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# DEHN FILLING, REDUCIBLE 3-MANIFOLDS, AND KLEIN BOTTLES

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Abstract. Let $M$ be a compact, connected, orientable, irreducible 3-manifold whose boundary is a torus. We announce that if two Dehn fillings create reducible manifold and manifold containing Klein bottle, then the maximal distance is three.

## 1. Introduction

Let $M$ be a compact, connected, orientable, irreducible 3-manifold such that $\partial M$ is a torus. The *slope* of an essential simple loop on $\partial M$ is its isotopy class, and if $\pi$ and $\gamma$ are two slopes on $\partial M$ then $\triangle = \triangle(\pi, \gamma)$ will denote their minimal geometric intersection number. Let $M(\pi)$ denote the manifold obtained from $M$ by $\pi$-Dehn filling, that is, by attaching a solid torus $V_\pi$ to $M$ along $\partial M$ so that the boundary of a meridian disk is identified with $\pi$, and similarly for $\gamma$.

There are many results on $\triangle(\pi, \gamma)$ for two distinct slopes $\pi$ and $\gamma$ on $\partial M$, for example [BZ1], [CGLS], [Go1], [GLu1], and [Wu1]. Especially Gordon and Luecke [GLu2] have shown that if both $M(\pi)$ and $M(\gamma)$ are reducible, then $\triangle \leq 1$, and Wu [Wu2] and Oh [Oh] proved independently that if $M$ is hyperbolic and $M(\pi)$ is reducible while $M(\gamma)$ contains an incompressible torus, then $\triangle \leq 3$. In this paper we consider the situation where $M(\gamma)$ contains an embedded Klein bottle.

**Theorem 1.1.** *Let $M$ be a hyperbolic 3-manifold. If $M(\pi)$ is reducible and $M(\gamma)$ contains a Klein bottle, then $\triangle(\pi, \gamma) \leq 3$.*

It is still unknown whether or not the bound 3 is best possible. This result gives us a partial improvement to Theorem 0.1(2) of [BZ2], dealing with reducible and finite Dehn fillings.

**Corollary 1.2.** *If $M$ is hyperbolic, $M(\pi)$ is reducible and $M(\gamma)$ is a Seifert fiber space over the 2-sphere with three exceptional fibers of orders $2, 2, n$, then $\triangle(\pi, \gamma) \leq 3$.*

We will be following closely the argument in [Oh] and will hereafter assume familiarity with this paper. To obtain a contradiction, we shall suppose that $\triangle(\pi, \gamma) \geq 4$. Let $\widehat{Q}$ be a reducing sphere in $M(\pi)$ which intersects $V_\pi$ in a family of meridian

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disks. We choose $\widehat{Q}$ so that $Q = \widehat{Q} \cap M$ has the minimal number, say $q$, of boundary components. Similarly we choose $\widehat{S}$, a Klein bottle in $M(\gamma)$ in such a manner.

By an isotopy of $Q$, we may assume that $Q$ and $S$ intersect transversely, and $Q \cap S$ has the minimal number of components. Then no circle component of $Q \cap S$ bounds a disk in $Q$ or $S$, and no arc of $Q \cap S$ is boundary parallel in $Q$ or $S$. The boundary of a regular neighborhood of $\widehat{S}$ is a torus $\widehat{T}$, meeting $V_\gamma$ in $t = 2|\widehat{S} \cap V_\gamma|$ points in $M(\gamma)$. Note that $\widehat{T}$ is not necessarily incompressible. Now we obtain a graph $G_Q$ in $\widehat{Q}$ by taking $\widehat{Q} \cap V_\pi$ as its fat vertices and the arcs in $Q \cap T$ as its edges. Similarly we obtain the graph $G_T$ in $\widehat{T}$. Note that each fat vertex of $G_Q$ ($G_T$) intersects each fat vertex of $G_T$ (respectively $G_Q$) exactly $\triangle$ times. Number the components of $\partial Q$ $1, 2, \ldots, q$ successively along $\partial M$, and similarly number the components of $\partial T$ $1, 2, \ldots, t$. In this way each end of each edge of $G_Q$ ($G_T$) has a label, namely the number of the corresponding fat vertex of $G_T$ (respectively $G_Q$). When traveling around a fat vertex of $G_Q$, the labels appear as $1, \ldots, t$ repeated $\triangle$ times and similarly for a fat vertex of $G_T$. Assigning orientations to $\widehat{Q}$ and $\widehat{T}$ allows us to refer to $+$ and $-$ vertices of $G_Q$ ($G_T$), according to the sign of the corresponding intersection with the core of $V_\pi$ (respectively $V_\gamma$). If two vertices have the same sign they are called *parallel,* otherwise *antiparallel.* The orientability of $Q$, $T$ and $M$ give us the following *parity rule* : an edge connects parallel vertices on one graph if and only if it connects antiparallel vertices on the other. As is done above, define the labelled graphs $G_Q^S$ in $\widehat{Q}$ and $G_S$ in $\widehat{S}$ coming from the intersection of $Q$ and $S$.

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## 2. Preliminaries

In this section we define the concept of a Scharlemann cycle and a $(k)\,x$-web, and introduce some combinatorial techniques developed in other articles.

Let $G$ be the graph $G_Q$ or $G_T$, and $x$ a label of $G$. An *$x$-edge* in $G$ is an edge with label $x$ at one endpoint. An *$x$-cycle* is a cycle of $x$-edges of $G$ such that all the vertices are parallel and all the edges can be oriented so that the tail of each edge has label $x$. A *Scharlemann cycle* is an $x$-cycle that bounds a disk face of $G$. A Scharlemann cycle with exactly two edges is called an *S-cycle.* Note that by construction, $G$ has no face with only one edge.

A $(k)\,$*$x$-web* is a connected subgraph $\Sigma$ of $G_Q$ such that all the edges of $\Sigma$ are $x$-edges, all the vertices of $\Sigma$ are parallel, and all but possibly $k$ edges with label $x$ at the vertices of $\Sigma$ connect them to the vertices of $\Sigma$. Such exceptional edges are called ghost edges, and their endpoints in $\Sigma$ are called ghost vertices. A *great* $(k)\,$*$x$-web* satisfies the additional condition that there is a component $U$ of $\widehat{Q} - \Sigma$ such that all the vertices of $G$ in $\widehat{Q} - U$ have the same sign.

We hereafter assume that $q \geq 3$ because of Lemma 2.3 of [BZ3]. We have the following lemma from the argument of Proposition 1.3 of [GLi].

**Lemma 2.1.** *$G_T$ (hence $G_S$) cannot have $q$ mutually parallel edges.* □

If we assume that $t \geq 4$, then we have the following two lemmas; Lemma 2.2 is Lemmas 2.6 and 2.7 of [Oh] and Lemma 2.3 can be obtained from Lemmas 2.1 - 2.4 of [Wu1].

**Lemma 2.2.** *(1) $G_Q$ has at most two S-cycles on disjoint label pairs.*

*(2) $G_Q$ has at most $t/2+2$ mutually parallel edges connecting parallel vertices. Furthermore, if $t \equiv 2 \ (mod\, 4)$, then $G_Q$ cannot have $t/2+2$ mutually parallel edges connecting parallel vertices.* □

**Lemma 2.3.** *(1) $G_T$ cannot contain two S-cycles on distinct label pairs.*

*(2) $G_T$ has at most $q/2+1$ mutually parallel edges connecting parallel vertices. Furthermore, if there are such $q/2+1$ edges, then first two or last two of these edges form an S-cycle in $G_T$.* □

## 3. Proof of Theorem 1.1 (the case $t \geq 6$)

In this section we prove Theorem 1.1 when $t \geq 6$. As the argument in the case $t = 2$ or 4 is quite different, we handle it separately in Section 4.

We assume familiarity with the terminology of [GLu1, Section 2.1] and the more generalized terminology discussed in [GLu2]. Proposition 3.1 of [Oh], which is the analog of Proposition 3.1 of [GLu2], is still true in our case, indeed it works for any $\triangle \geq 2$. Hence we get the following proposition;

**Proposition 3.1.** *Either $G_Q$ contains a great $(k)$ $x$-web, or for all $\{1, \dots, q\}$-types, $\tau$, there are at least $kt$ faces of $G_T(L)$ representing $\tau$.*

Let $k$ be the smallest number greater than $\triangle/2$. Now Theorem 1.1 is broken into two cases, which will be carried out in the rest of this section. For a graph $\Gamma$, the *reduced graph* $\overline{\Gamma}$ of $\Gamma$ is defined to be the graph obtained from $\Gamma$ by amalgamating each family of mutually parallel edges of $\Gamma$ to a single edge. Note that every family of mutually parallel edges of $G_Q$ has an even number of edges since $\widehat{T}$ is the boundary of a regular neighborhood of $\widehat{S}$.

**Theorem 3.2.** *$G_Q$ cannot contain a great $(k)$ $x$-web.*

*Proof.* Assume for contradiction that there is a great $(k)$ $x$-web $\Sigma$ in $G_Q$. We may assume that $\Sigma$ has no separating edge, for if $e$ is an edge of $\Sigma$ such that $\Sigma - e$ has two components then one of them is also a great $(k)$ $x$-web. Let $U$ be a component of $\widehat{Q} - \Sigma$ such that all the vertices of $G_Q$ in $\widehat{Q} - U$ have the same sign. Let $\Gamma_Q$ be the subgraph $G_Q - U$ of $G_Q$. Suppose that $U$ is an $n$-gon, i.e. $\overline{\Gamma}_Q$ has $n$ boundary vertices. Let $v, e$ and $f$ be the number of vertices, edges and faces of $\overline{\Gamma}_Q$ in $\widehat{Q}$. Since each face of $\overline{\Gamma}_Q$ is a disk with at least 3 sides, we have $2e \geq 3(f-1)+n$. Thus

$$2 = \chi(\widehat{Q}) = v - e + f \leq v - \frac{e}{3} + 1 - \frac{n}{3}.$$

Therefore $2e \leq 6v - 2n - 6$. We distinguish two cases

**(1)** Some interior vertex $y$ of $\overline{\Gamma}_Q$ has valency at most $\triangle+1$.

There are $\triangle t$ edges in $G_Q$ which are incident to $y$ and connect $y$ to parallel vertices. By Lemma 2.2(2), $\triangle t \leq (\triangle+1)(\frac{t}{2}+2)$, i.e. $t < 7$ and $t \neq 6$, a contradiction.

**(2)** All interior vertices of $\overline{\Gamma}_Q$ have valency at least $\triangle+2$.

Suppose some boundary vertex $y$ of $\overline{\Gamma}_Q$ which is not a ghost vertex has valency at most $\triangle - 1$. Then at least $(\triangle-1)t+2$ edges (must be even because $T$ is a double covering of $S$) incident to $y$ in $G_Q$ connect $y$ to parallel vertices. Again $(\triangle-1)t+2 \leq (\triangle-1)(\frac{t}{2}+2)$ by Lemma 2.2(2). Hence $t < 4$. Therefore all boundary vertices of $\overline{\Gamma}_Q$ which are not ghost vertices have valency at least $\triangle$. Similarly each boundary vertex of $\overline{\Gamma}_Q$ which is a ghost vertex with $i$ ghost edges has valency at least $\triangle - i$. Since $\overline{\Gamma}_Q$ has at most $k$ ghost edges, we get the following inequality (here $v-n$ is the number of interior vertices):

$$(\triangle + 2)(v - n) + \triangle n - k \leq 2e. \tag{1}$$

From two previous inequalities, we finally have $(\triangle-4)v + 6 \leq k$, which contradicts that $k \leq \triangle/2 + 1$.

It was pointed out by the referee that the great $(k)\, x$-web $\Sigma$ might have some separating vertices. Then after cutting along separating vertices, we can choose a subgraph of $\Sigma$ which is also a great $(k/2{+}\triangle)\, x$-web containing no separating vertex, because there are at least two subgraphs of $\Sigma$ containing only one separating vertex of $\Sigma$. Then we are able to use the following inequality instead of (1):

$$(\triangle + 2)(v - n) + \triangle n - (\frac{k}{2} + \triangle) \leq 2e.$$

Thus $(\triangle-4)v + 6 - \triangle \leq k/2$, which again contradicts that $k \leq \triangle/2 + 1$. □

**Lemma 3.3.** *$G_T$ contains a Scharlemann cycle.*

*Proof.* We separate two cases.

**(1)** Suppose that there is a vertex $x$ of $G_Q$ such that for all labels $y$ at least 2 edges incident to $x$ at $y$ connect $x$ to antiparallel vertices.

Then the conclusion follows from case (1) of the proof of Lemma 5.1 of [Oh].

**(2)** Here we assume the negation of (1). That is, for each vertex $x$ of $G_Q$ there is a label $y(x)$ such that at least $\triangle - 1$ edges incident to $x$ at $y(x)$ connect $x$ to parallel vertices.

Let $\Lambda_Q$ be an innermost connected component of the subgraph which is obtained from $G_Q$ by deleting all edges connecting antiparallel vertices and all separating families of mutually parallel edges. Without loss of generality, we assume that in $\Lambda_Q$, every vertex, except possibly one, called a ghost vertex, which is an endpoint of a separating family of mutually parallel edges, has valency at least $(\triangle-2)t{+}2$. Then $\Lambda_Q$ here is very similar to the graph $\Gamma_Q$ described in the proof of Theorem 3.2. The same argument as the proof of Theorem 3.2 shows that there is no interior vertex of $\Lambda_Q$ of valency at most $\triangle{+}1$ in $\overline{\Lambda}_Q$, and so there is a boundary vertex, $y$, of $\Lambda_Q$ which is not a ghost vertex and has valency at most $\triangle{-}1$ in $\overline{\Lambda}_Q$ (here $\overline{\Lambda}_Q$ has at most one ghost vertex). By using Lemma 2.2(2) twice, $(\triangle-2)t{+}2 \leq (\triangle-1)(\frac{t}{2}{+}2)$, i.e. $t \leq 4(\triangle-2)/(\triangle-3)$ and $t \neq 6$. Thus $t = 8$ when $\triangle = 4$. That is, $y$ has valency at least 18 in $\Lambda_Q$ and at most 3 in $\overline{\Lambda}_Q$. By Lemma 2.2(2) again, these 18 edges consist of 3 families each of which has exactly 6 mutually parallel edges and so contains 2 S-cycles on disjoint label pairs. And adjacent 2 families contain 4 S-cycles on disjoint label pairs, which contradicts Lemma 2.2(1). □

Now we are ready to prove Theorem 1.1 when $t \geq 6$.

*Proof of Theorem 1.1.* Using Proposition 3.1 and Theorem 3.2, we are able to conclude that for all $\{1, \ldots, q\}$-types, $\tau$, there are at least $kt$ faces, i.e. more than $(\triangle/2)t$ faces, of $G_T$ representing $\tau$. Recall that $\widehat{Q}$ is essential in $M(\pi)$ and $G_T$ contains a Scharlemann cycle. Now we can apply the same arguments in the context of Sections 4, 5 and 6 in [GLu2] where they use the facts that $\triangle = 2$ and for every face type $\tau$ there are more than $p$ (the number of boundary components of the other planar surface $P$) faces representing $\tau$. In our case we have more than $(\triangle/2)t$ faces representing $\tau$. In [GLu2] it is shown that there are a vertex $v$ of $G_Q$

and certain face types such that each face of $G_T$ representing such a type contains an edge incident to $v$, and this gives rise to too many edges in $G_Q$ incident to $v$. The theorem follows. □

## 4. Exceptional cases $t = 2$ or 4

This section will be devoted to proving Theorem 1.1 in the special cases $t = 2$ and $t = 4$. We will make frequent use of the combinatorics of Lemma 2.1 and Lemma 2.3 throughout this section to complete the proof of the main theorem.

**Lemma 4.1.** *If $t = 2$ then $\triangle \leq 3$.*

*Proof.* Since $G_S$ is a graph with one vertex, say 1, on a Klein bottle, the number of families of mutually parallel edges is at most 3, i.e. this vertex has valency at most 6 in $\overline{G}_S$. Hence $\triangle < 6$ by Lemma 2.1. Thus there are two cases.

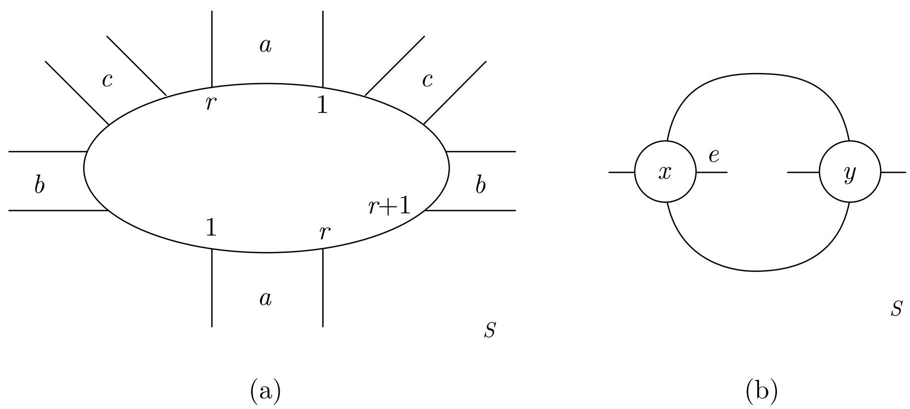


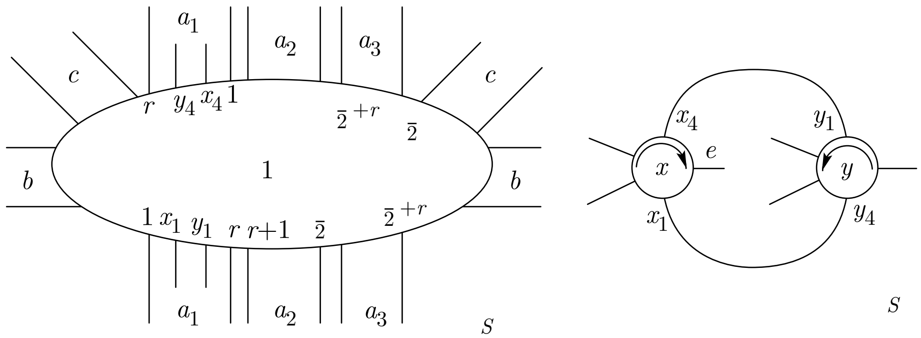

Figure 1

**(1)** When $\triangle = 4$; $G_S$ looks like Figure 1(a), where $1 < r < q$.

Here only the edges of family $a$ correspond to an orientation-preserving curve on $\widehat{S}$ and hence these edges connect antiparallel vertices of $G_Q^S$. Each pair of edges labelled $x$ and $r+1-x$ (where $x = 1, 2, \dots, r$) of family $a$ forms a cycle meeting vertices $x$ and $r+1-x$ in $G_Q^S$. Choose an innermost such cycle with vertices, say $x$ and $y = r+1-x$, which bounds a disk $D$ such that there is no edge connecting antiparallel vertices in the interior of $D$, as in Figure 1(b). Since in $G_S$ these edges are incident to 1 at non-adjacent occurrences of the label $x$, in $G_Q^S$ they are non-adjacent at vertex $x$. Let $e$ be the edge incident to $x$ that lies in $D$. Let $\Gamma$ be the subgraph of $G_Q^S$ which consists of all vertices parallel to $x$ in the interior of $D$ and their connecting edges except $e$. Then every vertex of $\Gamma$, except the one which $e$ is incident to, has even valency, namely 4. This contradicts a property of a graph.

**(2)** When $\triangle = 5$; We see $G_S$ as shown in Figure 2(a), where $1 < r < \frac{q}{2}$.

Again only the edges of families $a_1, a_2$ and $a_3$ correspond to the edges connecting antiparallel vertices of $G_Q^S$. As in case (1), each pair of edges of families $a_1$ and $a_3$ is a cycle meeting two related vertices in $G_Q^S$. Then there is an innermost cycle with vertices, say $x$ and $y$, which bounds a disk $D$ such that $D$ does not contain the vertices $1, \dots, r, \frac{q}{2}+1, \dots, \frac{q}{2}+r$. Now consider the edges incident to $x$ and $y$ in $D$. Since both boundary edges of $D$ are incident to 1 at labels $x$ (and $y$) and of the



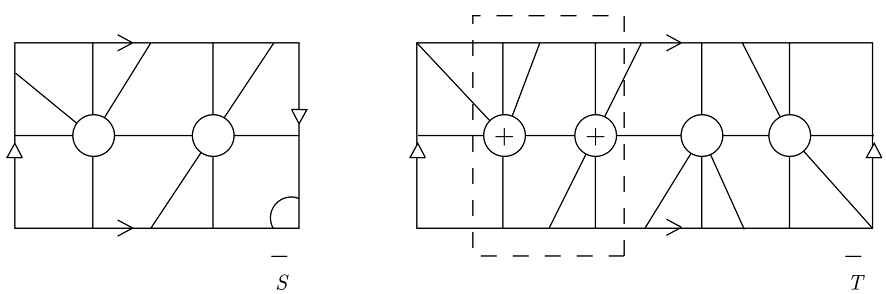

(a) (b)

FIGURE 2

same family in $G_S$, and $x$ and $y$ are antiparallel, if $n$ edges are incident to $x$ in the interior of $D$ then $3-n$ edges are incident to $y$ in the interior of $D$ where $n = 0, 1, 2$ or 3 (as shown in Figure 2(b) when $n = 1$). Thus either $x$ or $y$ has odd valency in the interior of $D$. Furthermore for each vertex $v$ in the interior of $D$, only one edge incident to $v$ connects $v$ to an antiparallel vertex (this edge is in family $a_2$ in $G_S$); i.e. exactly the other four edges connect $v$ to parallel vertices. As is done in case (1), we can define a subgraph of $G_Q^S$, only one of whose vertices has odd valency, a contradiction. □

**Lemma 4.2.** *If $t = 4$ then $\triangle \leq 3$.*

*Proof.* By an Euler characteristic count, $\overline{G}_S$ has at most 6 edges with 2 vertices as shown in Figure 3(a). Hence its double cover $\overline{G}_T$ is the graph illustrated in Figure 3(b) (or the same graph but with the vertices on the middle line labelled $+, -, -, +$) because the first and the third vertices come from the same vertex of $\overline{G}_S$, i.e. they are antiparallel.




(a) (b)

FIGURE 3

We may assume that $G_T$ contains a part of the graph as shown in Figure 4. Lemmas 2.1 and 2.3(2) imply the following inequality:

$$2(q-1) + 4(\frac{q}{2}+1) = 4q + 2 \geq \triangle q.$$

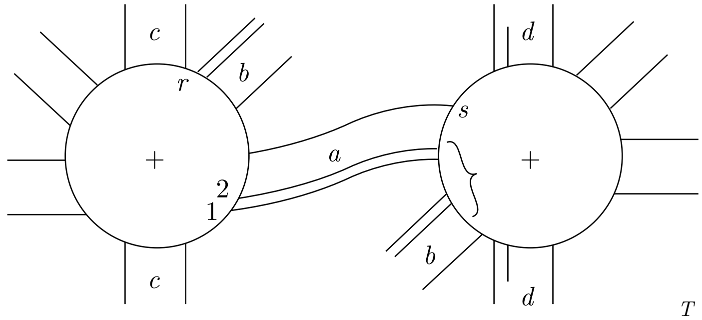



FIGURE 4

Hence $\triangle \leq 4$. Assume that $\triangle = 4$. In Figure 4 $r$ must be odd by the parity rule.

Suppose that families $a$ and $b$ have a total of $q+2$ edges, i.e. each family has $q/2{+}1$ edges. Lemma 2.3(2) implies that each family contains an S-cycle on the side. Thus family $a$ contains $\{1,2\}$ or $\{\frac{q}{2}, \frac{q}{2}{+}1\}$ S-cycle and family $b$ contains $\{\frac{q}{2}{+}2, \frac{q}{2}{+}3\}$ or $\{1,2\}$ S-cycle. If both families contain $\{1,2\}$ S-cycles, then we have labels 1,2,1 and 2 successively on mark $A$ indicated in Figure 4. It is impossible. Thus $G_T$ contains two S-cycles on distinct label pairs in each family, contradicting Lemma 2.3(1).

Therefore families $a$ and $b$ have $q$ edges, i.e. $r = 1$. Then each family $c$ and $d$ has $q/2{+}1$ edges. By Lemma 2.3(1) and (2) again, both families contain $\{\frac{q}{2}, \frac{q}{2}{+}1\}$ S-cycles on the side. If 2 edges on the left side of family $d$ form $\{\frac{q}{2}, \frac{q}{2}{+}1\}$ S-cycle, then families $a$ and $b$ must have $q-2$ edges. Thus 2 edges on the other side form $\{\frac{q}{2}, \frac{q}{2}{+}1\}$ S-cycle, i.e. $s = 1$. This implies that the middle 2 edges of family $a$ form an S-cycle which do not have labels $\{\frac{q}{2}, \frac{q}{2}{+}1\}$, a contradiction. □

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