}$, a contradiction.

$\mathfrak{p}_1 \leq \mathfrak{p}$: First suppose that $\mathfrak{p} = \mathfrak{t}$. Note that every tower of Borel functions is also filtered. Hence $\mathfrak{p}_1 \leq \mathfrak{t}_1 \leq \mathfrak{t} = \mathfrak{p}$.

By Theorem 2.1, it remains to consider the case where $\mathfrak{p} < \mathfrak{t}$. Let $\{A_\xi\}_{\xi<\mathfrak{p}} \subseteq \mathscr{P}(\mathbb{N})$ be a filter base with no infinite pseudo-intersection. For each $\xi < \mathfrak{p}$, define $f_\xi : \mathbb{R} \to \mathscr{P}(\mathbb{N})$ by $f_\xi(x) = A_\xi$ for all $x \in \mathbb{R}$. Obviously $\{f_\xi\}_{\xi<\mathfrak{p}}$ is a filtered family of Borel functions. Suppose towards a contradiction that $f : \mathbb{R} \to \mathscr{P}_\infty(\mathbb{N})$ is a pseudo-intersection of $\{f_\xi\}_{\xi<\mathfrak{p}}$. For each $\xi < \mathfrak{p}$, choose a comeager $G_\xi \subseteq \mathbb{R}$ such that $f(x) \subseteq^* A_\xi$ for all $x \in G_\xi$. By assumption and Theorem 2.3, $\mathfrak{p} < \mathrm{add}(\mathcal{M})$, whence there is a comeager $G \subseteq \bigcap_{\xi<\mathfrak{p}} G_\xi$. But if we take $x \in G$, then $f(x)$ is an infinite pseudo-intersection of $\{A_\xi\}_{\xi<\mathfrak{p}}$, giving a contradiction.

$\mathfrak{p} \leq \mathfrak{p}_1$, $\mathfrak{t} \leq \mathfrak{t}_1$: We take $\kappa < \mathfrak{p}$ ($\kappa < \mathfrak{t}$), and prove that $\kappa < \mathfrak{p}_1$ ($\kappa < \mathfrak{t}_1$). Note that $\mathbb{R}$ is homeomorphic to the unit interval $(0,1)$ via the standard homeomorphism. Consider the standard surjection $\Phi : \mathscr{C} \to [0,1]$, where $\Phi(x)$ is the base 2 expansion of x after the decimal point, for all $x \in \mathscr{C}$. Φ is a homeomorphism on a co-countable subset of $\mathscr{C}$. Hence, if we replace the domains of the functions in the formulations of $\mathfrak{p}_1$ and $\mathfrak{t}_1$ with $\mathscr{C}$, then we are proving an equivalent result.

Suppose that $f_\xi : \mathscr{C} \to \mathscr{P}(\mathbb{N})$ ($\xi < \kappa$) is a filtered family (tower) of Borel functions. (For $\{f_\xi\}_{\xi<\kappa}$ a tower, we assume that the enumeration respects the well-ordering of the tower.) For each $\xi < \kappa$, we can find a dense G_δ—and thus comeager—set $G_\xi \subseteq \mathscr{C}$ such that $f_\xi \restriction G_\xi$ is continuous. For each nonempty $F \in [\kappa]^{<\aleph_0}$, write $F = \{\xi_1 < \xi_2 < \cdots < \xi_n\}$. Then there is a comeager $G_F \subseteq \mathscr{C}$ such that $\left|\bigcap_{i=1}^n f_{\xi_i}(x)\right| = \aleph_0$ for all $x \in G_F$ (and for a tower, $f_{\xi_j}(x) \subseteq^* f_{\xi_i}(x)$ for all $1 \leq i < j \leq n$, for all $x \in G_F$).

By Theorem 2.3, there is a dense G_δ set $H \subseteq \mathscr{C}$ such that

$$H \subseteq \bigcap_{\xi<\kappa} G_\xi \cap \bigcap_{F \in [\kappa]^{<\aleph_0}} G_F.$$

Then

(1) for all $\xi < \kappa$, $f_\xi \restriction H$ is continuous, and

(2) for all $x \in H$, $\{f_\xi(x)\}_{\xi<\kappa}$ is a filter base,

(2′) for all $x \in H$, $\{f_\xi(x)\}_{\xi<\kappa}$ is a tower.

Let $\{x_n\}_{n=0}^{\infty}$ be an enumeration of a dense subset of H.

Since $\kappa < \mathfrak{p}$ ($\kappa < \mathfrak{t}$), by (2) (by (2′)), there is a sequence $\{d(n)\}_{n=0}^{\infty} \subseteq \mathscr{P}_{\infty}(\mathbb{N})$ such that

$$d(n) \subseteq^* f_\xi(x_n) \quad \text{for all } n \in \mathbb{N}, \text{ and all } \xi < \kappa. \tag{3}$$

For the remainder of the proof, we need only the fact that $\kappa < \mathfrak{b}$. By (3), for each $\xi < \kappa$, we can choose $g_\xi : \mathbb{N} \to \mathbb{N}$ so that $d(n) \setminus g_\xi(n) \subseteq f_\xi(x_n)$ for all $n \in \mathbb{N}$. Since $\kappa < \mathfrak{b}$, there is a $D : \mathbb{N} \to \mathbb{N}$ such that $g_\xi \leq^* D$ for all $\xi < \kappa$. For each $\xi < \kappa$, fix $m_\xi \in \mathbb{N}$ so that $g_\xi(n) \leq D(n)$ for all $n \geq m_\xi$. Then

$$d(n) \setminus D(n) \subseteq f_\xi(x_n) \quad \text{for all } \xi < \kappa, \text{ for all } n \geq m_\xi. \tag{4}$$

Now we measure the continuity of $f_\xi \restriction H$ at each x_n.

Notation. For $t \in 2^{<\mathbb{N}}$ and $A \subseteq \mathscr{C}$, $[t]_A = [t] \cap A$.

Claim 5. *There is a function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $\xi < \kappa$,*

$$f_\xi''[x_n \restriction F(n,\ell)]_H \subseteq [f_\xi(x_n) \restriction \ell] \quad \textit{for all but finitely many } (n,\ell) \in \mathbb{N} \times \mathbb{N},$$

i.e. $[f_\xi(x_n) \restriction \ell] = \{A \subseteq \mathbb{N} : f_\xi(x_n) \cap \ell \sqsubseteq A\}$, where $A \sqsubseteq B$ means that A is an initial segment of B for $A, B \subseteq \mathbb{N}$.

Proof. Let $\Phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection. For each $\xi < \kappa$, since $f_\xi \restriction H$ is continuous, there is a function $F_\xi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$f_\xi''[x_n \restriction F_\xi(n,\ell)]_H \subseteq [f_\xi(x_n) \restriction \ell]$$

for all $n, \ell \in \mathbb{N}$. For each $\xi < \kappa$, define $g_\xi : \mathbb{N} \to \mathbb{N}$ by $g_\xi(m) = F_\xi(\Phi(m))$ for all $m \in \mathbb{N}$. Since $\kappa < \mathfrak{b}$, there is an $h : \mathbb{N} \to \mathbb{N}$ so that $g_\xi \leq^* h$ for all $\xi < \kappa$. Define $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $F(n,\ell) = h(\Phi^{-1}(n,\ell))$ for all $n, \ell \in \mathbb{N}$. □

Let $\{s_i\}_{i=0}^{\infty}$ enumerate $2^{<\mathbb{N}}$. We define $\{n_i\}_{i=0}^{\infty}, \{k_i\}_{i=0}^{\infty} \subseteq \mathbb{N}$, and $\{t_i\}_{i=0}^{\infty} \subseteq 2^{<\mathbb{N}}$ by recursion on i so that for all $i \in \mathbb{N}$,

(a) $s_i \subseteq x_{n_i}$,
(b) $n_i \geq i$,
(c) $k_i \in d(n_i)$,
(d) $k_i \geq D(n_i)$,
(e) $k_{i+1} > k_i$,
(f) $t_i \subseteq x_{n_i}$, and
(g) $|t_i| \geq \max\big(F(n_i, k_i + 1), |s_i|\big)$.

Let $i \in \mathbb{N}$ be given. Since $\{x_n\}_{n=0}^{\infty}$ is dense, there are infinitely many x_n's extending s_i. Hence we can find $n_i \geq i$ such that $s_i \subseteq x_{n_i}$. Let $k_i \in d(n_i)$ be sufficiently large so that (d) and (e) hold. Then obviously we can find t_i as required.

Define $G^* \subseteq \mathscr{C}$ by

$$G^* = \bigcap_{m=0}^{\infty} \bigcup_{i=m}^{\infty} [t_i].$$

By (a), (f), and (g), $t_i \supseteq s_i$ for all $i \in \mathbb{N}$. It follows that G^* is a dense G_δ set. Hence $G = G^* \cap H$ is also dense G_δ. Now we define $f : G \to \mathscr{P}_{\infty}(\mathbb{N})$ by

$$f(x) = \{k_i : i \in \mathbb{N},\, t_i \subseteq x\}.$$

By (e), $f(x)$ is infinite for all $x \in G$, whence f is well-defined.

Claim 6. *For every $x \in G$, $f(x) \subseteq^* f_\xi(x)$ for all $\xi < \kappa$.*

Proof. Take $x \in G$ and $\xi < \kappa$. By Claim 5, there is an $\ell_\xi \in \mathbb{N}$ such that

$$f''_\xi[x_n \restriction F(n,\ell)]_H \subseteq [f_\xi(x_n) \restriction \ell] \quad \text{for all } n \in \mathbb{N}, \text{ and all } \ell > \ell_\xi.$$

We claim that

$$f(x) \setminus (\ell_\xi \cup \{k_i : i < m_\xi\}) \subseteq f_\xi(x).$$

Suppose that $i \in \mathbb{N}$ is such that $k_i \in f(x) \setminus (\ell_\xi \cup \{k_i : i < m_\xi\})$. By (g), $|t_i| \geq F(n_i, k_i + 1)$. Therefore, since $t_i \subseteq x_{n_i}$ and $k_i + 1 > \ell_\xi$,

$$f''_\xi[t_i]_H = f''_\xi[x_{n_i} \restriction |t_i|]_H \subseteq [f_\xi(x_{n_i}) \restriction k_i + 1].$$

Since $i \geq m_\xi$, $n_i \geq m_\xi$ by (b). Hence by (4), (c), and (d), $k_i \in f_\xi(x_{n_i})$. Thus $k_i \in A$ for all $A \in [f_\xi(x_{n_i}) \restriction k_i + 1]$. Since $t_i \subset x$ and $x \in H$, $x \in [t_i]_H$. Therefore $f_\xi(x) \in [f_\xi(x_{n_i}) \restriction k_i + 1]$, whence $k_i \in f_\xi(x)$. □

Claim 7. *f is continuous.*

Proof. Take $x \in G$. Given $\ell \in \mathbb{N}$, we need to find an $n \in \mathbb{N}$ such that $f''[x \restriction n]_G \subseteq [f(x) \restriction \ell]$. By (e), $n = \max\{|t_i| : k_i < \ell\}$ is finite. Fix $y \in [x \restriction n]_G$. Clearly, for all $k < \ell$, $k \in f(x)$ iff $k \in f(y)$. Thus $f(y) \in [f(x) \restriction \ell]$. □

Extend the domain of f by defining $f(x) = \mathbb{N}$ for all $x \notin G$. Then by Claim 7, and since G is G_δ, f is Borel. And by Claim 6, $f(x)$ is a pseudo-intersection of $\{f_\xi(x)\}_{\xi<\kappa}$ for all $x \in G$. Since G is comeager, $f \subseteq^* f_\xi$ for all $\xi < \kappa$. Therefore f is a pseudo-intersection of the family of functions $\{f_\xi\}_{\xi<\kappa}$. This completes the proof that $\kappa < \mathfrak{p}_1$ ($\kappa < \mathfrak{t}_1$). □

3. Adding a single Cohen real

Definitions and notation. The poset for adding a single Cohen real is viewed as the poset of finite partial functions from $\mathbb{N}$ into 2 which we denote by $\mathscr{C}_\omega$. If $f : \mathscr{C} \to \mathscr{P}(\mathbb{N})$ is a Borel function, then $\tilde{f}$ is a name for the decoding of f in the forcing extension. Fix $\mathscr{C}_\omega$-names $\dot{\mathfrak{p}}$ and $\dot{\mathfrak{t}}$ which are forced to be the values of $\mathfrak{p}$ and $\mathfrak{t}$ in the extension by one Cohen real, respectively. Let $\dot{c}$ be the canonical name for the Cohen real.

There is a canonical correspondence between names for reals in the Cohen extension and codes for Borel functions from $\mathscr{C}$ into $\mathscr{P}(\mathbb{N})$. We describe this by $\dot{x} \mapsto f_{\dot{x}}$, where $\mathscr{C}_\omega \Vdash \tilde{f}_{\dot{x}}(\dot{c}) = \dot{x}$ (see [3]). And in the other direction we have: $f \mapsto \dot{x}_f$, where $\mathscr{C}_\omega \Vdash \tilde{f}(\dot{c}) = \dot{x}_f \subseteq \check{\mathbb{N}}$. Moreover, it is an easy exercise to verify that given any two names $\dot{x}$ and $\dot{y}$ for reals,

$$\mathscr{C}_\omega \Vdash \dot{x} \subseteq^* \dot{y} \quad \text{iff} \quad f_{\dot{x}} \subseteq^* f_{\dot{y}}. \tag{3.1}$$

Also, for any finite sequence $\dot{x}_1, \ldots, \dot{x}_n$ of names for reals,

$$\mathscr{C}_\omega \Vdash \left|\bigcap_{k=1}^n \dot{x}_k\right| = \aleph_0 \quad \text{iff} \quad \left|\bigcap_{i=1}^n f_{\dot{x}_k}(y)\right| = \aleph_0 \text{ for almost all } y. \tag{3.2}$$

For example, if $\{\dot{x}_\xi\}_{\xi<\kappa}$ is a name for a tower, then by (3.1) and (3.2), $\{f_{\dot{x}_\xi}\}_{\xi<\kappa}$ is a tower of Borel functions. In this manner it is easily checked that

(3.3) $\mathscr{C}_\omega \Vdash \dot{\mathfrak{p}} = \check{\mathfrak{p}}_1$, and

(3.4) $\mathscr{C}_\omega \Vdash \dot{\mathfrak{t}} = \check{\mathfrak{t}}_1$.

In effect, we are viewing $\mathscr{C}_\omega$-names as Borel codes (e.g. the code for $f_{\dot{x}}$), with the above translation yielding $\dot{\mathfrak{p}} = \check{\mathfrak{p}}_1$ and $\dot{\mathfrak{t}} = \check{\mathfrak{t}}_1$. By (3.3) and (3.4), the following Corollaries are immediate from Theorems 2.4 and 2.5.

Corollary 3.5. $\mathscr{C}_\omega \Vdash \dot{\mathfrak{p}} = \check{\mathfrak{p}}$.

Corollary 3.6. $\mathscr{C}_\omega \Vdash \dot{\mathfrak{t}} = \check{\mathfrak{t}}$.

Remark 3.7. The fact that MA(σ-centered) is preserved under the addition of a single Cohen real is known as Roitman's Theorem [5]. By Bell's Theorem [1], Roitman's Theorem states that $\mathscr{C}_\omega \Vdash \dot{\mathfrak{p}} \geq \check{\mathfrak{p}}$.

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