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ON A CONVOLUTION INEQUALITY OF SAITOH

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ABSTRACT. Let $F_1, F_2, \ldots, F_j, \ldots$ be in the class $L_{\text{loc}}(\mathbb{R}_+)$ of locally integrable functions on $\mathbb{R}_+ = (0, \infty)$. Define the convolution product $\prod_{j=1}^m *F_j$ inductively by $[\prod_{j=1}^2 *F_j](x) = (F_1 * F_2)(x) = \int_0^x F_1(y)F_2(x-y)\,dy$ and $\prod_{j=1}^m *F_j = [\prod_{j=1}^{m-1} *F_j] * F_m$ for $m > 2$. The inequality

$$\int_0^\infty x^{-(m-1)(p-1)} \left| \left[\prod_{j=1}^m *F_j \right](x) \right|^p dx \le [(m-1)!]^{1-p} \prod_{j=1}^m \int_0^\infty |F_j(y)|^p\,dy$$

is obtained for each p, $1 < p < \infty$. Further, the constant $[(m-1)!]^{1-p}$ is shown to be the best possible, and the nonzero extremal functions are determined.

1. INTRODUCTION

Let $F_1, F_2, \ldots, F_j, \ldots$ be in the class $L_{\text{loc}}(\mathbb{R}_+)$ of complex-valued locally integrable functions on $\mathbb{R}_+ = (0, \infty)$ (i.e., they are integrable on $(0, r)$ for each $r > 0$). Define the convolution product $\prod_{j=1}^m *F_j$ by $[\prod_{j=1}^1 *F_j] = F_1$, $[\prod_{j=1}^2 *F_j](x) = (F_1 * F_2)(x) = \int_0^x F_1(y)F_2(x-y)\,dy$ and inductively for $m > 2$ by $\prod_{j=1}^m *F_j = [\prod_{j=1}^{m-1} *F_j] * F_m$. It is easy to check that each of the functions $\prod_{j=1}^m *F_j$ must also be in $L_{\text{loc}}(\mathbb{R}_+)$.

Our result here is

Theorem. *Fix p, $1 < p < \infty$. Then, for each positive integer m,*

$$\int_0^\infty x^{-(m-1)(p-1)} \left| \left[\prod_{j=1}^m *F_j \right](x) \right|^p dx \le [(m-1)!]^{1-p} \prod_{j=1}^m \int_0^\infty |F_j(x)|^p\,dy. \tag{1.1}$$

The constant $[(m-1)!]^{1-p}$ is best possible. Moreover the (nonzero) extremal functions are of the form $F_j(y) = C_j e^{-cy}$ a.e., where C_j are constants and $\operatorname{Re} c > 0$, $j = 1, \ldots, m$.

The above result is known in the case $p = 2$ and m even, where it was proved by Saitoh [5] using Aronszajn's theory of reproducing kernels.

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Our proof of the above theorem uses Hölder's inequality, Titchmarsh's convolution theorem, and the well-known functional equation for exponential functions.

2. The proof of the theorem

For $n \geq 1$ we have

$$\int_0^\infty x^{-n(p-1)} \left| \left[\prod_{j=1}^{n+1} *F_j \right] (x) \right|^p dx$$
$$= \int_0^\infty x^{-n(p-1)}\, dx \left| \int_0^x y^{(n-1)/p'} y^{(1-n)/p'} \left[\prod_{j=1}^{n} *F_j \right] (y) F_{n+1}(x-y)\, dy \right|^p$$

where $p' = \frac{p}{p-1}$. Applying Hölder's inequality to the inner integral we obtain that the above expression is dominated by

$$\int_0^\infty x^{-n(p-1)} \frac{x^{n(p-1)}}{n^{p-1}}\, dx \int_0^x y^{-(n-1)(p-1)} \left| \left[\prod_{j=1}^{n} *F_j \right] (y) \right|^p |F_{n+1}(x-y)|^p\, dy.$$

Applying Fubini's theorem to this integral we see that we have shown

$$\begin{aligned} &\int_0^\infty x^{-n(p-1)} \left| \left[\prod_{j=1}^{n+1} *F_j \right] (x) \right|^p dx \\ &\leq n^{1-p} \int_0^\infty y^{-(n-1)(p-1)} \left| \left[\prod_{j=1}^{n} *F_j \right] (y) \right|^p dy \int_0^\infty |F_{n+1}(t)|^p\, dt. \end{aligned} \tag{2.1}$$

We can now obtain (1.1) by induction. The case $m = 1$ is trivial. The case $m = 2$ is simply (2.1) with $n = 1$. If (1.1) holds for $m = n$, then (2.1) yields that

$$\begin{aligned} &\int_0^\infty x^{-n(p-1)} \left| \left[\prod_{j=1}^{n+1} *F_j \right] (x) \right|^p dx \\ &\leq n^{1-p} [(n-1)!]^{1-p} \prod_{j=1}^{n} \int_0^\infty |F_j(y)|^p\, dy \int_0^\infty |F_{n+1}(t)|^p\, dt \\ &= (n!)^{1-p} \prod_{j=1}^{n+1} \int_0^\infty |F_j(y)|^p\, dy, \end{aligned}$$

completing the proof of (1.1).

Now we determine under what conditions equality can hold in (1.1), apart from the obvious and trivial cases where $m = 1$ or $m > 1$ and one or more of the functions F_j vanish a.e. Equality in (1.1) implies that equality holds in (2.1) for each positive integer n with $n \leq m-1$. This happens only if equality holds in Hölder's inequality, i.e., only if for a.e. $x > 0$ there exists a number $k(x) \in \mathbb{C}$ such that

$$y^{(1-n)/p'} \left[\prod_{j=1}^{n} *F_j \right] (y) F_{n+1}(x-y) = k(x) y^{(n-1)/p} \tag{2.2}$$

for a.e. $y \in (0, x)$. It is convenient to rewrite this in the form:

$$\text{For a.e. } x \in \mathbb{R}_+ \ f(y)g(x-y) = k(x) \text{ for a.e. } y \in (0, x), \tag{2.3}$$

where $g = F_{n+1}$ and $f(y) = y^{1-n}[\prod_{j=1}^{n} *F_j](y)$.

Our next step is to prove that $k : \mathbb{R}_+ \to \mathbb{C}$ is a measurable function. This is not quite as obvious as it might seem at first. Observe that (2.3) is *not* automatically the same as saying $f(x)g(x-y) = k(x)$ for a.e. x in some interval, for various fixed values of y. (See Remark 2.7 below for further discussion of this matter.) Our proof will use an auxiliary function of the form

$$h(x, y) := \frac{U(x)}{x} \cdot \chi_{\{(x,y)|0<y<x\}}(x, y) \cdot V(\mathrm{Re}\{f(y)g(x-y)\}).$$

We need U to be integrable and strictly positive with $\int_0^\infty U(x)\,dx = 1$, and $V : \mathbb{R} \to \mathbb{R}$ must be continuous, strictly monotone, and bounded (e.g. take $U(x) = e^{-x}$ and $V(t) = \arctan t$). Clearly h is an integrable function on $\mathbb{R}_+ \times \mathbb{R}_+$. So, by Fubini's theorem, the function $H(x) := \int_0^\infty h(x, y)\,dy$ must be an integrable and thus measurable function of x on $\mathbb{R}_+$. But, by (2.3), we have $H(x) = U(x)V(\mathrm{Re}\,k(x))$ for a.e. $x \in \mathbb{R}_+$. Consequently $\mathrm{Re}\,k(x) = V^{-1}(\frac{H(x)}{U(x)})$ must be measurable. Similarly $\mathrm{Im}\,k(x)$ is also measurable, and therefore so is k.

Now we can deduce that the nonnegative function

$$\varphi(x, y) = |f(y)g(x-y) - k(x)| \cdot \chi_{\{(x,y)|0<y<x\}}(x, y)$$

must also be measurable on $\mathbb{R}_+ \times \mathbb{R}_+$. So we can apply Tonelli's theorem (i.e., Fubini's theorem for nonnegative but not necessarily integrable functions) to φ. Using (2.3) we obtain first that $f(y)g(x-y) = k(x)$ for a.e. (x, y) in $\{(x, y) : 0 < y < x\}$. Equivalently we have that

$$f(\alpha)g(\beta) = k(\alpha + \beta) \tag{2.4}$$

holds for a.e. (α, β) in the set $\mathbb{R}_+ \times \mathbb{R}_+$. We also obtain that for a.e. $y \in \mathbb{R}_+$, $f(y)g(x-y) = k(x)$ for a.e. $x \in (y, \infty)$. This implies that k must be locally integrable on $\mathbb{R}_+$ since g is.

We have excluded the case where g vanishes a.e. and we can also assume that f is nonzero on some subset of positive measure of $\mathbb{R}_+$ since otherwise, by Titchmarsh's theorem [6] (see also [2], [3]), at least one of the functions $F_1, F_2, \ldots, F_n$ would have to vanish a.e. Thus, by the Lebesgue differentiation theorem, there exist intervals $[\alpha_0, \alpha_1]$ and $[\beta_0, \beta_1]$ in $\mathbb{R}_+$ such that $\int_{\alpha_0}^{\alpha_1} f(t)\,dt$ and $\int_{\beta_0}^{\beta_1} g(t)\,dt$ are both nonzero. We deduce that $f(\alpha)$ coincides for a.e. $\alpha \in \mathbb{R}_+$ with the continuous function

$$f_1(\alpha) := \int_{\beta_0}^{\beta_1} k(\alpha + t)\,dt \Big/ \int_{\beta_0}^{\beta_1} g(t)\,dt,$$

and analogously $g(\beta)$ coincides for a.e. $\beta \in \mathbb{R}_+$ with the continuous function $g_1(\beta) := \int_{\alpha_0}^{\alpha_1} k(t + \beta)\,dt / \int_{\alpha_0}^{\alpha_1} f(t)\,dt$. Now for every positive t and r and for every positive α and β such that $t = \alpha + \beta$ we have

$$\frac{1}{r^2}\int_0^r \int_0^r k(t + x + y)\,dx\,dy = \frac{1}{r^2}\int_0^r \int_0^r f_1(\alpha + x)g_1(\beta + y)\,dx\,dy.$$

We deduce that the limit $k_1(t) := \lim_{r\to 0} \frac{1}{r^2}\int_0^r \int_0^r k(t+x+y)\,dx\,dy$ exists for every

$t > 0$ and satisfies

$$k_1(\alpha + \beta) = f_1(\alpha)g_1(\beta) \tag{2.5}$$

for all positive α and β. Clearly $k_1(t) = f_1(\frac{1}{2})g_1(\frac{t}{2})$ and so is a continuous function on $\mathbb{R}_+$.

We claim that all of the functions $f_1(t), g_1(t)$, and $k_1(t)$ are nonzero for every $t > 0$. Suppose not; then $k_1(\delta) = 0$ for some $\delta > 0$, and either $f_1(\frac{\delta}{2})$ or $g_1(\frac{\delta}{2})$ must also vanish. Then, for every $\gamma > \frac{\delta}{2}$ we have $k_1(\gamma) = f_1(\gamma - \frac{\delta}{2})g_1(\frac{\delta}{2}) = f_1(\frac{\delta}{2})g_1(\gamma - \frac{\delta}{2}) = 0$ and thus k_1 vanishes on the interval $[\frac{\delta}{2}, \infty)$. By reiterating this argument sufficiently many times we see that $k_1(t) = 0$ for each $t > 0$. As explained above, each of the functions f and g is nonzero on some set of positive measure. This implies that the same is true for f_1 and g_1. But this contradicts (2.5) for some values of α and β and so proves our claim.

It now follows, again using (2.5), that the limits $f_1(0+)$ and $g_1(0+)$ both exist and are nonzero. Hence the function $H(x) := f_1(x)/f_1(0+) = g_1(x)/g_1(0+)$ satisfies $H(x - y)H(y) = k_1(x)/f_1(0+)g_1(0+) = H(x)$ for all $0 < y < x$. This implies that $f_1(x) = f_1(0+)e^{-cx}$ and $g_1(x) = g_1(0+)e^{-cx}$, where c is a constant satisfying $\operatorname{Re} c > 0$. See [1], pp. 35–36.

In particular, setting $n = 1$, the above argument shows that

$$F_j(x) = C_j e^{-cx} \quad \text{a.e.} \tag{2.6}$$

for $j = 1, 2$. Now, if (2.6) holds for $j = 1, 2, \ldots, n$, then clearly

$$\left[\prod_{j=1}^{n} *F_j\right](y) = \text{const.}\, y^{n-1}e^{-cy}.$$

But also, again by the preceding argument, $[\prod_{j=1}^{n} *F_j](y) = \text{const.}\, y^{n-1}e^{-c'y}$ and $F_{n+1}(y) = \text{const.}\, e^{-c'y}$ for some constant c'. It follows that $c' = c$, which shows that (2.6) also holds for $j = n + 1$, and so, by induction, for all $j = 1, 2, \ldots, m$. □

Remark 2.7. The proof of measurability of k given above may seem somewhat indirect. Let us indicate some difficulties which are encountered if we attempt to give a more direct proof. Let $F(x, y) = f(y)g(x - y)\chi_T(x, y)$ and $K(x, y) = k(x)\chi_T(x, y)$, where $T = \{(x, y) : 0 < y < x\}$, and let $N = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : F(x, y) \neq K(x, y)\}$. For each $x > 0$ define the x-section $N_x = \{y > 0 : (x, y) \in N\}$, and for each $y > 0$ define the y-section $N^y = \{x > 0 : (x, y) \in N\}$. To show that k is measurable it would suffice to show that N^y has zero measure for each y in some sequence tending to zero. We know from (2.3) that N_x has zero measure for a.e. $x > 0$. If we knew that N were a measurable subset of $\mathbb{R}_+ \times \mathbb{R}_+$, then we could immediately apply Tonelli's theorem to obtain that N has zero planar measure and consequently N^y has zero measure for a.e. y. But to show that N is measurable we need to know what we are trying to prove, namely that k is measurable. As a further indication of the possible difficulties here, we mention the example due to Sierpinski (see [4], p. 167) of a nonmeasurable subset Q of $[0, 1] \times [0, 1]$ all of whose x- and y-sections are measurable subsets of $[0, 1]$. In fact, Q_x has measure 1 and Q^y has measure 0. So if we define $P = \{(x, y) : (y, x) \in Q\}$ and then $\widetilde{N} = \bigcup_{m\geq 0, n\geq 0} P + (m, n)$ we obtain that $\widetilde{N}^y$ has infinite measure for each $y > 0$ even though $\widetilde{N}_x$ has measure 0 for each $x > 0$.

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