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MIGRATION OF ZEROS FOR SUCCESSIVE DERIVATIVES OF ENTIRE FUNCTIONS

ARIE HAREL, SU NAMN, AND JACOB STURM

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Abstract. It is shown that if f is an entire function of order less than one, all of whose zeros are real, then the minimal root of $f^{(k)}$ is an increasing function of k which accelerates as k increases.

In his survey article on successive derivatives of analytic functions, G. Polya begins with the question, "How do the zeros of the n^{th} derivative $f^{(n)}(x)$ behave when n becomes very large?" ([P]). He asks if one can find "some definite trend" in this "migration of zeros". In that same paper, Polya describes his rather pleasing answer when f is a meromorphic function with at least one pole, in which case a complete description of the final set of f is provided.

There are many intersting results in the case where f is an entire function—generally speaking, as is pointed out in [P], "If the order of $f(z)$ is less than one, the differentiation tends to scatter the zeros; the zeros of $f^{(n)}(z)$ tend to move out to ∞ as n increases. If the order of $f(z)$ is greater than one, their distribution becomes denser" (see [B] for a survey of some results along these lines). In this article, we study the question in the setting where f is entire, of order less than one, and posessing only finitely many non-real zeros. Then, we know from [CCS] that for large n, the zeros of $f^{(n)}(z)$ are all real. We show that once this happens, the zeros start to scatter towards infinity in such a way that the zero free region grows in an *accelerating* fashion.

Let $\mathcal{F}$ be the set of entire functions f of one variable satisfying the following two conditions: All the roots of f are real, and the order of f is less than one. One easily sees that $f \in \mathcal{F}$ implies $f' \in \mathcal{F}$. We shall study the migration properties of the zeros of $f^{(k)}$ as a function of k. Our main result says that the minimal root of $f^{(k)}$ accelerates as k increases. To be precise, let

$$r_k = inf\{x \in \mathbf{R} : f^{(k)}(x) = 0\}$$

for all $k \geq 0$ such that $f^{(k)} \neq 0$.

Theorem 1. *The sequence $\{r_k\}$ has non-negative velocity and non-negative acceleration. That is, if we let $v_k = r_{k+1} - r_k$ and $a_k = v_{k+1} - v_k$, then $v_k \geq 0$ and $a_k \geq 0$ for $k \geq 0$.*

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Remark 1. It is natural to ask what happens if we relax the condition that f be of order less than one, and merely impose the condition that f and all its derivatives have no non-real zeros. This class of functions has been identified with the class of Laguerre-Polya functions ([HW1], [HW2]); such functions all have order less than two. It is easy to find functions in this class which violate the conclusion of Theorem 1. For example, if we let $f(z) = e^{-z}(z^2 - az)$ where $a \in \mathbf{R}$ and $a < 0$, then $r_k = min(0, a + 2k)$.

Remark 2. It follows from Theorem 1 that $\lim_{k\to\infty} r_k = \infty$ when f has a smallest zero. This fact is known and appears in [W] (see also [G], page 35).

Before proving the theorem, we first verify that $\mathcal{F}$ is closed under differentiation. Recall that the order of a function f is the smallest $\lambda = \lambda(f) \geq 0$ with the following property: For every $\epsilon > 0$ there exists $C > 0$ such that

$$sup\{|f(z)| : |z| = r\} < Ce^{r^{\lambda+\epsilon}} .$$

The Cauchy integral formula implies that $\lambda(f) \geq \lambda(f')$. Thus, to see that $\mathcal{F}$ is closed under differentiation, it suffices to observe that $f \in \mathcal{F}$ implies that all the roots of f' are real. Now $\lambda(f) < 1$ implies that f has a product expansion which converges uniformly on compact subsets of $\mathbf{C}$:

$$f(z) = Kz^m \prod_n \left(1 - \frac{z}{a_n}\right) \tag{1}$$

where $K \in \mathbf{C}$, m is a non-negative integer, and the a_n range over the non-zero roots of f (counted with multiplicity) ([A], Theorem 8, chapter 5). Thus f can be uniformly approximated on compact sets by polynomials with real zeros; since the set of such polynomials is closed under differentiation, we see that $\mathcal{F}$ is also closed under differentiation.

Next we observe that to prove the theorem, it suffices to consider the case where f is a polynomial. First we note that if $r_0 = -\infty$, then $r_k = -\infty$ for all k, in which case there is nothing to prove. Thus in (1) we may assume that $r_0 = a_0$ and that $a_n \leq a_{n+1}$ for all n. Let f_N be defined by (1) with the product restricted to those $n \leq N$. Then f_N converges to f uniformly on compact subsets, and thus it suffices to prove the theorem for each f_N.

Henceforth we shall assume that f is a polynomial of degree m. It is clear that r_k has non-negative velocity. Thus we are reduced to showing that the acceleration is positive. We may assume $m \geq 3$ (otherwise there is nothing to prove). We change notation slightly, and rephrase the main theorem: Let $f(x)$ be a polynomial of degree m with real coefficients. Assume $m \geq 3$ and that all the roots of f are real. Let $n = m - 1$ and let C be the "center" of f, that is, C is the average of the the roots of f . For $0 \leq k \leq n$ define

$$R_k = max\{x \in \mathbf{R} : f^{(k)}(x) = 0\},$$

$$r_k = min\{x \in \mathbf{R} : f^{(k)}(x) = 0\} .$$

It is clear that $r_0 \leq r_1 \leq ... \leq r_n = C = R_n \leq R_{n-1} \leq ...R_0$.

Theorem 2. *The sequences* $\{r_k\}$ *and* $\{R_k\}$ *accelerate towards the center. That is,*

$$r_k - r_{k-1} \leq r_{k+1} - r_k \quad \text{and} \quad R_{k-1} - R_k \leq R_k - R_{k+1}$$

for $1 \leq k \leq n - 1$.

Lemma 1. *Let $n \geq 2$ and let $\alpha_1, ..., \alpha_n$ be a decreasing sequence of real numbers. Let*

$$g(x) = \prod_{i=1}^{n}(x - \alpha_i) \ .$$

Let β be the largest root of $g'(x) = 0$. Then

$$\beta \geq \frac{(\alpha_1 + \alpha_2)}{2} \ .$$

Proof. If $\alpha_1 = \alpha_2$, then $\beta = \alpha_1 = \alpha_2$, and the result is clear. Thus we may assume that $\alpha_1 > \beta > \alpha_2$. Then

$$0 = \frac{g'(\beta)}{g(\beta)} = \sum_{i=1}^{n} \frac{1}{\beta - \alpha_i} \geq \frac{1}{\beta - \alpha_1} + \frac{1}{\beta - \alpha_2} \ ,$$

which implies the result.

We now return to the proof of the theorem. First we note that if $a, b, c \in \mathbf{R}$ with $ac \neq 0$, then it suffices to prove the theorem for $cf(ax + b)$. In fact, replacing $f(x)$ by $f(-x)$, we are reduced to proving that $\{R_k\}$ accelerates.

By induction on the degree of f, we need only show that $R_0 - R_1 \leq R_1 - R_2$. Replacing $f(x)$ by $f(ax + b)$ for appropriate $a, b \in \mathbf{R}$, we may assume that $R_0 = 2$ and $R_2 = 0$. Thus, our task is to prove $R_1 \geq 1$. If f has two or more non-negative roots, then the lemma implies that $R_1 \geq 1$. Thus we may assume that f can be written in the form:

$$f(x) = (x - 2) \prod_{i=1}^{n}(1 + \rho_i x) \ ,$$

with $\rho_i > 0$.

Let

$$G(\vec{\rho}\,) = G(\rho_1, ..., \rho_n) = \sum_{i=1}^{n} \rho_i - 2 \sum_{i<j} \rho_i \rho_j \ .$$

Then $R_2 = 0$ implies $G(\vec{\rho}\,) = 0$. To show $R_1 \geq 1$ it suffices to prove that $f'(1) \leq 0$. Since $f(1) < 0$, we are reduced to proving $f'(1)/f(1) \geq 0$, that is, if we define

$$F(\vec{\rho}\,) = \sum_{i=1}^{n} \frac{\rho_i}{1 + \rho_i} \ ,$$

then we must show that $G(\vec{\rho}\,) = 0$ implies $F(\vec{\rho}\,) \geq 1$.

Let

$$S = \{\vec{\rho} \in \mathbf{R}^n : G(\vec{\rho}\,) = 0, \ \ \rho_i \geq 0 \text{ for all } i \text{ and } \ \rho_i > 0 \text{ for some } i\}.$$

Lemma 2. *Let $0 < \epsilon < 1/2(n - 1)$.*

If $\vec{\rho} \in S$, then $\rho_i > \epsilon$ for some i.

If $\vec{\rho} \in S$ and $\rho_i > 0$ for some i, then $\rho_j > \epsilon$ for some $j \neq i$.

Proof. If $\rho_i < 1/(n-1)$ for all i, and if $\rho_i > 0$ for some i, then

$$G(\vec{\rho}\,) = \sum_{i=1}^{n} \rho_i \ (1 - \sum_{j \neq i} \rho_j) > 0 \ .$$

Now assume that $\rho_1 > 0$, and assume that $\rho_i < 1/2(n-1)$ for all $i > 1$. Then we have:

$$G(\vec{\rho}\,) = \rho_1 \ (1 - 2\sum_{i>1} \rho_i) + \sum_{i>1} \rho_i \ (1 - \sum_{j \neq 1,i} \rho_j) > 0 \ .$$

Lemma 3. *The function F achieves its minimum on the set S.*

Proof. Note that if $\rho_i = 1/(n-1)$ for all i, then $\vec{\rho} \in S$ and $F(\vec{\rho}\,) = 1$. Now choose K to be a large real number, and let

$$S(K) = \{\vec{\rho} \in S : \rho_i \leq K \text{for all i}\} \ .$$

Then, by Lemma 2, the set $S(K)$ is compact. Also, if $\vec{\rho} \in S$ and $\vec{\rho} \notin S(K)$, then

$$F(\vec{\rho}\,) > \frac{K}{K+1} + \frac{\epsilon}{\epsilon+1} > 1$$

for K sufficiently large (by Lemma 2). This proves Lemma 3.

Let S_{min} be the set of points where F acheives its minimum value. Let $T \subseteq S_{min}$ be the subset consisting of those points with the maximal number of components which are zero.

Lemma 4. *Let $\vec{a} \in T$ be a point with a minimum number of distinct non-zero entries. Then all the non-zero entries of $\vec{a}$ are equal and $F(\vec{a}\,) = 1$.*

Proof. Assume not: Then we may assume $a_1 \neq a_2$ and $a_1 a_2 \neq 0$. Let $A = a_3 + \cdots + a_n$ and $B = 2\sum_{3 \leq i < j} a_i a_j$, and let $x = \rho_1 + \rho_2$ and $y = 2\rho_1\rho_2$. Consider the function

$$g(x,y) = G(\rho_1, \rho_2, a_3, ..., a_n) = x(1-2A) - y + A - B \ .$$

The domain of g is

$$D = \{(x,y) \in \mathbf{R}^2 : x^2 \geq 2y \geq 0\} \ .$$

Let

$$f(x,y) = F(\rho_1, \rho_2, a_3, ..., a_n) = \frac{\rho_1}{1+\rho_1} + \frac{\rho_2}{1+\rho_2} + C = \frac{x+y}{1+x+y/2} + C$$

where C is a constant depending only on $a_3, ..., a_n$. Now our assumptions imply that if we restrict $f(x,y)$ to the line segment $g(x,y) = 0$, $(x,y) \in D$, then the minimum of f occurs in the interior of the line segment. But the function $f(x, x(1-2A)+A-B)$ is a non-constant linear fractional transformation, and therefore has no local minima (or maxima), a contradiction. Thus all the non-zero entries of $\vec{a}$ are equal, which yields $F(\vec{a}\,) = 1$. This proves the lemma, and hence the theorem.

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Department of Mathematics, National University of Singapore, Lower Kent Ridge Road, Singapore 119260

E-mail address: harel@math.nus.edu.sg

Department of Management Information Systems, Hannam University, Taejon, Korea

E-mail address: namn@eve.hannam.ac.kr

Department of Mathematics and Computer Science, Rutgers University, Newark, New Jersey 07102

E-mail address: sturm@andromeda.rutgers.edu