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ON THE GENERALIZED STEPANOV THEOREM

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Abstract. The generalized Stepanov theorem is derived from the Alexandrov theorem on the twice differentiability of convex functions. A parabolic version of the generalized Stepanov theorem is also proved.

In the first part of this note we provide a new proof of the generalized Stepanov theorem. This classical result is due to Calderón and Zygmund [3] (see also Oliver [12]), but is usually associated with Stepanov's name because it generalizes the Stepanov theorem (see e.g. [6]). The result we prove below (Theorem 1) constitutes a special case of a general theorem in [3]. Recently this particular version found applications in proving the twice differentiability a.e. of viscosity solutions of elliptic partial differential equations, see [11], [14] and [2]. The only complete proof of the generalized Stepanov theorem the authors are aware of is contained in [3], where Whitney's extension theorem is used. In this note the generalized Stepanov theorem will be proved by means of the Aleksandrov theorem on the twice differentiability of convex functions [1]; see also [5], [9], [10] or the appendix in [4] for more modern treatments.

In the second part of this note we show how to modify our proof to obtain a parabolic version of the generalized Stepanov theorem (Theorem 3). A result of this type is needed to prove the differentiability a.e. twice in x and once in t of viscosity solutions of parabolic equations. To the best of the authors' knowledge this result is original, though some relevant arguments appear in [16].

$|\cdot|$ and $\langle\cdot,\cdot\rangle$ will stand for the Euclidean norm and inner-product in $\mathbb{R}^n$, and $B_r(x)$ will denote the open ball in $\mathbb{R}^n$ of radius r centered at x. Given a measurable set A in an Euclidean space, $|A|$ will denote its Lebesgue measure.

Recall some notation from [3] (see also [17]). Let $u\colon \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, be bounded and $x \in \Omega$. We say that $u \in T^2_\infty(x)$ ($u \in t^2_\infty(x)$, resp.) if there exists an affine function P_x (a quadratic function Q_x) such that

$$\sup_{y\in B_r(x)\cap\Omega} |u(y) - P_x(y)| \le O(r^2)$$

$$\Big(\sup_{y\in B_r(x)\cap\Omega} |u(y) - Q_x(y)| \le o(r^2) \text{ as } r \downarrow 0, \text{ resp.}\Big).$$

Observe that $u \in t^2_\infty(x)$ if and only if u possesses a second order Taylor series expansion at x whose remainder behaves like $o(r^2)$. If this is the case we will say

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that u is twice differentiable at x. On the other hand, $u \in T^2_\infty(x)$ is equivalent to saying that u can be enclosed between two paraboloids meeting at x. In particular, if $\Omega \subset \mathbb{R}^n$ is open and $u \in T^2_\infty(x)$, then u is differentiable at x and $P_x(y) = u(x) + \langle Du(x), y - x\rangle$.

Theorem 1 (Calderón-Zygmund [3]). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and suppose that $u\colon \overline{\Omega} \to \mathbb{R}$ is bounded. If $u \in T^2_\infty(x)$ for a.e. $x \in \Omega$ then $u \in t^2_\infty(x)$ for a.e. $x \in \Omega$.*

Proof. By the assumption for a.e. $x \in \Omega$ there are $p_x \in \mathbb{R}^n$ and $M_x \geq 0$ such that

$$|u(y) - u(x) - \langle p_x, y - x\rangle| \leq M_x|y - x|^2 \text{ for all } y \in \Omega;$$

note that p_x is uniquely determined and we can assume that M_x is the smallest with this property. It follows that M_x is well defined and finite a.e., moreover, the mapping $x \mapsto M_x$ is measurable. For $M = 1, 2, \dots$ put

$$\Omega_M = \{x \in \Omega\colon\ M_x \leq M\};$$

then every Ω_M is measurable and $\cup_{M=1}^\infty \Omega_M$ is of full measure in Ω. Therefore it is enough to show that for every M

$$u \in t^2_\infty(x) \text{ for a.e. } x \in \Omega_M.$$

From now on let M be fixed. Note that for every $x \in \Omega_M$

$$u(y) - \langle p_x, y\rangle \leq u(x) - \langle p_x, x\rangle + M|y - x|^2 \text{ for all } y \in \Omega,$$

or

$$\tilde{u}(y) \leq \tilde{u}(x) + \langle q_x, y - x\rangle \text{ for all } y \in \Omega,$$

where $\tilde{u} = u - M|\cdot|^2$ and $q_x = p_x - 2Mx$. Denoting by $\hat{u}$ the upper concave envelope of $\tilde{u}$ on $\overline{\Omega}$, that is,

$$\hat{u}(x) = \inf\{p(x)\colon\ p \text{ is affine and } p \geq \tilde{u} \text{ on } \Omega\},$$

we obtain that $\tilde{u} = \hat{u}$ on Ω_M, or using the notation in [7], $\Omega_M \subset \Gamma$, where $\Gamma = \Gamma^+_{\tilde{u}} = \{\tilde{u} = \hat{u}\}$ is the upper contact set of $\tilde{u}$ on Ω. From the Aleksandrov theorem $\hat{u}$ is twice differentiable a.e., that is, there exists $F \subset \Omega$ of full measure such that $\hat{u} \in t^2_\infty(x)$ for every $x \in F$. Note that $D\tilde{u} = D\hat{u}$ on $\Omega_M \cap F$, which yields

$$|\tilde{u}(y) - \hat{u}(y)| \leq O(|y - x|^2) \text{ for every } x \in \Omega_M \cap F. \tag{1}$$

We will show that (1) implies that

$$|\tilde{u}(y) - \hat{u}(y)| \leq o(|y - x|^2) \text{ as } y \to x \text{ for a.e. } x \in \Omega_M. \tag{2}$$

Put $v = \hat{u} - \tilde{u}$ and for $N = 1, 2, \dots$ let

$$\Omega_{M,N} = \{x \in \Omega_M\colon\ |v(y)| \leq N|y - x|^2 \text{ for all } y \in \Omega\}.$$

To prove (2) it is enough to show that for every N

$$|v(y)| \leq o(|y - x|^2) \text{ as } y \to x \text{ for a.e. } x \in \Omega_{M,N}. \tag{3}$$

We will show that this holds for any point of density of $\Omega_{M,N}$. So let $x_0 \in \Omega_{M,N}$ be a point of density and let $1 > \epsilon > 0$. Then for all sufficiently small r, say $r < \delta$, where $B_\delta(x_0) \subset \Omega$,

$$\frac{|B_r(x_0) \setminus \Omega_{M,N}|}{|B_r(x_0)|} < \epsilon^n. \tag{4}$$

Suppose that $y \in B_{\delta(1-\epsilon)}(x_0)$ and let $r = |y - x_0|/(1-\epsilon) < \delta$. It follows that $B_{\epsilon r}(y) \subset B_r(x_0)$ and from (4) $B_{\epsilon r}(y) \cap \Omega_{M,N} \neq \emptyset$, say $x_1 \in B_{\epsilon r}(y) \cap \Omega_{M,N}$. Then

$$|v(y)| \leq N|y - x_1|^2 < N\epsilon^2 r^2 = \epsilon \frac{N\epsilon}{(1-\epsilon)^2}|y - x_0|^2,$$

and (3), and consequently (2), follows.

To finish the proof of the theorem it is enough to remark that if $\hat{u} \in t^2_\infty(x)$ and $|\tilde{u}(y) - \hat{u}(y)| \leq o(|y - x|^2)$ as $y \to x$, then the Taylor expansion for $\hat{u}$ works for $\tilde{u}$, and thus $\tilde{u} \in t^2_\infty(x)$, and consequently $u \in t^2_\infty(x)$.

Remark 2. Under the assumptions of Theorem 1 we proved that for a.e. $x_0 \in \Omega$ there exist $p(x_0) \in \mathbb{R}^n$ and a symmetric $n \times n$ matrix $A(x_0)$ such that

$$u(y) = u(x_0) + \langle p(x_0), y - x_0\rangle + \frac{1}{2}\langle A(x_0)(y - x_0), y - x_0\rangle + o(|y - x_0|^2) \text{ as } y \to x_0. \tag{5}$$

Clearly u is then differentiable at every such point x_0 with $Du(x_0) = p(x_0)$. A natural question arises whether $A(x)$ is the derivative of $Du(x)$. Denoting $F_1 = \{x \in \Omega\colon\ Du(x) \text{ exists}\}$ and $F_2 = \{x \in \Omega\colon\ u \in t^2_\infty(x)\} \subset F_1$, we would like to find out whether for a.e. $x_0 \in F_2$

$$Du(y) = Du(x_0) + \langle A(x_0), y - x_0\rangle + o(|y - x_0|) \text{ as } F_1 \ni y \to x_0. \tag{6}$$

By the C^2 version of the Aleksandrov theorem (see e.g. [10] or [4]) convex functions have this property, and therefore the proof of Theorem 1 shows that (6) holds in the approximate sense for a.e. $x_0 \in \Omega$. That is, there exists $F_3 \subset F_2 \subset \Omega$ of full measure such that for every $x_0 \in F_3$ and $\epsilon > 0$ the set

$$\{y \in F_1\colon\ |Du(y) - Du(x_0) - \langle A(x_0), y - x_0\rangle| < \epsilon|y - x_0|\}$$

has density 1 at x_0. In general, to claim (6) stronger assumptions on u are required; see e.g. Theorem 3.5.7 in [17].

We would like to emphasize that this paper is concerned with pointwise derivatives and in general in our setting one doesn't expect the existence of generalized derivatives. However, if $u \in t^2_\infty(x)$ for all $x \in \Omega$ with p and A as in (5) belonging to $L^p(\Omega)$, $1 \leq p < \infty$, then $u \in W^{2,p}(\Omega)$; see Theorem 3.9.5 in [17].

A modification of our approach leads to a proof of a parabolic version of the generalized Stepanov theorem. We are concerned with real-valued functions on $\mathbb{R}^{n+1}$. We will write points in $\mathbb{R}^{n+1}$ as (x, t), where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Given $(y, s), (x, t) \in \mathbb{R}^{n+1}$, define their parabolic distance d according to

$$d((y, s), (x, t)) = \sqrt{|x - y|^2 + |t - s|}$$

and their one-sided parabolic distance d_∞ by

$$d_\infty((y, s), (x, t)) = \begin{cases} d((y, s), (x, t)) \text{ if } s \leq t, \\ +\infty \quad \text{otherwise.} \end{cases}$$

Let $u\colon Q \to \mathbb{R}$, $Q \subset \mathbb{R}^{n+1}$, be bounded and $(x_0, t_0) \in Q$. We say that $u \in T^{2,1}_\infty(x_0, t_0)$ ($u \in t^{2,1}_\infty(x_0, t_0)$, resp.) if there exists an affine function P_{x_0,t_0} of variable

x (a quadratic in x and affine in t function Q_{x_0,t_0}) such that

$$|u(y,s) - P_{x_0,t_0}(y,s)| \le O\left(d_\infty^2((y,s),(x_0,t_0))\right) \text{ for } (y,s) \in Q$$
$$\left(|u(y,s) - Q_{x_0,t_0}(y,s)| \le o\left(d^2((y,s),(x_0,t_0))\right) \text{ as } Q \ni (y,s) \to (x_0,t_0), \text{ resp.}\right).$$

Note that in the definition of $t_\infty^{2,1}(x_0,t_0)$ an appropriate inequality holds for s both larger and smaller than t_0, while in the definition of $T_\infty^{2,1}(x_0,t_0)$ only the values $s \le t_0$ matter. $u \in t_\infty^{2,1}(x_0,t_0)$ roughly corresponds to the differentiability of u at (x_0,t_0), twice in x, once in t.

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $T > 0$ and suppose that $u\colon \overline{Q} \to \mathbb{R}$ is bounded, where $Q = \Omega \times (0,T)$. If $u \in T_\infty^{2,1}(x,t)$ for a.e. $(x,t) \in Q$ then $u \in t_\infty^{2,1}(x,t)$ for a.e. $(x,t) \in Q$.*

For $(x,t) \in \mathbb{R}^{n+1}$ and $r > 0$ put

$$P_r(x,t) = \{(y,s) \in \mathbb{R}^{n+1}\colon\ d((x,t),(y,s)) < r\},$$
$$Q_r(x,t) = \{(y,s) \in \mathbb{R}^{n+1}\colon\ d_\infty((x,t),(y,s)) < r\}.$$

The following proposition will be used in the proof of Theorem 3. It follows in a standard way from a version of the covering theorem of Vitali, which employs Q_r's instead of the Euclidean balls; see e.g. Remark I.3.1 in [8].

Proposition 4. *Let $A \subset \mathbb{R}^{n+1}$ be measurable. Then*

$$\lim_{r\downarrow 0} \frac{|Q_r(x,t) \setminus A|}{|Q_r(x,t)|} = 0 \textit{ for a.e. } (x,t) \in A.$$

Proof of Theorem 3. The proof of Theorem 3 parallels that of Theorem 1. By assumption for a.e. $(x,t) \in Q$ there are $p_{x,t} \in \mathbb{R}^n$ and $M_{x,t} \ge 0$ such that

$$|u(y,s) - u(x,t) - \langle p_{x,t}, y-x\rangle| \le M_{x,t}(|y-x|^2 + t - s) \text{ for all } y \in \Omega,\ s \in [0,t].$$

As before the mapping $(x,t) \mapsto M_{x,t}$ is measurable and putting for $M = 1,2,\dots$

$$Q_M = \{(x,t) \in Q\colon\ M_{x,t} \le M\}$$

gives that $\cup_{M=1}^\infty Q_M$ is of full measure in Q. Fix M and define $\tilde{u}(x,t) = u(x,t) - M(|x|^2 - t)$; it follows that for every $(x,t) \in Q_M$

$$\tilde{u}(y,s) \le \tilde{u}(x,t) + \langle q, y-x\rangle \text{ for all } y \in \Omega \text{ and } s \in [0,t], \tag{7}$$

with an appropriate $q \in \mathbb{R}^n$. In the parabolic context the upper concave envelope $\hat{u}$ of given function $\tilde{u}\colon Q \to \mathbb{R}$ is defined by (see [13] or [15])

$$\hat{u} = \inf\{v\colon\ v \ge \tilde{u} \text{ on } Q,\ v \text{ concave in } x \text{ and increasing in } t\},$$

and thus (7) shows that $\tilde{u} = \hat{u}$ on Q_M. A parabolic version of the Aleksandrov theorem (see Theorem 1, Appendix 2 in [9]) guarantees that there exists $F \subset Q$ of full measure such that $\hat{u} \in t_\infty^{2,1}(x,t)$ for every $(x,t) \in F$. It follows that

$$|\tilde{u}(y,s) - \hat{u}(y,s)| \le O\left(d_\infty^2((y,s),(x,t))\right) \text{ for every } (x,t) \in Q_M \cap F. \tag{8}$$

We will show that (8) implies that

$$|\tilde{u}(y,s) - \hat{u}(y,s)| \le o\left(d^2((y,s),(x,t))\right) \text{ for a.e. } (x,t) \in Q_M, \tag{9}$$

which will give the result as in the proof of Theorem 1. Put $v = \hat{u} - \tilde{u}$ and for $N = 1, 2, \dots$ let

$$Q_{M,N} = \{(x,t) \in Q_M : \ |v(y,s)| \le N d_\infty^2((y,s),(x,t)) \text{ for every } (y,s) \in Q\},$$

and suppose that $(x_0, t_0) \in Q_{M,N}$ is such that

$$\lim_{r \downarrow 0} \frac{|Q_r(x_0,t_0) \setminus Q_{M,N}|}{|Q_r(x_0,t_0)|} = 0;$$

by Proposition 4 a.e. $(x_0, t_0) \in Q_{M,N}$ will do. Let $0 < \epsilon < 1$. For all sufficiently small r, say $r < \delta$,

$$\frac{|Q_r(x_0,t_0) \setminus Q_{M,N}|}{|Q_r(x_0,t_0)|} < \epsilon^{n+2}. \tag{10}$$

Suppose that $(y,s) \in P_{\delta(1-\epsilon)}(x_0,t_0)$ and let $r = d((y,s),(x_0,t_0))/(1-\epsilon) < \delta$. It follows that $P_{\epsilon r}(y,s) \subset P_r(x_0,t_0)$ and from (10) $Q_{\epsilon r}(y,s) \cap \ Q_{M,N} \neq \emptyset$, say $(x_1,t_1) \in Q_{\epsilon r}(y,s) \cap Q_{M,N}$. In particular $t_1 \ge s$ and therefore

$$|v(y,s)| \le N(|y - x_1|^2 + t_1 - s) < N\epsilon^2 r^2 = \epsilon \frac{N\epsilon}{(1-\epsilon)^2} d^2((y,s),(x_0,t_0)).$$

Thus (9) is proved for a.e. $(x,t) \in Q_{M,N}$ for every N, and consequently for a.e. $(x,t) \in Q_M$.

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