PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 2, February 1994

CONSTRUCTION OF FUNCTORS CONNECTING HOMOLOGY AND HOMOTOPY THEORIES

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(Communicated by Frederick R. Cohen)

ABSTRACT. For each manifold class $\mathscr{F}$ it is given a functor $\Theta^{\mathscr{F}}$ satisfying the Eilenberg and Steenrod axioms except the excision axiom. It provides a nice unification of geometric treatments of homology and homotopy theories.

INTRODUCTION

The geometric treatments of homology and homotopy are very different. Nevertheless, from the axiomatic viewpoint, both give functors satisfying the first six axioms of Eilenberg and Steenrod (E.S.).

Their different behaviour under an excision is essentially the chief distinguishing feature of homology and homotopy.

The present note exhibits a nice interpretation of this formal resemblance by constructing a geometric theory which, generalizing the objects and the relations, allows for the unification of the treatments and also provides for make new functors.

The unifying notion is that of $\mathscr{F}$-singular sphere for the objects, while it is that of $\mathscr{F}$-cobordism for the relations. By $\mathscr{F}$ we mean a manifold class, whose definition is reproduced in §1.

For each manifold class $\mathscr{F}$, we construct a functor $\Theta^{\mathscr{F}}$, from the category of pointed pairs of topological spaces to the category of graded groups, sastisfying the first six axioms of E.S. $\Theta^{\mathscr{F}}$ agrees with the homology functor H if $\mathscr{F}$ is the class $\mathscr{C}$ of the geometric cycles without boundary, and it agrees with the homotopy functor Π if $\mathscr{F}$ is the class $\mathscr{PL}$ of the standard PL-spheres. Moreover, if $\mathscr{F}' \subset \mathscr{F}$, there is a homomorphism

$$\Psi_{\mathscr{F}',\mathscr{F}}: \Theta^{\mathscr{F}'}(X, A, x_0) \to \Theta^{\mathscr{F}}(X, A, x_0),$$

and it is shown that $\Psi_{\mathscr{PL},\mathscr{F}}$ and $\Psi_{\mathscr{F},\mathscr{C}}$ give a factorization of Hurewicz homomorphism for each manifold class $\mathscr{F}$.

Received by the editors July 30, 1990 and, in revised form, June 4, 1992.
1991 *Mathematics Subject Classification*. Primary 57Q20, 57P99, 55P65.
Key words and phrases. Manifold class, cobordism, homotopy functor.
Research supported by M.U.R.S.T. (40%).

1. Preliminaries

Let $\mathscr{F} = \{\mathscr{F}_n\}_{n \geq 0}$, where each $\mathscr{F}_n$ is a class of compact n-polyhedra (closed under PL-isomorphisms) satisfying:

(a) $S^0 \in \mathscr{F}_0$;
(b) $\forall \Sigma \in \mathscr{F}_n$ and $\forall x \in \Sigma$, $Lk(x, \Sigma) \in \mathscr{F}_{n-1}$;
(c) $\forall \Sigma \in \mathscr{F}_n$ and $\forall \Sigma' \in \mathscr{F}_m$, $\Sigma * \Sigma' \in \mathscr{F}_{n+m+1}$;
(d) $\forall \Sigma \in \mathscr{F}_n$ and $\forall x \in \Sigma$, $\Sigma - \overset{\circ}{st}(x, \Sigma) \notin \mathscr{F}_n$.

The set $\mathscr{F}$ is called a manifold class. The elements Σ of $\mathscr{F}_n$ are called $\mathscr{F}_n$-spheres. The cone $c * \Sigma$ on an $\mathscr{F}_{n-1}$-sphere Σ is called an $\mathscr{F}_n$-disc, and a polyhedron of the form $\Sigma - \overset{\circ}{st}(x, \Sigma)$, $\Sigma \in \mathscr{F}_n$ and $x \in \Sigma$, is called an $\mathscr{F}_n$-pseudodisc. Note that an $\mathscr{F}_n$-disc is an $\mathscr{F}_n$-pseudodisc and that the suspension of an $\mathscr{F}_n$-pseudodisc is an $\mathscr{F}_{n+1}$-pseudodisc.

An $\mathscr{F}$-manifold of dimension n is a polyhedron M^n such that each link is either an $\mathscr{F}_{n-1}$-sphere or an $\mathscr{F}_{n-1}$-pseudodisc. The boundary of M, ∂M, consists of points whose links are $\mathscr{F}_{n-1}$-pseudodiscs. The polyhedron $M - \partial M$ will be denoted by $\overset{\circ}{M}$.

As an immediate consequence of the definition we observe that the boundary of an $\mathscr{F}$-manifold of dimension n is itself an $\mathscr{F}$-manifold of dimension $n-1$ without boundary.

A manifold class $\mathscr{F}$ is said to be connected if, for each pair P_1, P_2 of $\mathscr{F}_n$-pseudodiscs with $\partial P_1 \approx_f \partial P_2$, $P_1 \cup_f P_2 \in \mathscr{F}_n$.

If $\mathscr{F}$ is connected, the cylinder and the cone on an $\mathscr{F}_n$-pseudodisc are $\mathscr{F}_{n+1}$-pseudodiscs (see [4]). Moreover, it is easy to prove that, if P_1, P_2 are $\mathscr{F}_n$-pseudodiscs and $x_1 \in \partial P_1$, $x_2 \in \partial P_2$ such that $st(x_1, \partial P_1) \approx_g st(x_2, \partial P_2)$, then the polyhedron $P_1 \cup_g P_2$ is an $\mathscr{F}_n$-pseudodisc. Details about manifold classes can be found in [1, 4].

From now on we will sometimes omit the prefix $\mathscr{F}$, if no ambiguity arises, and all the manifold classes are assumed to be connected and such that $\mathscr{F}_0 = \{S^0\}$.

The hypothesis $\mathscr{F}_0 = \{S^0\}$ implies that any $\mathscr{F}$-manifold M of dimension n is a geometric n-cycle, so it makes sense to define M to be orientable if M is orientable as geometric cycle.

The following manifold classes satisfy the above conditions:

$\mathscr{PL} = \{\text{standard PL-spheres}\}$,
$\mathscr{H} = \{\text{homology spheres}\}$,
$h = \{\text{homotopy spheres}\}$,
$\mathscr{C} = \{\mathscr{C}_n\}$, where $\mathscr{C}_0 = \{S^0\}$ and $\mathscr{C}_n = \{$compact geometric n-cycles without boundary$\}$ if $n > 0$.

A $\mathscr{PL}$-manifold is simply a PL-manifold; an $\mathscr{H}$-manifold is usually called a homology manifold; an h-manifold is a homotopy manifold; and a $\mathscr{C}$-manifold is a geometric cycle.

Evidently for each manifold class $\mathscr{F}$ we have $\mathscr{PL} \subset \mathscr{F} \subset \mathscr{C}$.

Observe that a closed $\mathscr{C}$-manifold of dimension $n > 0$ is a $\mathscr{C}$-sphere. This property characterizes the manifold class $\mathscr{C}$ according to

Theorem 1.1. *Let $\mathscr{F}$ be a manifold class such that each closed $\mathscr{F}_n$-manifold, $n > 0$, is an $\mathscr{F}$-sphere. Then $\mathscr{F} = \mathscr{C}$.*

Proof. Since $\mathscr{F}_0 = \{S^0\}$, we can proceed by induction. Suppose that $\mathscr{F}_{n-1} = \mathscr{C}_{n-1}$. Then, by the previous observations, it suffices to prove that $\mathscr{C}_n \subset \mathscr{F}_n$. Let X be a compact geometric n-cycle without boundary. Since $Lk(x, X) \in \mathscr{C}_{n-1} = \mathscr{F}_{n-1}$, for each $x \in X$, X is an $\mathscr{F}$-manifold without boundary and hence an $\mathscr{F}$-sphere. □

2. The functor $\Theta^{\mathscr{F}}$

From now on all $\mathscr{F}$-manifolds are assumed to be orientable. If M denotes an oriented $\mathscr{F}$-manifold, $-M$ will denote the same manifold with the opposite orientation.

Definition 2.1. An $\mathscr{F}$-cobordism between two oriented $\mathscr{F}$-spheres Σ_1 and Σ_2 is an oriented $\mathscr{F}$-manifold W such that:

(a) ∂W is the disjoint union of Σ_1 and $-\Sigma_2$, and
(b) $W \cup c_1 * \Sigma_1 \cup c_2 * \Sigma_2$ is an $\mathscr{F}$-sphere.

An $\mathscr{F}$-cobordism between two oriented pseudodiscs P_1, P_2 is an oriented $\mathscr{F}$-manifold W such that:

(a′) $\partial W = P_1 \cup -P_2 \cup W_0$, where W_0 is a cobordism between ∂P_1 and ∂P_2; and
(b′) $W \cup c_1 * P_1 \cup c_2 * P_2$ is an $\mathscr{F}$-pseudodisc.

Observe that if X in an oriented $\mathscr{F}$-sphere or an oriented $\mathscr{F}$-pseudodisc, $X \times I$ realizes a corbordism between X and X. If W is a cobordism between X_1 and X_2, and W' is a cobordism between X_2 and X_3 (X_i, $i = 1, 2, 3$, is an oriented $\mathscr{F}$-sphere or an oriented $\mathscr{F}$-pseudodisc), then it is a little troublesome to prove that $W \cup_{X_2} W'$ is a cobordism between X_1 and X_3. Furthermore, a cobordism between pseudodiscs is itself a pseudodisc. This follows using essentially the property of $\mathscr{F}$ to be connected and the fact that the cone on a pseudodisc in a pseudodisc (see [4]).

Remark 2.2. If $\mathscr{F} = \mathscr{C}$, (b) follows from (a) and (b′) follows from (a′).

Remark 2.3. If $\mathscr{F} = \mathscr{PL}$, a cobordism between spheres is a cylinder and a cobordism between pseudodiscs is a PL-disc.

Definition 2.4. Let (X, x_0) be a pointed topological space. A singular $\mathscr{F}$-sphere of (X, x_0) is a triple (Σ, D, f), where Σ is an oriented $\mathscr{F}$-sphere, $D \subset \Sigma$ is a top-dimensional simplex, and $f\colon (\Sigma, D) \to (X, x_0)$ is a continuous map. Two singular $\mathscr{F}$-spheres (Σ_1, D_1, f_1), (Σ_2, D_2, f_2) of (X, x_0) are $\mathscr{F}$-cobordant if there exists a triple (W, W', g), called $\mathscr{F}$-cobordism, where W is an $\mathscr{F}$-cobordism between Σ_1 and Σ_2, $W' \subset W$ is a $\mathscr{PL}$-cobordism between D_1 and D_2, and $g\colon (W, W') \to (X, x_0)$ is a continuous map, so that the following conditions hold:

(1) $W' \cap \partial W = D_1 \cup D_2$;
(2) $g/\Sigma_i = f_i$, $i = 1, 2$.

From the previous observation it follows that the $\mathscr{F}$-cobordism relation between singular $\mathscr{F}$-spheres is an equivalence relation.

Theorem 2.5. *Let* (Σ, D_1, f), (Σ, D_2, f) *be singular spheres of* (X, x_0). *If there exists a connected PL-manifold* $M \subset \Sigma$, *containing* D_1 *and* D_2, *such that* $f(M) = x_0$, *then* (Σ, D_1, f) *and* (Σ, D_2, f) *are cobordant.*

Proof. Without loss of generality, we can suppose that Σ is triangulated so that $D_i = v_i * \partial D_i$, $i = 1, 2$. Choose a simplicial path s in $M \times I$, without loops, from $v_1 \times \{0\}$ to $v_2 \times \{1\}$, such that $s(I) \cap M \times \{0\}$ and $s(I) \cap M \times \{1\}$ are singletons. Let U be a regular neighborhood (nbd) of $s(I)$ in $M \times I$. Then the triple $(\Sigma \times I, U, f \times \mathrm{id})$ is a cobordism between (Σ, D_1, f) and (Σ, D_2, f). □

Given a singular sphere (Σ, D, f), Theorem 2.5 allows us to assume, up to a cobordism, that ∂D has a collar C in $\Sigma - \mathring{D}$ and f is constant on C. One can see that such singular spheres are cobordant iff there is a cobordism (W, W', g) between them such that W' has a regular nbd N which is a PL-manifold and g/N is constant.

Let $\Theta_n^{\mathscr{F}}(X, x_0)$ denote the set of $\mathscr{F}$-cobordism classes of singular $\mathscr{F}_n$-spheres of (X, x_0). In order to define an addition in $\Theta_n^{\mathscr{F}}(X, x_0)$, $n > 1$, it is convenient to consider only singular spheres and cobordisms satisfying the above conditions.

Now given two singular n-spheres (Σ_1, D_1, f_1), (Σ_2, D_2, f_2), $n > 1$, let

$$(\Sigma_1, D_1, f_1) + (\Sigma_2, D_2, f_2) = (\Sigma, D, f),$$

where Σ is the oriented sphere obtained by gluing $\Sigma_1 - \mathring{D}_1$ and $\Sigma_2 - \mathring{D}_2$, $f = f_1 \cup f_2$, and D is a top-dimensional simplex chosen in an open bicollar N on $\partial D_1 \approx \partial D_2$ in Σ, on which f is constant. Being $n > 1$, N is a connected $\mathscr{PL}$-manifold; so by Theorem 2.5, the cobordism class of (Σ, D, f) does not depend on the choice of D in N.

Theorem 2.6. *The cobordism class of* $(\Sigma_1, D_1, f_1) + (\Sigma_2, D_2, f_2)$ *depends only on the classes of* (Σ_1, D_1, f_1) *and* (Σ_2, D_2, f_2).

Proof. The statement follows on observing that we can add the cobordisms, likewise the spheres. The $\mathscr{PL}$-cobordism contained in a cobordism plays the role of the simplex contained in a sphere. □

The previous theorem allows an addition in $\Theta_n^{\mathscr{F}}(X, x_0)$, $n > 1$, by taking

$$[(\Sigma_1, D_1, f_1)] + [(\Sigma_2, D_2, f_2)] = [(\Sigma_1, D_1, f_1) + (\Sigma_2, D_2, f_2)].$$

We shall say that a singular n-sphere (Σ, D, f) is cobordant to zero (0-cobordant) if there is a triple (P, Δ, g), where P is an oriented $(n+1)$-pseudodisc, $\Delta \subset P$ is a top-dimensional simplex, and $g\colon (P, \Delta) \to (X, x_0)$ is a continuous map such that $\partial P = \Sigma$, $\Delta \cap \Sigma = D$, and $g/\Sigma = f$. The triple (P, Δ, g) is called a cobordism to zero of (Σ, D, f).

All the spheres (Σ, D, f) are cobordant to zero provided f is constant.

Let D_0 denote a PL n-disc contained in the standard oriented n-sphere S^n. Then we have

Lemma 2.7. *A singular n-sphere is cobordant to zero iff it is cobordant to* (S^n, D_0, f_0), *where* f_0 *is constant.*

Proof. Assume (Σ, D, f) is cobordant to (S^n, D_0, f_0), and let (W, W', G) be a cobordism between them. According to the definition of cobordism, $W_0 =$

$W \cup c * S^n$ is a pseudodisc and $\partial W_0 = \Sigma$. An extension G' of G to W_0 can be defined by $G'/c * S^n =$ constant map. Let $D' = W' \cup c * D_0$; the triple (W_0, D', G') is the required cobordism to zero.

Conversely, assume (P, Δ, g) is a cobordism to zero of (Σ, D, f). Let $D' \subset \mathring{\Delta}$ be a PL $(n+1)$-disc, and let $D'' \subset \partial D'$ be a PL n-disc; it is easily checked that the polyhedron $P - \mathring{D}'$ determines a cobordism between (Σ, D, f) and $(\partial D', D'', f_0)$. If $\partial D'$ coincides with S^n, up to orientation-preserving PL-homeomorphisms, the statement is proved. Otherwise, gluing $P - \mathring{D}'$ and $S^n \times I$ by an orientation-reversing PL-homeomorphism between $\partial D'$ and $S^n \times \{0\}$, we have a polyhedron which determines a cobordism between (Σ, D, f) and (S^n, D_0, f_0). □

By the above lemma, the 0-cobordant singular spheres belong to the same cobordism class. Such a class is the zero element of $\Theta_n^{\mathscr{F}}(X, x_0)$. Moreover, for each (Σ, D, f), the pseudodisc $(\Sigma - \mathring{D}) \times I$ determines, in a natural way, a cobordism to zero of $(\Sigma, D, f) + (-\Sigma, D, f)$. Hence, each element of $\Theta_n^{\mathscr{F}}(X, x_0)$ has an inverse.

Finally one can see that the addition in $\Theta_n^{\mathscr{F}}(X, x_0)$ is associative and commutative, so we have

Theorem 2.8. *$\Theta_n^{\mathscr{F}}(X, x_0)$ is an abelian group for each $n > 1$.*

The graded group $\{\Theta_n^{\mathscr{F}}(X, x_0)\}_{n>1}$ will be denoted by $\Theta^{\mathscr{F}}(X, x_0)$.

Like the homotopy groups, $\Theta^{\mathscr{F}}(X, x_0)$ does not depend on the choice of x_0 in X, provided that X is path-connected.

Theorem 2.9. *Let $\mathscr{F}' \subset \mathscr{F}$ be manifold classes. Then there exists a canonical homomorphism $\Psi_{\mathscr{F}', \mathscr{F}}: \Theta_n^{\mathscr{F}'}(X, x_0) \to \Theta_n^{\mathscr{F}}(X, x_0)$, for each $n > 1$.*

Proof. Since an $\mathscr{F}'$-sphere (cobordism) is also an $\mathscr{F}$-sphere (cobordism), it makes sense to define

$$\Psi_{\mathscr{F}', \mathscr{F}}: [(\Sigma, D, f)]_{\mathscr{F}'} \in \Theta_n^{\mathscr{F}'}(X, x_0) \to [(\Sigma, D, f)]_{\mathscr{F}} \in \Theta_n^{\mathscr{F}}(X, x_0).$$

Obviously $\Psi_{\mathscr{F}', \mathscr{F}}$ is a homomorphism. □

Let (X, A) be a pair of topological spaces and x_0 is a point of A. By relative $\mathscr{F}_n$-sphere of (X, A, x_0) we mean a triple (P, Δ, f), where P is an oriented $\mathscr{F}_n$-pseudodisc, $\Delta \subset P$ is a top-dimensional simplex meeting ∂P in a top-dimensional simplex, and $f: (P, \Delta) \to (X, x_0)$ is a map which carries ∂P to A.

Given a relative $\mathscr{F}_n$-sphere (P, Δ, f) of (X, A, x_0), $(\partial P, \Delta \cap \partial P, f/)$ is a singular $\mathscr{F}_{n-1}$-sphere of (A, x_0) which will be denoted by $\partial(P, \Delta, f)$.

Two relative $\mathscr{F}$-spheres (P_i, Δ_i, g_i), $i = 1, 2$, of (X, A, x_0) are called $\mathscr{F}$-cobordant if there exists a triple (V, V', G) where V is an $\mathscr{F}$-cobordism between P_1 and P_2, $V' \subset V$ is a $\mathscr{PL}$-cobordism between Δ_1 and Δ_2, and $G: (V, V') \to (X, x_0)$ is a continuous map, such that the following conditions hold:

(1) $V' \cap P_i = \Delta_i$, $i = 1, 2$.

(2) Let $W = \partial V - (\mathring{P}_1 \cup \mathring{P}_2)$ and $W' = W \cap V'$. Then $(W, W', G/)$ is an $\mathscr{F}$-cobordism between $\partial(P_1, \Delta_1, g_1)$ and $\partial(P_2, \Delta_2, g_2)$ with $G(W) \subset A$.

The $\mathscr{F}$-cobordism between relative spheres is an equivalence relation. Imitating the techniques of Theorem 2.5, it is easily seen that the following holds.

Theorem 2.10. *Let (P, Δ_1, f) and (P, Δ_2, f) be relative $\mathscr{F}$-spheres of (X, A, x_0). If there exists a connected $\mathscr{PL}$-manifold $M \subset P$ containing Δ_1 and Δ_2 such that $M \cap \partial P$ is a connected $\mathscr{PL}$-manifold and $f(M) = x_0$, then (P, Δ_1, f) and (P, Δ_2, f) are $\mathscr{F}$-cobordant.*

Given a relative $\mathscr{F}$-sphere (P, Δ, f), the previous theorem allows us to assume, up to a cobordism, that the frontier of Δ in P has a collar C in $P - \operatorname{int}(\Delta)$ and f is constant on C. Let $\Theta_n^{\mathscr{F}}(X, A, x_0)$ denote the set of the $\mathscr{F}$-cobordism classes of relative $\mathscr{F}_n$-spheres of (X, A, x_0). We can introduce in $\Theta_n^{\mathscr{F}}(X, A, x_0)$ $(n > 2)$ an addition by setting:

$$[(P_1, \Delta_1, f_1)] + [(P_2, \Delta_2, f_2)] = [(P, \Delta, f)]$$

where P is the oriented $\mathscr{F}_n$-pseudodisc obtained by gluing P_1 and P_2 by a PL-homeomorphism $g\colon \Delta_1 \cap \partial P_1 \to \Delta_2 \cap \partial P_2$, $f = f_1 \cup f_2$, and Δ is a top-dimensional simplex chosen in an open bicollar N on $\Delta_1 \cap \partial P_1 \approx \Delta_2 \cap \partial P_2$ in P on which f is constant. As for the singular $\mathscr{F}$-spheres, one can see that the above operation is well defined if $n > 2$, and it induces in $\Theta_n^{\mathscr{F}}(X, A, x_0)$ an abelian group structure. The zero element is the class of the triple $(\Delta^n, \overline{\Delta}^n, f_0)$, where Δ^n is the standard n-simplex, $\overline{\Delta}^n$ is an n-simplex of the first barycentric subdivision of Δ^n, and f_0 is the constant map to x_0.

Remark 2.11. A relative $\mathscr{F}_n$-sphere (P, Δ, f) is cobordant to $(\Delta^n, \overline{\Delta}^n, f_0)$, that is, it determines the zero element of $\Theta_n^{\mathscr{F}}(X, A, x_0)$, if and only if there exists a triple (Q, D, F) where Q is an $\mathscr{F}_{n+1}$-pseudodisc, D is an $(n+1)$-simplex of Q, and $F\colon (Q, D) \to (X, x_0)$ is a continuous map such that:

(a) $P \subset \partial Q$, $D \cap P = \Delta$;

(b) $F(\partial Q - \overset{\circ}{P}) \subset A$; and

(c) $(\partial Q - \overset{\circ}{P}, D \cap (\partial Q - \overset{\circ}{P}), F)$ is a cobordism to zero of $\partial(P, \Delta, f)$.

This follows by using Lemma 2.7 and reasoning as in itself.

From the definition of $\mathscr{F}$-cobordism between relative $\mathscr{F}_n$-spheres it follows that a cobordism between (P_1, Δ_1, f_1) and (P_2, Δ_2, f_2) determines a cobordism between the singular $\mathscr{F}_{n-1}$-spheres $\partial(P_1, \Delta_1, f_1)$ and $\partial(P_2, \Delta_2, f_2)$ of (A, x_0). This implies that one can define a map $\partial\colon \Theta_n^{\mathscr{F}}(X, A, x_0) \to \Theta_{n-1}^{\mathscr{F}}(A, x_0)$ by setting $\partial([(P, \Delta, f)]) = [\partial(P, \Delta, f)]$.

It is easy to prove the following.

Theorem 2.12. *∂ is a homomorphism.*

The groups $\Theta_n^{\mathscr{F}}(X, x_0)$ $(n > 2)$ appear as special case of $\Theta_n^{\mathscr{F}}(X, A, x_0)$, with $A = \{x_0\}$. More precisely, we have

Theorem 2.13. *There exists a canonical isomorphism $\phi\colon \Theta_n^{\mathscr{F}}(X, x_0) \to \Theta_n^{\mathscr{F}}(X, \{x_0\}, x_0)$.*

Proof. Let (Σ, D, f) be a singular $\mathscr{F}_n$-sphere of (X, x_0). We consider the relative $\mathscr{F}_n$-sphere $(\Sigma - \overset{\circ}{D}, D', f/)$, where D' is an n-simplex of the collar C on ∂D in $\Sigma - \overset{\circ}{D}$ on which f is constant. If (Σ, D, f) is cobordant to

$(\overline{\Sigma}, \overline{D}, \overline{f})$ and $(W, W', F/)$ is a cobordism between them, then $W - \operatorname{int} W'$ realizes trivially a cobordism between $(\Sigma - \mathring{D}, D', f/)$ and $(\overline{\Sigma} - \mathring{\overline{D}}, \overline{D}', \overline{f}/)$. So it makes sense to define a map $\phi\colon \Theta_n^{\mathscr{F}}(X, x_0) \to \Theta_n^{\mathscr{F}}(X, \{x_0\}, x_0)$, by setting $\phi([(\Sigma, D, f)]) = [(\Sigma - \mathring{D}, D', f/)]$.

An observation is necessary here, in order to simplify the proof. Let (P, Δ, f) be a relative $\mathscr{F}_n$-sphere of $(X, \{x_0\}, x_0)$. Being $f(\partial P) = x_0$, by Lemma 2.7 there exists an $\mathscr{F}$-cobordism (W, W', g) between $\partial(P, \Delta, f)$ and (S^{n-1}, D_0, f_0). Then the triple $(W \cup_{\partial P} P, D_0', g \cup f)$, where D_0' is an n-simplex such that $D_0' \cap S^{n-1} = D_0$ and $g(D_0') = x_0$, is a relative $\mathscr{F}_n$-sphere cobordant to (P, Δ, f). Then for any element of $\Theta_n^{\mathscr{F}}(X, \{x_0\}, x_0)$, we can take a representative triple (P, Δ, f) such that $\partial P = S^{n-1}$. It follows that it makes sense to define a map $\Psi\colon \Theta_n^{\mathscr{F}}(X, \{x_0\}, x_0) \to \Theta_n^{\mathscr{F}}(X, x_0)$, by setting $\Psi([(P, \Delta, f)]) = ([(P \cup c * \partial P, \Delta, \hat{f})])$, where $\partial P = S^{n-1}$ and $\hat{f}$ is the extension of f by the constant map to x_0. It is readily verified that Ψ is the inverse map of ϕ. So ϕ is a bijection.

We only need to prove that ϕ is a homomorphism. Let (Σ_i, D_i, f_i), $i = 1, 2$, be singular $\mathscr{F}_n$-spheres of (X, x_0), and let (Σ, D, f) be a representative element of $[(\Sigma_1, D_1, f_1)] + [(\Sigma_2, D_2, f_2)]$. It is easy to see that $\phi([(\Sigma, D, f)])$ is represented by the triple (P, D, f), where $P = (\Sigma_1 - \mathring{D}_1) \cup (\Sigma_2 - \mathring{D}_2)$ is obtained by identifying a top-dimensional simplex of $\partial(\Sigma_1 - \mathring{D}_1)$ with a simplex of $\partial(\Sigma_2 - \mathring{D}_2)$, D is an appropriate n-simplex of P, and $f = f_1 \cup f_2$. On the other hand, (P, D, f) is also a representative element of $\phi[(\Sigma_1, D_1, f_1)] + \phi[(\Sigma_2, D_2, f_2)]$. The proof of theorem is now complete. □

Given a continuous map $f\colon (X, A, x_0) \to (Y, B, y_0)$, we can define, for each $n > 2$, a homomorphism $\Theta^{\mathscr{F}}(f)\colon \Theta_n^{\mathscr{F}}(X, A, x_0) \to \Theta_n^{\mathscr{F}}(Y, B, y_0)$ by setting

$$\Theta^{\mathscr{F}}(f)([(P, \Delta, g)]) = [(P, \Delta, f \circ g)].$$

Thus the process above described allows us to build a covariant functor $\Theta^{\mathscr{F}}$ from the category of the pointed pairs of topological spaces to the category of graded groups.

Theorem 2.14. *The functor $\Theta^{\mathscr{F}}$ satisfies the following*:

(1) *Homotopy axiom. If $f_1, f_2\colon (X, A, x_0) \to (Y, B, y_0)$ are homotopic, then $\Theta^{\mathscr{F}}(f_1) = \Theta^{\mathscr{F}}(f_2)$.*

(2) *Dimension axiom. $\Theta^{\mathscr{F}}(\{x_0\}, x_0) = 0$.*

(3) *Exactness axiom. For any pointed pair (X, A, x_0), there is an exact sequence*

$$\cdots \to \Theta_{n+1}^{\mathscr{F}}(X, A, x_0) \xrightarrow{\partial} \Theta_n^{\mathscr{F}}(A, x_0) \xrightarrow{i} \Theta_n^{\mathscr{F}}(X, x_0) \xrightarrow{j} \Theta_n^{\mathscr{F}}(X, A, x_0) \to \cdots$$

where, for simplicity, the letter i denotes the inclusion map $(A, x_0) \subset (X, x_0)$ and at the same time the induced map $\Theta^{\mathscr{F}}(i)$ and j denotes the map obtained by composing ϕ with the inclusion $(X, \{x_0\}, x_0) \subset (X, A, x_0)$.

Proof. (1) Homotopy axiom. Let $F\colon (X \times I, A \times I, \{x_0\} \times I) \to (Y, B, y_0)$ be a homotopy between f_1 and f_2, and let (P, Δ, f) be a relative $\mathscr{F}_n$-sphere of (X, A, x_0). Then the triple $(P \times I, \Delta \times I, (F \circ f) \times \mathrm{id})$ is a $\mathscr{F}$-cobordism between $(P, \Delta, f_1 \circ f)$ and $(P, \Delta, f_2 \circ f)$. So $\Theta^{\mathscr{F}}(f_1) = \Theta^{\mathscr{F}}(f_2)$.

(2) Dimension axiom. It follows immediately from the fact that a singular $\mathscr{F}_n$-sphere (Σ, D, f) is cobordant to zero provided that f is constant.

(3) Exactness axiom. $i \circ \partial = 0$. This follows readily by observing that a relative sphere (P, Δ, f) of (X, A, x_0) is a cobordism to zero of $i \circ \partial(P, \Delta, f)$.

$\operatorname{Ker} i \subset \operatorname{Im} \partial$. Let (P, Δ, g) be a cobordism to zero of $i(\Sigma, D, f)$, then we have $(\Sigma, D, f) = \partial(P, \Delta, g)$.

$j \circ i = 0$. Let $[(\Sigma, D, f)]$ be an element of $\Theta_n^{\mathscr{F}}(A, x_0)$. Then

$$j \circ i([(\Sigma, D, f)]) = [(\Sigma - \mathring{D}, D', f/)] \in \Theta_n^{\mathscr{F}}(X, A, x_0).$$

Since $f(\Sigma - \mathring{D}) \subset A$, the relative sphere $(\Sigma - \mathring{D}, D', f/)$ is cobordant to zero.

$\operatorname{Ker} j \subset \operatorname{Im} i$. Let $[(\Sigma, D, f)]$ be an element of $\operatorname{Ker} j$, and let (Q, Δ, f) be a 0-cobordism of $(\Sigma - \mathring{D}, D', f/) = j(\Sigma, D, f)$. There exists an $\mathscr{F}_n$-pseudodisc P such that $\partial Q = P \cup \Sigma - \mathring{D}$ and $F(P) \subset A$. So the singular sphere $i(P \cup c * \partial P, \Delta \cap P, F/)$ is cobordant to (Σ, D, f).

$\partial \circ j = 0$. Let (Σ, D, f) be a singular $\mathscr{F}_n$-sphere of (X, x_0). Then we have $\partial \circ j([(\Sigma, D, f)]) = \partial([(\Sigma - \mathring{D}, D', f/)]) = [(\partial D, D' \cap D, f/)]$. Since f is constant on D, the last class is zero.

$\operatorname{Ker} \partial \subset \operatorname{Im} j$. Let (P, Δ, f) be a relative sphere of (X, A, x_0) such that $\partial(P, \Delta, f)$ is cobordant to zero, and let $(\overline{P}, \overline{\Delta}, g)$ be a 0-cobordism of $\partial(P, \Delta, f)$. The triple $(P \cup \overline{P}, \Delta, f \cup g)$ is a singular sphere of (X, x_0) such that $j([(P \cup \overline{P}, \Delta, f \cup g)]) = [(P, \Delta, f)]$. $\square$

3. The functors $\Theta^{\mathscr{C}}$ and $\Theta^{\mathscr{PL}}$

We now show that the groups $\Theta_n^{\mathscr{F}}(X, A, x_0)$ are actually the homology groups if $\mathscr{F} = \mathscr{C}$, while they coincide with homotopy groups if $\mathscr{F} = \mathscr{PL}$.

For the sake of simplicity, we will prove it only in the case of pointed spaces.

Theorem 3.1. *$\Theta_n^{\mathscr{C}}(X, x_0)$ is isomorphic to $H_n(X, x_0)$ for each $n > 1$.*

Proof. Denote by $\Psi_1 \colon \Theta_n^{\mathscr{C}}(X, x_0) \to H_n(X, x_0)$ the map described as follows. Let α be any element of $\Theta_n^{\mathscr{C}}(X, x_0)$. Choose a representative triple (Σ, D, f) of α, and consider the singular cycle $(\Sigma - \mathring{D}, f)$ of (X, x_0) and its homology class α'. If (Σ, D, f) is cobordant to (Σ', D', f') and (W, W', F) is a cobordism between them, it is easy to see that the singular cycle $(W - \operatorname{int} W', F/)$ is a homology between $(\Sigma - \mathring{D}, f)$ and $(\Sigma' - \mathring{D}', f')$. Hence, we may define Ψ_1 by taking $\Psi_1(\alpha) = \alpha'$.

Ψ_1 is onto. Let (Z, g) be a singular cycle of (X, x_0) and α' its homology class. The polyhedron $\Sigma = Z \cup c * \partial Z$ is a geometric cycle without boundary, that is, a $\mathscr{C}$-sphere. Denote by g' the extension of g to Σ by the constant map and fix a top-dimensional simplex D in ∂Z. If α is the cobordism class of $(\Sigma, c*D, g')$ we have $\Psi_1(\alpha) = \alpha'$; that is, $(\Sigma - (c*D)^0, g'/)$ is homologous to (Z, g). Let $A = \partial Z - \mathring{D}$ and $W = Z \times I \cup_{A \times I} c * (A \times I)$. Then we have that W is a geometric cycle with boundary

$$\begin{aligned} \partial W &= Z \times \{0\} \cup Z \times \{1\} \cup D \times I \cup c * \partial(A \times I) \\ &= [Z \times \{0\} \cup c * (A \times \{0\})] \cup (Z \times \{1\}) \\ &\quad \cup [D \times I \cup c * (A \times \{1\}) \cup c * (\partial D \times I)]. \end{aligned}$$

Define G on W to be $g \times \mathrm{id}$ on $Z \times I$ and the constant map elsewhere. Thus (W, G) realizes the required homology.

Ψ_1 is a homomorphism. Let α_i be an element of $\Theta_n^{\mathscr{C}}(X, x_0)$, $i = 1, 2$, and (Σ_i, D_i, f_i) a representative element of α_i. Given an orientation-preserving PL-homeomorphism h between two top-dimensional simplexes D_1' and D_2' of ∂D_1 and ∂D_2 respectively, we let $C = (\Sigma_1 - \mathring{D}_1) \cup_h (\Sigma_2 - \mathring{D}_2)$. So we have that $(C, g_1/ \cup g_2/)$ is a representative element of the homology class $\Psi_1(\alpha_1 + \alpha_2)$. Now consider the geometric cycle W obtained by gluing the cylinders $(\Sigma_i - \mathring{D}_i) \times [0, 1]$ along $\partial D_i' \times [1/2, 1]$. The relative cycle (W, G), where $G = (g_1/ \times \mathrm{id}) \cup (g_2/ \times \mathrm{id})$, realizes a homology between $(C, g_1/ \cup g_2/)$ and the disjoint union of $(\Sigma_1 - \mathring{D}_1, g_1/)$ and $(\Sigma_2 - \mathring{D}_2, g_2/)$.

Ψ_1 is injective. Let (Σ, D, g) be a singular sphere such that $(\Sigma - \mathring{D}, g/)$ bounds a geometric cycle (P, G), that is, $\partial P = (\Sigma - \mathring{D}) \cup Z$, where Z is a cycle with boundary ∂D, and $G_{/Z}$ is constant. If F denotes the extension of G to $P' = P \cup c * Z$ by the constant map, it is easy to take a top-dimensional simplex Δ in $c * Z$ such that (P', Δ, F) realizes a cobordism to zero of (Σ, D, g). □

Theorem 3.2. *$\Theta_n^{\mathscr{PL}}(X, x_0)$ is isomorphic to $\Pi_n(X, x_0)$ for each $n > 1$.*

Proof. Denote by $\Psi_2 \colon \Pi_n(X, x_0) \to \Theta_n^{\mathscr{PL}}(X, x_0)$ the map described as follows. Let β be any element of $\Pi_n(X, x_0)$. Choose a representative triple (S^n, N, f) of β, where N is the north pole of S^n. Without loss of generality, we can suppose that f is constant on a top-dimensional simplex D_0 containing N. Having chosen an orientation of S^n, the triple (S^n, D_0, f) is a singular sphere of (X, x_0) and hence determines an element β' of $\Theta_n^{\mathscr{PL}}(X, x_0)$. Observing that a homotopy between $(S^n, D_0' f)$ and (S^n, D_0, g) is itself a $\mathscr{PL}$-cobordism between the same elements taken as singular spheres, we conclude that β' does not depend on the choice of the representative element of β. It follows that we can define Ψ_2 by setting $\Psi_2(\beta) = \beta'$.

Ψ_2 is onto. Let (Σ, D, f) be a singular sphere of (X, x_0) and $\varphi \colon S^n \to \Sigma$ an orientation-preserving PL-homeomorphism such that $\varphi(D_0) = D$. In order to show that the singular spheres $(S^n, D_0, f \circ \varphi)$ and (Σ, D, f) are cobordant, we consider a PL-homeomorphism $h \colon (S^n \times I, S^n \times \{0\}, S^n \times \{1\}) \to (C_\varphi, S^n, \Sigma)$ (C_φ is the simplicial mapping cylinder of φ) such that $h_{/S^n \times \{0\}} = \mathrm{id}$ and $h_{/S^n \times \{1\}} = \varphi$. This PL-homeomorphism exists (see [2]). The map $g = [(f \circ \varphi) \times \mathrm{id}] \circ h^{-1} \colon C_\varphi \to X$ agrees with $f \circ \varphi$ and f respectively. Since C_φ is a $\mathscr{PL}$-manifold with boundary $S^n \cup \Sigma$, let $C' = h(D_0 \times I)$; then we have that (C_φ, C', g) is a $\mathscr{PL}$-cobordism between $(S^n, D_0, f \circ \varphi)$ and (Σ, D, f).

Ψ_2 is a homomorphism. To show this it suffices to observe that the sum of the homotopy classes of (S^n, D_0, f) and (S^n, D_0, g) coincides with the homotopy class of $(S^n - \mathring{D}_0 \cup_{\partial D_0} S^n - \mathring{D}_0, D, f/ \cup g/)$, provided we regard $(S^n - \mathring{D}_0 \cup S^n - \mathring{D}_0, D)$ as (S^n, D_0) up to PL-homeomorphisms.

Ψ_2 is injective. Let $[(S^n, D_0, f)]$ be an element of $\operatorname{Ker} \Psi_2$, that is, (S^n, D_0, f) $\mathscr{PL}$-cobordant to zero. Then, using Lemma 2.7, we can extend the map f to a PL-disc. Hence f is homotopic to zero. □

By Theorems 3.1 and 3.2, we can affirm that the homomorphism $\Psi_{\mathscr{PL}, \mathscr{C}}$ of

Theorem 2.9 is the Hurewicz homomorphism. Then for any manifold class $\mathscr{F}$ we have a factorization of the Hurewicz homomorphism. For, since $\mathscr{PL} \subset \mathscr{F} \subset \mathscr{C}$, the following diagram is commutative:

$$(*) \qquad \begin{array}{ccc} \Theta_n^{\mathscr{PL}}(X, x_0) & \xrightarrow{\Psi_{\mathscr{PL},\mathscr{C}}} & \Theta_n^{\mathscr{C}}(X, x_0) \\ & {\scriptstyle \Psi_{\mathscr{PL},\mathscr{F}}} \searrow \quad \nearrow {\scriptstyle \Psi_{\mathscr{F},\mathscr{C}}} & \\ & \Theta_n^{\mathscr{F}}(X, x_0) & \end{array}$$

4. The groups $\Theta_2^{\mathscr{F}}(X, A, x_0)$ and $\Theta_1^{\mathscr{F}}(X, x_0)$

In this section we wish to give a group structure even in $\Theta_2^{\mathscr{F}}(X, A, x_0)$ and $\Theta_1^{\mathscr{F}}(X, x_0)$ so that the resulting groups enjoy the same properties that we have already established for $\Theta_n^{\mathscr{F}}(X, A, x_0)$ and $\Theta_n^{\mathscr{F}}(X, x_0)$. Let $[(\Sigma_i, D_i, f_i)]$ $(i = 1, 2)$ be elements of $\Theta_1^{\mathscr{F}}(X, x_0)$. We may define $\Sigma = (\Sigma_1 - \mathring{D}_1) \cup (\Sigma_2 - \mathring{D}_2)$ and $f = f_1 / \cup f_2 / : \Sigma \to X$ as in the case $n > 1$.

The difficulty lies in the choice of D in the regular nbd of ∂D_i in Σ on which f is constant.

Because ∂D_i is not connected, neither are its regular nbds; therefore, the class of (Σ, D, f) generally depends on D and there are essentially two possible choices of D. Then we need a rule to determine D.

Observe that, being $\mathscr{F}_0 = S^0$, each $\mathscr{F}_1$-sphere is a finite disjoint union of standard 1-spheres. Let S_i be the component of Σ_i containing D_i and $g_i : [0, 1] \to S_i - \mathring{D}_i$ be an orientation-preserving PL-homeomorphism. Then we define $[(\Sigma_1, D_1, f_1)] + [(\Sigma_2, D_2, f_2)] = [(\Sigma, D, f)]$, where D is chosen containing $g_1(0)$.

It is easily checked that $\Theta_1^{\mathscr{F}}(X, x_0)$ is a group under the operation defined above. It need not be abelian, because the choice of D depends on the order of the addenda.

To give a group structure in $\Theta_2^{\mathscr{F}}(X, A, x_0)$, the trouble is again the choice of the simplex Δ and it can be removed reasoning as in $\Theta_1^{\mathscr{F}}(X, x_0)$.

Thus we can define a new functor $\Theta^{\mathscr{F}}$ by setting $\Theta^{\mathscr{F}}(X, A, x_0) = \{\Theta_n^{\mathscr{F}}(X, A, x_0)\}_{n \geq 1}$ ($\Theta^{\mathscr{F}}(X, A, x_0) = \Theta_1^{\mathscr{F}}(X, x_0)$ if $A = \{x_0\}$; it is simply a set otherwise). It satisfies the same properties of the previous one, but the graded group $\Theta^{\mathscr{F}}(X, A, x_0)$ need not be abelian.

In particular, $\Theta^{\mathscr{F}}$ is in fact the homology functor H if $\mathscr{F} = \mathscr{C}$, and it is the homotopy functor Π if $\mathscr{F} = \mathscr{PL}$.

5. Functors $\Theta^{\mathscr{F}}$ different from H and Π

An example of functor $\Theta^{\mathscr{F}}$ different from H and Π can be obtained by taking $\mathscr{F}$ such that $\mathscr{F}_1 = \{S^1\}$ and $\mathscr{F}_2$ is generated by a given orientable surface of genus $g \geq 1$ and the standard 2-sphere under the property to be connected. For, in this case, it is not too hard to show that $\Theta_1^{\mathscr{F}}(X, x_0)$ is abelian, so $\Theta^{\mathscr{F}}$ is not Π. Moreover, if X is not connected, $\Theta_1^{\mathscr{F}}(X, x_0)$ coincides with $\Theta_1^{\mathscr{F}}(K, x_0)$ where K is the path-component of X containing x_0. It follows that $\Theta_n^{\mathscr{F}}$ is not H.

The following example proves that there exist manifold classes $\mathscr{F}$ such that

the groups $\Theta_n^{\mathscr{F}}(X, x_0)$ are different from homology and homotopy groups even if X is a connected topological space.

For each positive integer m, let $\mathscr{F}'(m)$ be a manifold class such that:

$$\Sigma \in \mathscr{F}'(m)_n \Leftrightarrow \begin{cases} \Sigma = S^n & \text{if } n < m; \\ \Sigma \text{ is a finite disjoint union} & \\ \quad \text{of standard } m\text{-spheres} & \text{if } n = m; \\ \Sigma = S^0 * \Sigma',\ \Sigma' \in \mathscr{F}'(m)_m & \text{if } n = m+1. \end{cases}$$

Now we denote by $\mathscr{F}(m)$ the manifold class generated by $\mathscr{F}'(m)$ under the property to be connected. An $\mathscr{F}(m)$-sphere Σ of dimension $m+1$ satisfies the following properties:

(1) Σ is a connected polyhedron.
(2) If Σ is not S^{m+1}, Σ has at least two singular points.
(3) Σ is a finite union of standard $(m+1)$-spheres such that the intersection of any two of them is either the empty set or a finite set.

It is easy to prove the above properties by observing that:

(a) they trivially hold for the spheres of $\mathscr{F}'(m)$ of dimension $m+1$.

(b) if P_1 and P_2 are $\mathscr{F}(m)$-pseudodiscs of dimension $m+1$, such that the spheres $P_i \cup c * \partial P_i$ $(i = 1, 2)$ satisfy (1), (2), and (3), and $\partial P_1 \approx_f \partial P_2$, then the sphere $P_1 \cup_f P_2$ satisfies (1), (2), and (3).

We now show that if X is a topological space such that $\Pi_m(X, x_0) \neq 0$, then the map $\Psi_{\mathscr{PL}, \mathscr{F}(m)}\colon \Theta_m^{\mathscr{PL}}(X, x_0) \to \Theta_m^{\mathscr{F}(m)}(X, x_0)$ is not surjective.

Let $\alpha \neq 0$ be any element of $\Theta_m^{\mathscr{PL}}(X, x_0)$, choose a representative triple (S^m, D, f) of α, and consider the singular $\mathscr{F}(m)$-sphere (Σ, D, g), where $\Sigma = S_1 \dot{\cup} S_2$ $(S_i \approx S^m,\ i = 1, 2)$ and $g = f \cup f$. The class $[(\Sigma, D, g)]$ does not lie in $\operatorname{Im} \Psi_{\mathscr{PL}, \mathscr{F}(m)}$. For, if (Σ, D, g) is cobordant to (S^m, D', h) and (W, W', F) is a cobordism between them, then $\Sigma' = W \cup c * \Sigma \cup c' * S^m$ is an $(m+1)$-dimensional $\mathscr{F}(m)$-sphere, which contains, by (3), a PL-disc Δ such that $\partial\Delta$ coincides with a path-component of Σ and c, c' do not belong to Δ. Then we can suppose $\Delta \subset W$, and hence $F_{/\Delta}$ is an extension of f to a PL-disc. This contradicts the hypothesis on f.

The last assertion shows that the functor $\Theta^{\mathscr{F}(m)}$ is different from the functors H and Π. For, let X be an $(m-1)$-connected topological space $(m > 1)$ such that $\Pi_m(X, x_0)$ is a finite group. By the Hurewicz theorem and the commutativity of the diagram $(*)$, the map $\Psi_{\mathscr{PL}, \mathscr{F}(m)}\colon \Theta_m^{\mathscr{PL}}(X, x_0) \to \Theta_m^{\mathscr{F}(m)}(X, x_0)$ is a monomorphism and it is not onto by the previous arguments. Then, $\Pi_m(X, x_0)$ being a finite group, $\Theta_m^{\mathscr{F}(m)}(X, x_0)$ is isomorphic neither to $\Pi_m(X, x_0)$ nor to $H_m(X, x_0)$.

We close this section observing that there is not an analogue of the Hurewicz theorem for any two functors $\Theta^{\mathscr{F}}$, $\Theta^{\mathscr{F}'}$ with $\mathscr{F}' \subset \mathscr{F}$.

From the above observation it follows that there is not a generalization of the Hurewicz theorem to the functors $\Theta^{\mathscr{PL}}$, $\Theta^{\mathscr{F}(m)}$. This also does not occur for the functors $\Theta^{\mathscr{F}(m)}$, $\Theta^{\mathscr{C}}$. For, let X be an $(m-1)$-connected topological space $(m > 1)$ such that $\Pi_m(X, x_0) \neq 0$. As an immediate consequence of the definition of $\mathscr{F}(m)$, we have $\Theta_h^{\mathscr{PL}}(X, x_0) \approx \Theta_h^{\mathscr{F}(m)}(X, x_0)$ for any X and for any $h < m-1$, and it is not difficult to see that this relation holds even if $h = m-1$. Then, by hypothesis on X, $\Theta_h^{\mathscr{F}(m)}(X, x_0) = 0$ for any

$h \leq m-1$. Since $\Psi_{\mathscr{PL},\mathscr{C}}: \Theta_m^{\mathscr{PL}}(X, x_0) \to \Theta_m^{\mathscr{C}}(X, x_0)$ is an isomorphism and $\Psi_{\mathscr{PL},\mathscr{F}(m)}: \Theta_m^{\mathscr{PL}}(X, x_0) \to \Theta_m^{\mathscr{F}(m)}(X, x_0)$ is not surjective, the commutativity of the diagram $(*)$ assures that $\Psi_{\mathscr{F}(m),\mathscr{C}}: \Theta_m^{\mathscr{F}(m)}(X, x_0) \to \Theta_m^{\mathscr{C}}(X, x_0)$ is not an isomorphism.

References

1. G. A. Anderson, *Resolution of generalized polyhedral manifolds*, Tôhoku Math. J. (2) **31** (1979), 495–517.
2. S. Buoncristiano, C. P. Rourke, and B. J. Sanderson, *A geometric approach to homology theory*, Cambridge Univ. Press, London and New York, 1976.
3. M. M. Cohen, *Simplicial structures and transverse cellularity*, Ann. of Math. (2) **85** (1967), 218–245.
4. S. Dragotti, R. Esposito, and G. Magro, *$\mathscr{F}$-varietà e funzioni dual cellulari*, Ricerche Mat. **39** (1990), 21–33.
5. S. T. Hu, *Homotopy theory*, Academic Press, New York and London, 1959.
6. C. P. Rourke and B. J. Sanderson, *Introduction to PL topology*, Ergeb. Math. Grenzgeb. (3), bd. 69, Springer-Verlag, Berlin and New York, 1972.
7. ——, *A geometric approach in homology theory*, notes, Warwick Univ., Coventry, 1971.
8. H. Seifert and W. Threlfall, *Lehrbuch der topologie*, Teubner Verlagsgesellschaft, Leipzig, 1934.
9. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

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