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ON VOICULESCU'S DOUBLE COMMUTANT THEOREM

C. A. BERGER AND L. A. COBURN

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ABSTRACT. For a separable infinite-dimensional Hilbert space H, we consider the full algebra of bounded linear transformations $B(H)$ and the unique non-trivial norm-closed two-sided ideal of compact operators $\mathcal{K}$. We also consider the quotient C^*-algebra $\mathcal{C} = B(H)/\mathcal{K}$ with quotient map

$$\pi\colon B(H) \to \mathcal{C}.$$

For $\mathcal{A}$ any C^*-subalgebra of $\mathcal{C}$, the relative commutant is given by $\mathcal{A}' = \{C \in \mathcal{C}\colon CA = AC \text{ for all } A \text{ in } \mathcal{A}\}$. It was shown by D. Voiculescu that, for $\mathcal{A}$ any *separable* unital C^*-subalgebra of $\mathcal{C}$,

(VDCT) $$\mathcal{A}'' = \mathcal{A}.$$

In this note, we exhibit a *non-separable* unital C^*-subalgebra $\mathcal{A}_0$ of $\mathcal{C}$ for which (VDCT) fails.

1. Introduction

For a separable infinite-dimensional Hilbert space H, we consider the full algebra of bounded linear transformations $B(H)$ and the unique non-trivial norm-closed two-sided ideal of compact operators $\mathcal{K}$. We also consider the quotient C^*-algebra $\mathcal{C} = B(H)/\mathcal{K}$ with quotient map

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For $\mathcal{A}$ any C^*-subalgebra of $\mathcal{C}$, the relative commutant is given by $\mathcal{A}' = \{C \in \mathcal{C}\colon CA = AC \text{ for all } A \text{ in } \mathcal{A}\}$. It was shown by D. Voiculescu in [7] that, for $\mathcal{A}$ any *separable* unital C^*-subalgebra of $\mathcal{C}$,

(VDCT) $$\mathcal{A}'' = \mathcal{A}.$$

In this note, we exhibit a *non-separable* unital C^*-subalgebra $\mathcal{A}_0$ of $\mathcal{C}$ for which (VDCT) fails.

The construction of $\mathcal{A}_0$ involves Berezin-Toeplitz operators on the Segal-Bargmann Hilbert space of Gaussian square-integrable entire functions on the complex plane $\mathbf{C}$. The necessary analysis was done in [1, 2] and we simply put the pieces together here in order to construct the desired example.

The analytic preliminaries are discussed in §2. In §3, we exhibit the algebra $\mathcal{A}_0$ and show that (VDCT) fails. In §4, we discuss examples of non-separable C^*-subalgebras of $\mathcal{C}$ for which (VDCT) holds. In §5, there are some additional remarks.

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2. Preliminary results

As customary, for $\mathcal{B}$ a subalgebra of $B(H)$, we write $\mathcal{B}' = \{T \in B(H)\colon TB = BT$ for all B in $\mathcal{B}\}$. We will consider the particular Hilbert space

$$H^2 = H^2(\mathbf{C}^n, d\mu)$$

where $d\mu(z) = (2\pi)^{-n} e^{-|z|^2/2}\, dv(z)$ is Gaussian measure ($dv(z)$ is ordinary Lebesgue measure on $\mathbf{C}^n$) and H^2 consists of all the $d\mu$ square-integrable entire functions. This space is a Bergman space, with reproducing kernel functions $e^{z\cdot a/2}$. Here,

$$z \cdot a = z_1\overline{a}_1 + \cdots + z_n\overline{a}_n$$

for $z = (z_1, \ldots, z_n)$, $a = (a_1, \ldots, a_n)$ in $\mathbf{C}^n$ and

$$|z|^2 = z \cdot z.$$

The kernel functions have the property that

$$h(a) = \langle h, e^{z\cdot a/2}\rangle \equiv \int h(z) e^{a\cdot z/2} d\mu(z)$$

for all h in H^2 and a in $\mathbf{C}^n$. H^2 is a closed subspace of $L^2 = L^2(\mathbf{C}^n, d\mu)$ and the orthogonal projection from L^2 onto H^2 is given by

$$(Pg)(a) = \langle g, e^{z\cdot a/2}\rangle$$

for all g in L^2.

For f in $L^\infty(\mathbf{C}^n)$, the full algebra of bounded measurable functions, we can define a bounded Berezin-Toeplitz operator on H^2 by

$$(T_f h)(a) = P(fh)(a) = \langle f(z)h(z), e^{z\cdot a/2}\rangle.$$

In [1, 2], a detailed study of these operators was carried out. In this connection, two C^*-subalgebras of L^∞ are especially noteworthy: the algebras AP and ESV. AP consists of uniform limits of finite linear combinations of characters

$$\chi_a(z) = e^{i\operatorname{Im}(z\cdot a)}$$

($\operatorname{Im}(z\cdot a) = (z\cdot a - a\cdot z)/2i$) while ESV consists of all f in L^∞ for which (ignoring sets of measure zero)

$$(*) \qquad \operatorname*{Lim}_{R\to\infty} \sup_{\{z\colon |z|\geq R\}} \sup_{\{w\colon |z-w|\leq 1\}} \{|f(z) - f(w)|\} = 0.$$

The condition $(*)$ is uniformly closed and says that the function f is "slowly varying at infinity". The algebras AP and ESV have only the constant functions in common. ESV contains, for example, all functions

$$\hat{f}(z) = f(z/|z|), \qquad z \neq 0,$$

where f is continuous on the unit sphere S^{2n-1}.

We denote by $\tau(AP)$, $\tau(ESV)$, $\tau(L^\infty)$ the C^*-algebras on H^2 generated, respectively, by all Berezin-Toeplitz operators T_f with f in AP, ESV, L^∞. The algebra $\tau\{AP(\mathbf{C}^n)\}$ was shown in [1] to be exactly the "canonical commutation relation" algebra $CCR(\mathbf{C}^n)$ described in [3, pp. 19–22]. It follows that $\tau(AP)$ is a *simple* C^*-algebra.

In the discussion which follows, we will need to recall that an element A of $B(H)$ is *Fredholm* if and only if $\pi(A)$ is invertible in $\mathcal{C}$. Since $\tau(AP)$ is *simple*, π restricted to $\tau(AP)$ must be a $*$-isomorphism and it follows easily that the only Fredholm elements in $\tau(AP)$ must be *invertible*. For $\tau\{ESV(\mathbf{C})\}$, on the other hand, it was checked in [1] that, for

$$\theta(z) = \begin{cases} z, & |z| \le 1, \\ z/|z|, & |z| \ge 1, \end{cases}$$

T_θ is Fredholm with

$$\text{index}(T_\theta) = -1 = \dim \ker(T_\theta) - \dim\ \text{coker}(T_\theta).$$

It follows from standard operator theory that T_θ is neither invertible nor, even, a compact perturbation of an invertible. In fact, $T_\theta + K$ is Fredholm with

$$\text{index}(T_\theta + K) = -1$$

for all K in $\mathcal{K}$.

3. Main result

We can now demonstrate the failure of (VDCT) in the non-separable case.

Theorem. (VDCT) *fails for* $\mathcal{A}_0 = \pi\tau\{AP(\mathbf{C})\}$.

Proof. By [2, Theorem D],

$$\{\pi\tau(AP)\}' = \pi\tau(ESV).$$

Moreover, by [2, Proposition A and Theorem B],

$$\pi\tau(L^\infty) \subset \{\pi\tau(ESV)\}'.$$

It follows that

$$\pi\tau(AP) \subset \pi\tau(L^\infty) \subset \{\pi\tau(AP)\}''.$$

To show that (VDCT) fails, we need only check that

$$\pi\tau(AP) \neq \pi\tau(L^\infty).$$

For $n = 1$ ($\mathbf{C}^n = \mathbf{C}$), this is easy. Suppose that

$$\pi\tau\{AP(\mathbf{C})\} = \pi\tau\{L^\infty(\mathbf{C})\}.$$

Then, since $\mathcal{K} \subset \tau\{L^\infty(\mathbf{C})\}$ [2, Theorem 16], we must have

$$\tau\{L^\infty(\mathbf{C})\} = \tau\{AP(\mathbf{C})\} + \mathcal{K}.$$

But, by the discussion at the end of §2, T_θ is in $\tau\{L^\infty(\mathbf{C})\}$ and is Fredholm with $\text{index}(T_\theta) = -1$ while

$$T_\theta = A_\theta + K_\theta$$

for some A_θ in $\tau\{AP(\mathbf{C})\}$ and K_θ in $\mathcal{K}$. It follows that $A_\theta = T_\theta - K_\theta$ must be Fredholm with

$$\text{index}(A_\theta) = \text{index}(T_\theta - K_\theta) = -1.$$

This is contradicted by the observation that A_θ is invertible.

Remark. In fact, (VDCT) fails for $\pi\tau\{AP(\mathbf{C}^n)\}$ for all n. One needs to consider $n \times n$ systems as in [4] to show that an index obstruction yields

$$\tau\{L^\infty(\mathbf{C}^n)\} \otimes M_n \neq \tau\{AP(\mathbf{C}^n)\} \otimes M_n + \mathcal{K} \otimes M_n.$$

4. Positive results

It turns out that (VDCT) holds for many non-separable C^*-subalgebras in $\mathcal{C}$. Following up earlier work of [5], it was shown in [6] that, for $\mathcal{B}$ *any* von Neumann algebra in $B(H)$,

$$\pi(\mathcal{B})' = \pi(\mathcal{B}')$$

with the evident corollary (since $\mathcal{B} = \mathcal{B}''$) that

$$\pi(\mathcal{B})'' = \pi(\mathcal{B}')' = \pi(\mathcal{B}'') = \pi(\mathcal{B}).$$

We do not know of any other large class of non-separable C^*-subalgebras of $\mathcal{C}$ for which (VDCT) holds.

5. Additional remarks

The proof of our main result provides some additional interesting information. Since $\pi\tau(L^\infty)$ is contained in $\{\pi\tau(ESV)\}'$ it is clear that $\pi\tau(ESV)$ is in $\{\pi\tau(L^\infty)\}'$. Since $\pi\tau(AP)$ is contained in $\pi\tau(L^\infty)$, we have

$$\{\pi\tau(L^\infty)\}' \subset \{\pi\tau(AP)\}'.$$

Recalling that $\{\pi\tau(AP)\}' = \pi\tau(ESV)$, we finally get

$$\{\pi\tau(L^\infty)\}' \subset \{\pi\tau(AP)\}' = \pi\tau(ESV) \subset \{\pi\tau(L^\infty)\}'$$

so that

$$\{\pi\tau(L^\infty)\}' = \{\pi\tau(AP)\}' = \pi\tau(ESV).$$

Thus, we have exhibited two *distinct* unital C^*-subalgebras of $\mathcal{C}$ with the same relative commutant.

Moreover, suppose $\{\pi\tau(AP)\}'' = \mathcal{C}$. Since $\mathcal{C}' = \mathbf{C}1$ by the remarks in §4, we must have

$$\pi\tau(ESV) = \{\pi\tau(AP)\}' = \mathbf{C}1.$$

This conclusion is clearly false by [2, Theorem E]. It follows, since

$$\pi\tau(L^\infty) \subset \{\pi\tau(ESV)\}' = \{\pi\tau(AP)\}'',$$

that $\pi\{\tau(L^\infty)\} \neq \mathcal{C}$ and so $\tau(L^\infty) \neq B(H^2)$.

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Department of Mathematics and Computer Science, Herbert H. Lehman College, City University of New York, Bronx, New York 10468

Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14214