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THE p^n THEOREM FOR SEMISIMPLE HOPF ALGEBRAS

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ABSTRACT. We give an algebraic version of a result of G. I. Kac, showing that a semisimple Hopf algebra A of dimension p^n, where p is a prime and $n > 0$, over an algebraically closed field of characteristic 0 contains a non-trivial central group-like. As an application we prove that, if $n = 2$, A is isomorphic to a group algebra.

INTRODUCTION

Throughout the paper we work over an algebraically closed field k of characteristic 0.

Let p be a prime. Recently Y. Zhu [Z] proved that a Hopf algebra of dimension p is isomorphic to the group algebra kC_p of the cyclic group C_p of order p. For this, he reformulated G. I. Kac's Theorem [K, Theorem 2] on 'ring groups'. In this paper, first we give an algebraic version of another result [K, Corollary 2] of Kac to show that a semisimple Hopf algebra of dimension p^n with a positive integer n contains a non-trivial central group-like. Secondly, applying the first result, we classify all semisimple Hopf algebras of dimension p^2. Namely we prove that such a Hopf algebra is isomorphic to the group algebra kC_{p^2} or $k(C_p \times C_p)$.

THE p^n THEOREM

Let A be a finite-dimensional Hopf algebra with antipode S. Suppose that A is semisimple as an algebra, or equivalently cosemisimple as a coalgebra, or equivalently involutory, namely $S \circ S = \mathrm{id}$ (See [LR1, Theorem 3.3]; [LR2, Theorems 1, 3]).

Denote by A^* the dual Hopf algebra $\mathrm{Hom}_k(A, k)$ of A. Let λ be an integral in A^* such that $\lambda(1) = (\dim A)1$. Then it follows by [LR2, Proposition 1] that λ is the character of the regular representation of A, that is, $\lambda(a)$ for $a \in A$ equals the trace of the right (or left) multiplication $b \mapsto ba$ ($b \in A$) by a. Regard A^* as a right A-module with the action $\leftharpoonup$ determined by

$$(f \leftharpoonup a)(b) = f(bS(a)) \quad (f \in A^*, a, b \in A).$$

Then one sees from [Sw, Theorem 5.1.3] that

$$\alpha\colon\ A \to A^*, \qquad \alpha(a) = \lambda \leftharpoonup a$$

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gives a right A-linear and left A^*-linear isomorphism, where A has the left A^*-module structure arising from the natural right A-comodule structure.

Lemma. *α gives linear isomorphisms between each pair in* (a), (b) *below.*

(a) *The subalgebra $kG(A)$ of A spanned by the group-likes $G(A)$ and the sum of the* 1*-dimensional ideals of A^*.*

(b) *The center $Z(A)$ of A and the subalgebra $C_k(A)$ of A^* spanned by the characters of A.*

Proof. For (a), one has only to see that α gives a 1-1 correspondence between the 1-dimensional left A^*-submodules of A and of A^*. (Note that a 1-dimensional left ideal of A^* is actually two-sided since A^* is semisimple.)

For (b), we see first that the antipode S gives a permutation of the primitive central idempotents in A, which form a k-basis of $Z(A)$. Let e be a primitive central idempotent in A and let χ be the irreducible character corresponding to e (that is, the character of the representation $A \to eA$). Then it follows that $\alpha(S(e)) = \chi(1)\chi$ ($\neq 0$), where $\chi(1)$ equals the degree of χ. From these we have that $Z(A) \simeq C_k(A)$. □

Corollary. *There is a 1-1 correspondence $g \leftrightarrow I$ between the central group-likes g in A and the* 1*-dimensional ideals I of A^* included in $C_k(A)$, such that $1 \leftrightarrow k\lambda$.*

Theorem 1. *Suppose that the dimension* $\dim A = p^n$*, where p is a prime and n is a positive integer. Then there is a non-trivial central group-like in A.*

Proof. By the Corollary it suffices to prove that there is a 1-dimensional ideal $I \neq k\lambda$ of A^* which is included in $C_k(A)$.

Let $e_1 = \frac{1}{\dim A}\lambda$, $e_2, \ldots, e_m$ be orthogonal primitive idempotents in $C_k(A)$ whose sum is 1. Then we have

$$A^* = k\lambda \oplus e_2 A^* \oplus \cdots \oplus e_m A^*.$$

Since each $\dim e_i A^*$ divides p^n by [Z, Theorem 1], it follows by counting dimensions that there is $2 \leq i \leq m$ such that $\dim e_i A^* = 1$. This $e_i A^*$ is the required I. □

This theorem is an algebraic (and hopefully accessible) version of a result [K, Corollary 2] of G. I. Kac on 'ring groups'.

The p^2 theorem

Theorem 2. *A semisimple Hopf algebra of dimension p^2 with a prime p is isomorphic to the group algebra kC_{p^2} or $k(C_p \times C_p)$, where C_n is the cyclic group of order n.*

Proof. Let A be a semisimple Hopf algebra of dimension p^2. It suffices to show that A is commutative and cocommutative.

It follows from Theorem 1 and [NZ, Theorem 7] that there is a group G of group-likes in A such that $G \subset Z(A)$ and the order $|G| = p$. Since $G \subset Z(A)$, the Hopf subalgebra $K = kG$ is normal, so that we have an extension [M1, Definition 1.3]

$$1 \to K \to A \to H \to 1$$

of finite-dimensional Hopf algebras. Since this is *cleft* (roughly $A \simeq K \otimes_k H$) by [S, Theorem 2.4], it follows that $\dim H = p$, so that $H \simeq kC_p$ by Zhu's Theorem [Z, Theorem 2] cited in the Introduction. One sees immediately from [DT, Theorem 11] that, as an algebra including K, A is a crossed product of H over K. In the terminology of group theory, A is a crossed product $K * C_p$ [P, p. 2] of C_p over K. Here the associated action of C_p on K is trivial, since K is central in A. Hence A is a twisted group ring $K^t[C_p]$ [P, p. 4], a cyclic extension of a central subalgebra. This is trivially commutative. Thus A is commutative. By applying the result to A^*, it follows that A is cocommutative. □

Using this result we classify all semisimple Hopf algebras of dimension p^3 with an odd prime p in the final version of [M3], while such Hopf algebras of dimension $8 = 2^3$ are classified in [M2].

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