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ESTIMATES FOR THE WAVE OPERATOR ON THE TORUS Π^n

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Abstract. We prove $L^{p'} \to L^p$ bounds for the wave operator on the torus for large time. The new feature is the distribution of the singularities of the wave kernel, which can be understood by making use of Hardy-Littlewood method for exponential sums.

1. Introduction

We consider the standard homogenuous wave equation on the torus Π^n :

$$\begin{aligned} \Box\Phi(T,x) &= 0, \qquad x \in \Pi^n, \\ \Phi(0,x) &= 0, \qquad \partial_T\Phi(0,x) = f(x), \end{aligned} \tag{1}$$

and prove analogues of the inequalities obtained by Strichartz [4], for large values of the time T.

Namely it is proved that

Theorem 1. *Let $n \geq 5$, $2 \leq p < \infty$, $f \in L^p(\Pi^n)$ such that $\hat{f}(0) = 0$.*

If we introduce the operators $\Lambda = (-\triangle)^{\frac{1}{2}}$, $\Phi(T,) = W_T f = K_T * f$, then we have

$$\| \Lambda^{1+(n+1)(\frac{1}{p}-\frac{1}{2})} W_T f \|_{L^p(\Pi^n)} \leq c_n T^{(n-3)(\frac{1}{2}-\frac{1}{p})} (\log T)^{\delta_p} \| f \|_{L^{p'}(\Pi^n)} . \tag{2}$$

We remark that $\Phi(T,x) \equiv T$ if $f \equiv 1$; hence there is no loss of generality by assuming $\hat{f}(0) = 0$. In the proof we use a variant of the Littlewood-Paley decomposition for the torus which we describe below.

Let R_L, $L = 2^l$, $l \in Z$, be the usual multipliers used for R^n, i.e.: $\eta \in C_0^\infty(R)$, supp $\eta \subset (1/2, 2)$, and

$$\widehat{R_L f}(\xi) = \hat{f}(\xi)\,\eta(|\xi|/L), \qquad \sum_{l\in Z} \eta(|\xi|/L) = 1.$$

The same decomposition can be used for $f \in C(\Pi^n)$, and if $\hat{f}(0) = 0$ then obviously

$$f = \sum_{l=0}^{\infty} R_L f.$$

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For the square function $Sf(x)^2 = \sum_{l=0}^{\infty} |R_L f(x)|^2$ one has the analogues estimate for $1 \le p < \infty$:

$$c_p \parallel f \parallel_p \le \parallel Sf \parallel_p \le C_p \parallel f \parallel_p . \tag{3}$$

The proof of inequality (3) is just an imitation of that for the continuous case, if one observes that $\parallel R_L \parallel_{L^p(\Pi^n)\to L^p(\Pi^n)} \le \parallel R_L \parallel_{L^p(R^n)\to L^p(R^n)}$. This last inequality remains true for more general multipliers, as is shown by Stein [2, ch. 7]. Hence by using standard arguments (2) follows from the uniform estimates

$$\parallel \Lambda^{1+(n+1)(\frac{1}{p}-\frac{1}{2})} R_L W_T f \parallel_p \le T^{\frac{n-3}{2}(\frac{1}{2}-\frac{1}{p})} (\log T)^{\delta_p} \parallel f \parallel_{p'}, \tag{4}$$

where the constant $c_{n,p}$ is independent of L. Indeed, consider the identity operator between the mixed norm spaces :

$$I : \ L^p(l^2) \to l^2(L^p), \tag{i}$$

which is bounded for $p \ge 2$ by Minkowski's inequality; hence on taking the dual it follows that

$$I : \ l^2(L^{p'}) \to L^{p'}(l^2) \tag{ii}$$

is also bounded. We remark that the L^p norm of the square function Sf is the $L^p(l^2)$ mixed norm of the sequence $\{R_L f \, ; \, l = 0, 1, 2, \dots\}$; hence we have

$$\parallel f \parallel_p^2 \le c_p \parallel Sf \parallel_p^2 \le c_p \sum_L \parallel R_L \parallel_p^2 \quad \text{for} \, p \ge 2, \tag{I}$$

$$\parallel f \parallel_{p'}^2 \ge c_{p'} \parallel Sf \parallel_{p'}^2 \ge c_{p'} \sum_L \parallel R_L f \parallel_{p'}^2 . \tag{II}$$

Suppose M is a multiplier on Π^n for which the estimate $\parallel R_L M \parallel_{p'\to p} \le C_M$ holds uniformly in L . Then using almost orthogonality we have

$$\begin{aligned} \parallel Mf \parallel_p^2 &\le c_p \sum_L \parallel R_L M f \parallel_p^2 \\ &\le 3c_p \sum_L (\parallel R_L M R_{L/2} f \parallel_p^2 + \parallel R_L M R_L f \parallel_p^2 + \parallel R_L M R_{2L} f \parallel_p^2) \\ &\le 9c_p C_M^2 \sum_L \parallel R_L f \parallel_{p'}^2 \le 9c_p c_{p'} C_M^2 \parallel f \parallel_{p'}^2 . \end{aligned}$$

Observe that inequality (4) is trivial when $p = 2$, and the crucial point is the following:

Lemma 1. *Let $n \ge 5$, $L = 2^l$, $l = 0, 1, 2, \dots$. We have*

$$\parallel K_{T,L} \parallel_{L^\infty(\Pi^n)} \le c_{n,\gamma} T^{\frac{n-3}{2}} \log T, \tag{5}$$

where the constant $c_{n,\gamma}$ is independent of L .

Here for simplicity we denoted by $K_{T,L}$ the kernel of the operator

$$\Lambda^{-(n-1)/2+i\gamma} R_L W_T,$$

and we remark that the imaginary part of phase γ plays no role in any of our computations; it is needed in order to use analytic interpolation.

The proof of Lemma 1 is based on two different approaches. On the one hand we use the properties of the wave kernel for R^n and periodization, and on the other hand estimates are obtained by making use of the Fourier transform and methods for

exponential sums. In both cases the Hardy-Littlewood "circle" method, described in [1], will play an essential role.

2. Estimates in space

We start by showing a localization property of the kernel of the standard wave operator in R^n; $K^c_{T,L}$ is defined by

$$\hat{K}^c_{T,L}(\xi) = \hat{K}_{T,L}(\xi) \,=\, |\xi|^{-\frac{n-1}{2}+i\gamma}\eta(\xi/L)\frac{e^{iT|\xi|}}{|\xi|}, \tag{6}$$

where as usual we replaced the term $|\xi|^{-1}\,\sin(T|\xi|)$ by $|\xi|^{-1}\,e^{\pm iT|\xi|}$ in the Fourier transform of the wave kernel $K^c_T(x)$.

Proposition 1. *Let $TL \geq 1$. Then for every $N > 0$*

$$|K^c_{T,L}\,(x)| \,\leq\, c_{n,N,\gamma}\,T^{-\frac{n-1}{2}}\,(1 + ||x| - T|\cdot L)^{-N}. \tag{7}$$

Proof. First we deal with the case $|x| > T/2$. Integrating in polar coordinates, one has

$$K^c_{T,L}(x) \,=\, c_n \int e^{irT}\hat{\sigma}_1(r|x|)\,\eta(r/L)\,r^{\frac{n-3}{2}+i\gamma}\,dr,$$

where $\hat{\sigma}_1(\lambda)$ is the Fourier transform of the measure concentrated on the unit sphere. Using the method of stationary phase, one has

$$\hat{\sigma}_1(\lambda) \,=\, \lambda^{\frac{-(n-1)}{2}}e^{-i\lambda}\,R_-(\lambda) \,+\, \lambda^{\frac{-(n-1)}{2}}e^{i\lambda}\,R_+(\lambda), \tag{8}$$

where $|D^N_\lambda\,R_\pm(\lambda)| \,\leq\, c_N\lambda^{-N}\;\;\forall N \in \mathbf{N}$ if $\lambda \geq 1/2$.

Let us consider one of the terms of formula (8). After a change of variables we have

$$K^c_{T,L,-}(x) \,=\, |x|^{-\frac{n-1}{2}}\int e^{irL(T-|x|)}\eta(r)R_-(rL|x|)r^{-1+i\gamma}\,dr.$$

Since supp $\eta \subset (1/2, 2)$, using the decay property of the derivatives of $R_-(\lambda)$ it is easy to see that

$$(d/dr)^N(\eta(r)\,R_-(rL|x|)\,r^{-1+i\gamma}) \,\leq C_{n,N,\gamma}$$

and the proposition follows after integrating by parts N times. The other term can be handled the same way.

When $|x| \leq T/2$, on introducing $\xi = r\xi_1$, $|\xi_1| = 1$ and integrating in polar coordinates we have

$$K^c_{T,L}(x) \,=\, \int_{S^{n-1}} I(T,\xi_1)\,d\sigma(\xi_1),$$

where

$$I(T,\xi_1) \,=\, L^{\frac{n-1}{4}}\int e^{irL(T+x\xi_1)}\,\eta(r)\,r^{\frac{n-3}{2}+i\gamma}\,dr.$$

After integrating by parts $2N$ times we obtain the estimate

$$|I(T,\xi_1)| \,\leq\, C_N L^{\frac{n-1}{4}}\,|L(T+x\xi_1)|^{-2N} \,\leq\, c_{N,\gamma}\,(LT)^{-N}$$

using the fact that $|T + x\xi_1| \geq T/2$. This implies the same estimate for $K^c_{T,L}(x)$, and the proposition follows. □

Using the fact that

$$K_{T,L}(x) \,=\, \sum_{m\in Z^n} K^c_{T,L}(x-m)$$

and the localization shown above, we need upper bounds for the number of lattice points in a thin spherical shell centered at an arbitrary point. Since we have not seen these in the literature we present a proof by using a variant of the circle method, but will be brief and refer to the analysis in [1].

Proposition 2. *Let $\chi \in R^n$ be arbitrary but fixed. Then for $n \geq 5, T \geq 1, L \geq 1$ we have*

$$S_{T,L}(\chi) \,=\, |\{x \in Z^n\,; ||x-\chi| - T\,| \,\leq\, L^{-1}\,\}| \,\leq c_n\, T^{n-1} \max(T^{-1}, L^{-1}),$$

where the constant c_n is independent of the point χ.

Proof. It is enough to prove the inequality for $T = L$, and then it is easy to see that

$$S_{T,L}(\chi) \,\leq\, c\,|\{x;\, ||x-\chi|^2 - T^2| \leq 3\}| \,\leq\, c_n \sum_{x\in Z^n} \phi(|x-\chi|^2 - T^2)\, e^{-2\pi\epsilon|x-\chi|^2},$$

where ϕ is a strictly positive smooth function such that supp $\hat{\phi} \,\subset\, (-1,1)$, $\hat{\phi} \leq 1$ and $\epsilon = T^{-2}$. Hence, using the Fourier transform, one has

$$S_{T,L}(\chi) \,\leq\, c_n| \int_R \theta(t,\chi) e^{-2\pi i T^2 t}\, \hat{\phi}(t)\, dt\,| \,\leq 2c_n \int_0^1 |\theta(t,\chi)|\, dt.$$

Here $\theta(t,\chi) = \sum_x e^{2\pi i\, |x-\chi|^2 (t+i\epsilon)}$ and we used the fact that $|\,\theta(t,\chi)| = |\theta(-t,\chi)|$. It is enough to estimate the theta type function in one dimension, which can be done by Poisson summation as follows:

Let $t = p/q + \tau$, $x = rq + s$, $0 \leq s < q$ and $s_{\tau,\chi}(x) = e^{2\pi i\,(x-\chi)^2(\tau+i\epsilon)}$. Then one has

$$\begin{aligned}\theta(t,\chi) \,&=\, e^{2\pi i\,\chi^2 p/q} \sum_{s=0}^{q-1} e^{2\pi i s^2 p/q} \sum_{r\in Z} e^{-2\pi i\,\chi(rq+s)}\, s_{\tau,\chi}(rq+s)\\ &=\, e^{2\pi i\,\chi^2 p/q} \sum_{l\in Z} (\frac{1}{q}\sum_{s=0}^{q-1} e^{2\pi i\,(s^2 p/q\, - sl/q)}\,)\, \tilde{s}_{\tau,\chi}(l/q + 2\chi),\end{aligned}$$

where

$$\tilde{s}_{\tau,\chi}(\xi) \,=\, \int_R s_{\tau,\chi}(x) e^{-2\pi i x\xi}\, dx \,=\, e^{2\pi i\chi\xi}(\tau+i\epsilon)^{-1/2} e^{-\pi/2\frac{|\xi|^2}{\epsilon - i\tau}}$$

is the continuous Fourier transform.

On a major arc centered at p/q , one has $q \leq T$ $\tau \leq T^{-1}q^{-1}$, cf. [1]; hence it is easy to show that

$$\frac{\epsilon}{q^2(\epsilon^2+\tau^2)} \,\geq\, \frac{1}{2}.$$

This yields the one dimensional estimate

$$\theta(t,\chi) \,\leq\, cq^{-1/2}\,(\epsilon + |\tau|)^{-1/2} \sum_{l\in Z} e^{-\pi/4(l+2q\chi)^2} \leq cq^{-1/2}|\tau + i\epsilon|^{-1/2},$$

using the properties of Gaussian sums.

Taking the n-th power of this inequality and integrating over the major arcs yields

$$\int_0^1 |\theta(t,\chi)|\,dt \;\le\; \sum_{p,q} q^{-n/2} \int_0^\infty |\tau + i\epsilon|^{-n/2}\,d\tau$$
$$\le\; c_n\,\epsilon^{-n/2+1} \sum_{q=1}^{\infty} q^{-n/2} \;\le\; c_n\,T^{n/2-1}$$

because $-n/2+1<-1$ for $n\ge 5$, and Proposition 2 follows. □

This implies the following.

Lemma 2. *Suppose* $T, L \ge 1$. *Then we have*

$$|K_{T,L}(x)| \;\le\; c_n T^{\frac{n-1}{2}} \max(T^{-1}, L^{-1}). \tag{9}$$

Proof. Let us introduce the notations $Z_0 = |\{x \in Z^n\,;\, ||x-\chi|-T| \le L^{-1}\}|$ and $Z_j = |\{x\in Z^n\,;\, 2^{j-1}L^{-1} \le ||x-\chi|-T| < 2^j L^{-1}\}|$ for $j=1,2,\dots$. From Proposition 1 it follows that

$$|K^c_{T,L}(\chi)| \;\le\; c_{n,N} T^{-\frac{n-1}{2}} 2^{-Nj} \qquad \text{if} \quad x\in Z_j,$$

and then Proposition 2 implies

$$|K_{T,L}(\chi)| \;\le\; c_{n,N} T^{\frac{n-1}{2}} (\max(T^{-1},L^{-1}) + \sum_{j=1}^{\log_+ L/T} (T+2^jL^{-1})^{n-2} 2^{-Nj}$$
$$+\; c_{n,N} T^{-\frac{n-1}{2}} \sum_{j>\log_+(L/T)} (T+2^j\,L^{-1})\,2^jL^{-1}\,2^{-Nj}$$
$$=\; c_{n,N} T^{-\frac{n-1}{2}}\,(T^{n-1}\max(T^{-1},L^{-1}) \;+ S_1 + S_2)$$

where $\log_+(s) = \max(\log(s),0)$ and $S_1 = 0$ if $L\le T$.

It is easy to see that $S_2 \le c_n T^{n-1}\,L^{-1}$ and $S_1 \le c_n T^{n-2}$ for $L\ge T$. Lemma 2 follows. □

3. Direct estimates in Z^n

One can explicitely write down the kernel $K_{T,L}$ as a finite sum as follows:

$$K_{T,L}(\chi) \;=\; \sum_{\xi\in Z^n} e^{2\pi i\,\chi\xi}\,\eta(\xi/L)\,|\xi|^{-\frac{n+1}{2}+i\gamma}\,e^{iT|\xi|}, \tag{10}$$

which leads the trivial upper bound $|K_{T,L}(\chi)| \le c_n\,L^{\frac{n-1}{2}}$. This estimate together with Lemma 2 already gives a uniform upper bound in χ and L showing the possibility of the Strichartz type estimates for the torus. Now we show how to obtain essentially sharp uniform estimates using theta type functions and partial summation.

Lemma 3. *Let* $n\ge 5$. *For* $T<L/2$ *we have*

$$|K_{T,L}(\chi)| \;\le c_n\,L^{\frac{n-1}{2}}\,T^{-1}\,\log L. \tag{11}$$

For $L/2 \le T \le L^2$ *we have*

$$|K_{T,L}(\chi)| \;\le c_n\,L^{\frac{n-1}{2}-2}\,T\,\log L. \tag{12}$$

Proof. Let $N_2 = \{l^{1/2}\,; l \in N, L/2 \le l < 2L\}$. Then one can write

$$K_{T,L}(\chi) = \sum_{l\in N_2} \Big(\sum_{|\xi|=l} e^{2\pi i\chi\xi}\Big)\,\eta(l/L)l^{-\frac{n+1}{2}+i\gamma}\,e^{iTl}.$$

The inner sum is just the the Fourier coefficient of the theta function $\theta(\chi, t+i\epsilon) = \sum_\xi e^{i(\chi\xi+|\xi|^2(t+i\epsilon))}$. Hence one has

$$K_{T,L}(\chi) = \int_0^{2\pi} \theta(\chi, t+i\epsilon)\Big(\sum_{l\in N_2} e^{i(Tl-tl^2)}\,a(L,l)\Big)\,dt, \tag{13}$$

where $a(L,l) = \eta(l/L)l^{-\frac{n+1}{2}+i\gamma}e^{\epsilon l^2}$, where we have chosen $\epsilon = L^{-2}$.

We will estimate the inner sum uniformly in t by using partial summation. Since N_2 consists of the square roots of the integers, the difference between two consecutive elements $l < l'$ is bounded by cL^{-1}; hence it is easy to see that

$$|a(L,l)| \le c_n L^{-\frac{n+1}{2}} \text{ and } |a(L,l') - a(L,l)| \le c_n L^{-\frac{n+1}{2}-2}.$$

Introducing the notation

$$S_l = \sum_{l_1\in N_2, l_1\le l} e^{i(T\,l_1 - t\,l_1^2)},$$

we get, by partial summation,

$$\Big|\sum_{l\in N_2} e^{i(Tl-tl^2)}\,a(L,l)\Big| \le c_n L^{-\frac{n+1}{2}-2}\sum_{l\in N_2}|S_l|. \tag{14}$$

We now decompose these partial sums S_l into shorter sums, whose terms are between two consecutive integers. Let us introduce the notations

$$S_K^* = \sum_{K\le l<K+1} e^{i(Tl-tl^2)} \quad \text{and} \quad S_{K,\mu}^* = \sum_{l=K}^{\mu} e^{i(Tl-tl^2)},$$

where K is integer $l, \mu \in N_2$, $L/2 \le K < L$ and $K \le \mu < K+1$. It is not hard to show that

$$\sum_{l\in N_2}|S_l| \le |N_2|\sum_K |S_K^*| + \sum_l |S_{[l],l}^*| \le 4L^2\sum_K |S_K^*| + 16L^3, \tag{15}$$

where we used the facts that $|N_2| \le 4L^2$ and $|S_{[l],l}^*| \le 2K \le 4L$.

Let us remark that at this point the proof of the lemma is reduced to obtaining nontrivial estimates for the sum $\sum_K |S_K^*|$. In order to do this we make us of the Taylor formula to approximate the terms by simpler ones as follows. We can write

$$S_K^* = \sum_{j=0}^{2K} e^{i(T\sqrt{K^2+j}-t(K^2+j))} = e^{i(TK-tK^2)}\sum_{j=0}^{2K} e^{i(TK(\sqrt{1+jK^{-2}}-1)-tj)}.$$

Using linear approximations for the exponents in the sum, we have

$$\sqrt{1+jK^{-2}} - 1 = 1/2\,jK^{-2} + r(jK^{-2}),$$

where

$$|r((j+1)K^{-2}) - r(jK^{-2})| \le c\,K^{-2}\,jK^{-2} \le cL^{-3}.$$

Partial summation yields

(16)
$$|S_K^*| = |\sum_{j=0}^{2K} e^{ij(T/2K-t)} e^{iTKr(jK^{-2})}| \le c(\max_{0\le\mu<2K} |T_\mu|)TL^{-1} + T_{2K},$$

where $T_\mu = \sum_{j=0}^{\mu} e^{i\,j(T/2K-t)}$ is the sum of a finite geometric progression. One can use the simple uniform upper bound

$$\max_{0\le\mu\le 2K} |T_\mu| \le c\,\min(L, \{T/2K-t\}^{-1}), \tag{17}$$

where $\{\alpha\} = \min\{|\alpha - n|;\, n \text{ is an integer}\}$ denotes the fractional part of the number α. The crucial point is that the average over K of the right side of inequality (17) is essentially smaller than L.

First we deal with the case when $T < L/2$. Let us consider the set $Z_T = \{T/2K;\ L/2 \le K < 2L\}$. We remark that $Z_T \subset (0,1)$ and that the distance between two neighbouring points is at least TL^{-2}. It is not hard to see that

$$\begin{aligned}\sum_{x\in Z_T} \min(L, \{x-t\}^{-1}) &\le \sum_{x\in Z_T} \min(L, |x-t|^{-1} + (1-|x-t|)^{-1}) \\ &\le c(L + \sum_{j=1}^{2K} L^2T^{-1}j^{-1}) \le cT^{-1}L^2 \log L.\end{aligned}$$

The case when $L/2 \le T < L^2$ is slightly more complicated, however we can apply the previous argument for the sets $Z_{T,n} = Z_T \bigcap [n, n+1]$, where $n \le T/L$ is an integer. Hence in this case we have

$$\sum_{x\in Z_T} \min(L, \{x-t\}^{-1}) \le cT/L\,(L + T^{-1}L^2 \log L) \le cT \log L. \tag{18}$$

If we put together inequalities (14)–(18) we obtain the crucial estimates

$$|\sum_{l\in N_2} e^{i(Tl-tl^2)}\, a(L,l)| \le c\,L^{-\frac{n+1}{2}+2}T^{-1} \log L \quad \text{for } T < L/2 \tag{19}$$

and

$$|\sum_{l\in N_2} e^{i(Tl-tl^2)}\, a(L,l)| \le c\,L^{-\frac{n+1}{2}}T \log L \quad \text{for } L/2 \le T \le L^2. \tag{20}$$

Lemma 3 follows if one remarks that the "circle" method implies the estimate for $n \ge 5$, $\epsilon = L^{-2}$

$$\int_0^{2\pi} |\theta(\chi, t+i\epsilon)|\, dt \le c_n\, L^{n-2}. \quad \square$$

Optimizing between estimates (9) and (12) implies (5) for $L \ge T^{1/2}$, and for $L \le T^{1/2}$ we have the trivial estimate $|K_{T,L}(\chi)| \le c_n L^{\frac{n-1}{2}} \le c_n T^{\frac{n-3}{2}}$ when $n \ge 5$. Hence Lemma 1 follows. $\square$

Finally using the (energy) estimate

$$\| \Lambda^{1+i\gamma} R_L K_T f \|_{L^2} \le \| f \|_{L^2}, \tag{21}$$

which is obvious since the Fourier transform of the left side is $|\xi|^{i\gamma}\eta(\xi/L)e^{iT|\xi|}\hat{f}(\xi)$, estimate (4) follows immediately by interpolating between (9) and (21), which implies Theorem 1 using the Littlewood-Paley theory as we already explained in the introduction.

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