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# LEAST AREA TORI IN 3-MANIFOLDS

G. P. SCOTT AND G. A. SWARUP

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ABSTRACT. We consider incompressible maps of the torus into a 3-manifold that have least possible area among all such maps. We show that such a map must be an embedding in many cases.

Let $M$ be a compact orientable irreducible 3-manifold equipped with a Riemannian metric such that the boundary of $M$ is sufficiently convex. (See [HS, p. 110] for the definition of this term. This is Condition C of [MY2].) Any manifold admits such a metric. The purpose of this condition is simply to ensure the existence of least area surfaces in $M$. Let $F$ be a compact orientable surface, not the 2-sphere. A proper map $f: F \to M$ is *incompressible* if it induces an injective map of fundamental groups. An incompressible map of $F$ into $M$ is called *essential* if it is not properly homotopic into the boundary of $M$. (There are two different ideas of essentiality, of which this is the simpler one. For the purposes of this paper, the given definition will be convenient.) The condition of sufficient convexity on the boundary of $M$ ensures that any incompressible map of a closed surface and any essential map of the annulus into $M$ is properly homotopic to a smooth map of least possible area and that such a map is an immersion [SY, MY3, N1]. A map of $F$ into $M$ that is least area in its proper homotopy class is called simply *least area*. A map of $F$ into $M$ that has least area among all incompressible maps of $F$ will be called *absolutely least area*. A map of $F$ into $M$ that has least area among all essential maps of $F$ will be called an *absolutely least area essential* map. In this paper, we consider the questions of when an absolutely least area map or absolutely least area essential map of the torus into $M$ must be an embedding.

In [N2] Nakauchi considered the analogous question when the surface involved is an annulus. Nakauchi proved that any absolutely least area essential map of the annulus into $M$ is an embedding or a double covering of a 1-sided embedded Moebius band. Unfortunately, the corresponding statement for absolutely least area maps of the torus is false. For there are non-Haken Seifert fibre spaces that admit essential maps of the torus, but such a manifold does not admit any incompressible embedding of any surface. There are also other counterexamples. We are grateful to Max Neumann-Coto for pointing out the following example. Let $T$ denote the 2-torus and let $M$ be obtained from

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$T \times I$ by glueing the ends by a periodic homeomorphism of order 3 or 6. The product fibration of $T \times I$ by intervals yields a Seifert fibration of $M$ in which $T_0 = T \times \{0\}$ is a horizontal embedded incompressible torus and any incompressible torus embedded in $M$ is homotopic to $T_0$. In particular, no vertical torus in $M$ can be embedded. Choose a metric on $T$ invariant under the periodic homeomorphism and choose a product metric on $T \times I$. By choosing the $I$ factor to be very short, we can ensure that $M$ contains a (nonembedded) vertical torus of less area than $T_0$ so that $T_0$ is not absolutely least area. Thus an absolutely least area torus in $M$ cannot be embedded. But by choosing the $I$ factor to be very long, we can ensure that $T_0$ is absolutely least area in $M$, so that the answer to our question on the embeddedness of an absolutely least area torus in such a manifold depends on the choice of metric.

We obtain two positive results on this question.

**Theorem 3.** *Let $M$ be a compact orientable irreducible 3-manifold equipped with a Riemannian metric such that the boundary of $M$ is sufficiently convex. Suppose that $M$ is not a closed Seifert fiber space. Then an absolutely least area map of the torus into $M$ must be an embedding or a double covering of a 1-sided embedded Klein bottle.*

**Theorem 4.** *Let $M$ be a compact orientable irreducible 3-manifold equipped with a Riemannian metric such that the boundary of $M$ is sufficiently convex. Suppose that $M$ is not a Seifert fiber space. Then an absolutely least area essential map of the torus into $M$ must be an embedding or a double covering of a 1-sided embedded Klein bottle.*

If $M$ admits no incompressible map of the torus then both results are vacuously true. If $M$ does admit an incompressible map of the torus then either $M$ admits an embedded incompressible torus or $M$ is a Seifert fiber space. This is a consequence of the recent completion of the characterisation of Seifert fiber spaces by Casson and Jungreis [CJ] and Gabai [G], which was based on earlier work of Mess [M] and the author [Sc2]. It also uses the version of the Torus Theorem proved by Scott in [Sc1]. Casson [C] has given a simple proof of this result using least area surfaces. This version of the Torus Theorem implies that if $M$ admits an incompressible map of the torus, then either $M$ admits an embedded incompressible torus or $M$ is closed and $\pi_1(M)$ has an infinite cyclic normal subgroup. The work of Mess [M], combined with the work of Casson and Jungreis [CJ] or Gabai [G], implies that $\pi_1(M)$ is isomorphic to the fundamental group of a Seifert fiber space $M'$, and the author's work [Sc2] shows that $M$ and $M'$ must be homeomorphic, completing the proof that $M$ must be a Seifert fiber space.

It follows from the above discussion that, in both the above theorems, we can assume that $M$ has nonempty characteristic submanifold $V(M)$ and that $V(M)$ has nonempty boundary. In particular, $M$ is Haken. For each component of $\partial V(M)$ choose a least area map of the torus into $M$ homotopic to the inclusion map of the component. Let $K$ denote the collection of these least area tori. Theorem 5.1 of [FHS] shows that each torus in $K$ is embedded in $M$ or double covers a 1-sided Klein bottle. The second case can only occur if the corresponding component of $\partial V(M)$ bounds the orientable twisted $I$-bundle over the Klein bottle. Further Theorem 6.2 of [FHS] shows that two tori in

$K$ are disjoint or coincide. Thus, if $N$ denotes a regular neighborhood of the union of the tori in $K$, the components of $M - \text{int}(N)$ include components isotopic to each component of $V(M)$ except possibly for those components of $V(M)$ homeomorphic to $T \times I$ or to the orientable twisted $I$-bundle over the Klein bottle. Now any incompressible map of the torus into $M$ is homotopic into $V(M)$ [JS, Jo]. Thus any incompressible map of the torus into $M$ can be homotoped to be disjoint from $\partial V(M)$. Theorem 6.2 of [FHS] now implies that any least area torus in $M$ is disjoint from the tori in $K$ or coincides with one of them. It follows that any least area torus in $M$ either lies in a component of $M$-$K$ homeomorphic to a component of $V(M)$ or is homotopic to a covering of a component of $K$. Hence, in proving Theorems 3 and 4, we need only prove results about least area tori in Seifert fibre spaces. In fact, the following technical result implies both theorems.

**Theorem 2.** *Let $M$ be a compact orientable Seifert fibre space with nonempty boundary $\partial M$, equipped with a Riemannian metric such that $\partial M$ is sufficiently convex. Let $\Sigma$ be a (possibly empty) union of components of $\partial M$ such that $\Sigma$ is not equal to $\partial M$. If $f\colon T \to M$ is least area among all incompressible maps of $T$ into $M$ that are not homotopic into $\Sigma$ then $f$ is an embedding or a double covering of a 1-sided embedded Klein bottle.*

Theorem 3 follows by applying Theorem 2 to the Seifert fibre space $V(M)$ in the case where $\Sigma$ is empty. Note that with the metric that $V(M)$ inherits from $M$ cut along $K$, its boundary is minimal and hence sufficiently convex. Theorem 4 follows by applying Theorem 2 to $V(M)$ with $\Sigma$ consisting of those components of $\partial V(M)$ that are boundary components of $M$. In both cases $\Sigma$ will not equal $\partial V(M)$.

*Remarks.* There are exactly analogous results for each of the above three theorems if one triangulates $M$ and uses PL-area [JR] instead of smooth area.

Theorem 2 is easy to prove when the metric on $M$ is geometric in the sense of Thurston. See [Sc3]. This is discussed in more detail in §1.

As a final remark, we note that our results do not answer the analogous question in the case when $M$ is nonorientable. This is because the theory of 1-sided least area surfaces is not nearly so well understood as that for 2-sided least area surfaces. See [HR] for the theory of 1-sided shortest curves on surfaces.

## 1. The case of a geometric metric

In this section, we justify the claim made above that Theorem 2 is easy to prove when $M$ has a geometric metric. To say that a manifold $M$ without boundary has a geometric metric means that $M$ has a geometric structure modelled on one of the eight geometries $H^3$, $E^3$, $S^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, $\mathrm{SL}_2\,\mathbb{R}$, or Sol. See [Sc3]. This in turn means that $M$ is the quotient of one of these eight Riemannian manifolds by the action of a discrete group of isometries acting as a covering group. If $M$ has boundary, then a geometric metric on $M$ is a metric obtained by embedding $M$ in its interior $N$, using an embedding that is the identity outside a collar of $\partial M$, and choosing a geometric structure on $N$. If $M$ is a Seifert fiber space without boundary then it cannot have a geometric structure modelled on $H^3$ or Sol unless its fundamental group is $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. In this case, $M$ also admits a geometric structure modelled on one of the other geometries. If $M$ has a geometric structure modelled on $S^3$, it must have finite fundamental group as it is covered by $S^3$. This means that there cannot be an incompressible map of the torus into $M$. Thus in the proof of Theorem 1 below, we can assume that the geometric structure on $M$ is modelled on one of $E^3$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, $\mathrm{SL}_2\,\mathbb{R}$. Note that Theorem 1 is stronger than Theorem 2.

**Theorem 1.** *Let $M$ be a compact orientable Seifert fibre space equipped with a geometric metric such that $\partial M$ is sufficiently convex. Let $\Sigma$ denote some union of the components of $\partial M$. Let $f\colon T \to M$ be least area among all incompressible maps of $T$ into $M$ that are not homotopic into $\Sigma$.*

*Then one of the following occurs*:

(a) *$f$ is an embedding or a double covering of a* 1*-sided embedded Klein bottle.*

(b) *$\Sigma$ equals $\partial M$ and $M$ has base orbifold, which is one of the following*:

(i) *$S^2$ with three cone points,*
(ii) *$D^2$ with two cone points,*
(iii) *the annulus with one cone point,*
(iv) *the disc with two holes.*

*Remarks.* In case (b) any incompressible torus embedded in $M$ is parallel to a component of $\partial M$. Thus cases (a) and (b) are mutually exclusive.

The examples of Neumann-Coto can all be given a flat metric. They will give rise to case (b)(i) of Theorem 1.

*Proof.* Hass [H] showed that any least area torus in $M$ must be horizontal or vertical. Suppose first that our least area torus $f$ is vertical. Vertical means that the torus is a union of fibres of $M$ and hence has no general position triple points. Note that the loop in the base orbifold $B$ of $M$ which is the image of $f(T)$ must be orientation preserving. For an orientation reversing loop in $B$ has a vertical Klein bottle above it in $M$. If $f$ is not a covering of an embedded surface in $M$, choose a double curve $C$ of $f$ and consider the possible cut and paste operations along $C$. One such operation yields a single vertical torus in $M$. The other operation yields either two vertical tori in $M$ or two vertical Klein bottles in $M$. Note that a vertical Klein bottle in $M$ must be incompressible. For otherwise, there is an orientation reversing loop in $B$ of finite order, which implies that $B$ is the projective plane with no cone points and hence that $M$ has finite fundamental group. But this would contradict our hypothesis that $M$ admits an incompressible map of the torus.

First we will show that the case when two Klein bottles are obtained cannot occur. After rounding the corner along $C$, we will obtain two Klein bottles $K_1$ and $K_2$, possibly singular, whose area together is strictly less than the area of $f$. It follows that one of them, say $K_1$, has area strictly less than half the area of $f$, and hence that the area of the torus $T_1$ which double covers $K_1$ is strictly less than that of $f$. Now $K_1$ and hence $T_1$ must be incompressible, so that $T_1$ must be homotopic into $\Sigma$. In particular, $T_1$ is homotopic into the boundary $\partial M$ of $M$. Let $M_1$ denote the cover of $M$ with $\pi_1(M_1) = \pi_1(K_1)$. Then $K_1$ lifts into $M_1$ and the lift of $T_1$ into $M_1$ is homotopic into $\partial M_1$. Since $\pi_1(T_1)$ is the maximal abelian subgroup of $\pi_1(M_1)$, it follows that $\pi_1(T_1)$ equals the fundamental group of a boundary component $T_2$ of $M_1$. Now $\pi_1(T_2)$ is not a maximal surface group in $\pi_1(M_1)$ since $\pi_1(M_1)$ is isomorphic to the fundamental group of the Klein bottle. It follows from [Sc4, Sc5] that $T_2$ bounds a compact submanifold $X$ of $M_1$ which is homeomorphic to the twisted $I$-bundle over the Klein bottle. As $T_2$ is a boundary component of $M_1$, we must have $M_1$ equal to $X$. Hence the projection map $\partial M_1$ to $\partial M$ has degree one. This implies that the projection map $M_1$ to $M$ has degree one, and so $M$ also is homeomorphic to the twisted $I$-bundle over the Klein bottle. Now it follows that any incompressible torus in $M$ is homotopic to a cover of the embedded one-sided Klein bottle in $M$ and hence that any absolutely least area torus in $M$ must be embedded or double cover a one-sided Klein bottle. This yields the desired contradiction.

Now we consider the case when both the cut and paste operations along $C$ yield tori. Each of these tori has less area than $f$ after rounding the corner along $C$. Thus each must be compressible in $M$ or homotopic into $\Sigma$. As each is vertical, it determines an element in the fundamental group of the base orbifold $B$ and this element is of finite order or is peripheral. As $f$ is incompressible, the element determined by $f$ has infinite order. Thus we have elements $x$, $y$, and $z$ in the fundamental group of $B$ such that $xy = z$, $xy^{-1}$ has infinite order, and each of $x$, $y$, and $z$ has finite order or is peripheral. The subgroup generated by $x$, $y$, and $z$ determines an orbifold covering space $B_1$ of $B$, and the conditions on $x$, $y$, and $z$ imply that $B_1$ is $S^2$ with three cone points, or $D^2$ with two cone points, or the annulus with one cone point, or the disc with two holes or that $B_1$ has infinite cyclic fundamental group. In the first four cases, one can show easily, by using the orbifold Euler number, that $B$ must also be one of these four types of orbifold, so that we have case (b) of Theorem 1. Suppose now that $B_1$ has cyclic fundamental group. As one of $x$ and $y$ must be nontrivial, it follows that one of the tori obtained from $f$ by cut and paste along $C$ is incompressible and so must be homotopic into $\Sigma$. But as $B_1$ has cyclic fundamental group, this torus and the torus $f$ must both be coverings of a single torus in $M$. It follows that $f$ is homotopic into $\Sigma$, which contradicts our hypothesis on $f$ and so completes the proof of Theorem 1 in the case when $f$ is vertical.

Now suppose that $f$ is horizontal. This means that $f$ is transverse to the fibres of $M$. The only Seifert fibre spaces that admit a horizontal torus are those that are modelled on $E^3$ and so have a flat metric.

If $M$ is the 3-torus then any incompressible torus in $M$ is homotopic to a cover of an embedded flat torus. Also any flat torus $T$ in $M$ must be least area. This can be seen by considering the cover $M_T$ of $M$ with $\pi_1(M_T) = \pi_1(T)$ and the lift $T'$ of $T$ into $M_T$ and by using the fact that the orthogonal projection $M_T \to T'$ decreases the area of any torus homotopic to $T'$ unless it is parallel to $T'$ and hence also flat. It follows that any least area torus in $M$ is a cover of an embedded flat torus. Hence our absolutely least area map $f$ must be an embedding and we are in case (a).

If $M$ is not the torus, then the base orbifold of $M$ must either be $S^2$ with three cone points so that we are in case (b)(i), or it must be $S^2$ with four cone points each with cone angle $\pi$ or the projective plane $P^2$ with two cone points each with cone angle $\pi$. In the case of $S^2$ with four cone points, $M$ is the quotient of Euclidean 3-space $E^3$ by a group of isometries whose elements are translations and screw motions with rotation angle equal to $\pi$. Furthermore, the screw motions have parallel axes. Let $v$ denote a vector in the direction of these axes. Thus any line in $E^3$ parallel to $v$ projects to a fibre of $M$. Now any least area torus in $M$ is flat by the same argument as we used to show that any least area torus in the 3-torus is flat. Hence the preimage in $E^3$ of a least area torus in $M$ consists of flat planes. A plane orthogonal to $v$ projects to an embedded horizontal flat torus $T_0$ in $M$ and any other nonvertical plane that projects to a torus in $M$ projects to one of area strictly greater than that of $T_0$. As $f$ is an absolutely least area torus in $M$, it follows that $f$ must be a flat torus parallel to $T_0$ and, in particular, that $f$ is an embedding. A similar argument shows that $f$ must be an embedding in the case when the base orbifold of $M$ is $P^2$ with two cone points. This completes the proof of Theorem 1.

## 2. The general case

Before embarking on the proof of Theorem 2, it is worth remarking that the methods of §1 apply to any situation in which the least area map being considered is vertical and has no general position triple points. In this section, we show how to reduce to this case by using a tower argument. Note that if the least area map were horizontal then $M$ would be closed, a case that is excluded by the assumptions of Theorem 2.

*Proof of Theorem* 2. Choose a Seifert fibration of $M$ and let $K$ denote the cyclic subgroup of $\pi_1(M)$ carried by any regular fibre. We start by observing that the methods of Lemmas 1.3 and 1.6 of [FHS] show that we can assume that the smooth immersion $f$ is in general position. Thus we can take a regular neighborhood $N$ of $f(T)$ in $M$. Let $X$ be a component of the closure of $M$-$N$. Then $X$ is irreducible, as $M$ is irreducible and $f(T)$ cannot lie in a ball in $M$. Lemma 1.4(i) of [HRS] implies that the natural map $\pi_1(X) \to \pi_1(M)$ is injective. Note that we cannot deduce that $X$ is a handlebody, as in Lemma 1.4(ii) of [HRS], because $M$ is Haken as $\partial M$ is nonempty. There are two cases. If the image of $\pi_1(X)$ in $\pi_1(M)$ intersects $K$ nontrivially, then $\pi_1(X)$ has an infinite cyclic normal subgroup. This implies that $X$ must be a Seifert fibre space [W2, GH]. Otherwise the image of $\pi_1(X)$ projects injectively into the quotient group $\pi_1(M)/K$, which is the fundamental group of the base orbifold of $M$. Now a torsionfree subgroup of the fundamental group of a 2-dimensional orbifold with nonempty boundary must be free. Thus it follows in this second case that $\pi_1(X)$ is free, so that $X$ must be a handlebody. Note that the solid torus is both a handlebody and a Seifert fibre space.

Let $\widehat{N}$ denote the union of $N$ and of all those components of the closure of $M$-$N$ that are handlebodies. (This includes any components that are balls.) Each component $X$ of the closure of $M$-$\widehat{N}$ is a Seifert fibre space that cannot be a solid torus. Hence $\partial X$ is incompressible in $X$. As $\pi_1(X)$ injects into $\pi_1(M)$, it follows that each component of $\partial X$ is incompressible in $M$. Hence each component of $\partial\widehat{N}$ is incompressible in $M$. It follows that each component of $\partial\widehat{N}$ is isotopic to a horizontal or vertical torus in $M$ [W3, H]. As $M$ is not closed, by assumption, it cannot contain a horizontal torus. Hence $\widehat{N}$ itself is isotopic to a sub-Seifert fibre space of $M$ and the natural map $\pi_1(\widehat{N}) \to \pi_1(M)$ is injective.

For each component $S_i$ of $\partial N$, we consider the area $A_i$ of the projection of $S_i$ onto $f(T)$. Let $A(f)$ denote the area of $f(T)$. Then the sum of the $A_i$'s must equal exactly $2A(f)$. If $f$ is not an embedding, we can choose $N$ thin enough so that the area of each $S_i$ is strictly less than $A_i$ [MY3]. We will assume that this has been done. We will prove later that $A_i \leq A(f)$ for each $i$. For the moment, we will assume that this is the situation.

*Case* 1. Each $A_i \leq A(f)$. It follows that either $f$ is an embedding or each component of $\partial \widehat{N}$ has area strictly less than $A(f)$. If $f$ is not an embedding, we deduce that each component of $\partial \widehat{N}$ is parallel to a component of $\Sigma$ by our least area hypothesis on $f$. This implies that either $\widehat{N}$ lies in a collar neighborhood of a component of $\Sigma$ or that $\Sigma$ equals all of $\partial M$. Now we assume that $f$ was not homotopic into $\Sigma$ and that $\Sigma$ was not equal to $\partial M$, so the assumption that $f$ is not embedded leads to a contradiction. Thus, in Case 1, we deduce that $f$ must be an embedding as required.

It remains to show that $A_i \leq A(f)$, for each $i$.

*Case* 2. $A_i > A(f)$ for some $i$. If $A_i > A(f)$ then the projection of $S_i$ to $f(T)$ must be noninjective. In fact, there must be a component $D$ of $f(T)$ with its double curves removed such that $S_i$ projects onto $D$ twice. Let $a$ and $b$ be points of $S_i$ with the same image in the interior of $D$. Then there is a short path from $a$ to $b$ that intersects $D$ transversely in one point. We can also join them by a path in $S_i$. Taken together these paths yield a loop in $\widehat{N}$ that meets $f(T)$ transversely in one point. Thus the natural map $H_1(T) \to H_1(\widehat{N})$ is not onto, where we will use $\mathbb{Z}_2$ coefficients throughout. Note that any element of $H_1(\widehat{N})$ that lies in the image of $H_1(T)$ must have intersection number with $f(T)$, which is zero $\bmod 2$. See [FHS, Lemma 2.2].

It will be convenient to relabel $f$, $M$, and $N$ as $f_0$, $M_0$, and $N_0$. Let $M_1$ be a double cover of $\widehat{N}_0$ to which $f_0$ lifts. As $\widehat{N}_0$ is a Seifert fibre space, so is $M_1$. We denote the lift of $f_0$ by $f_1$ and let $N_1$ be a regular neighborhood of $f_1(T)$ in $M_1$. We construct a sub-Seifert fibre space $\widehat{N}_1$ of $M_1$ in the same way that we constructed $\widehat{N}_0$. We repeat this construction, stopping when we obtain $f_k\colon T \to M_k$ such that each component of $\partial N_k$ has area less than or equal to $A(f_k)$. This must eventually happen by the usual argument [He, 48] as the double cover $M_i \to \widehat{N}_{i-1}$ must induce a nontrivial double cover of $N_{i-1}$. This is because the natural map $H_1(N_{i-1}) \to H_1(\widehat{N}_{i-1})$ is onto, which is because $\widehat{N}_{i-1}$ consists of $N_{i-1}$ with handlebodies attached. Let $p_i$ denote the composite projection $\widehat{N}_i \to M_0$. This map induces an injective map on fundamental groups, for each $i$.

Now we consider the top of the tower. We will argue as in Case 1. Suppose that $f_k$ is not an embedding. As $f_k$ is at the top of the tower, it follows that each component of $\partial \widehat{N}_k$ has area strictly less than $A(f_k)$, which equals $A(f_0)$. Hence when projected into $M_0$, each component of $\partial \widehat{N}_k$ is homotopic into $\Sigma$. It follows that $p_k$ is homotopic to a proper map. As $p_k$ injects $\pi_1(\widehat{N}_k)$ into $\pi_1(M_0)$, it follows that either $p_k$ is homotopic into $\Sigma$ or that $p_k$ is homotopic to a covering map [W1]. Thus either $p_k(\widehat{N}_k)$ can be homotoped into $\Sigma$ or $p_k(\partial \widehat{N}_k)$ equals $\partial M_0$. As in Case 1, this implies that either $f_0$ is homotopic into $\Sigma$ or that $\Sigma$ equals $\partial M_0$, which is impossible. We conclude that $f_k$ must be an embedding, as required.

Now we will consider $f_{k-1}$ and show that Case 2 leads to a contradiction. As $f_{k-1}$ is not at the top of the tower, there is a loop $\lambda$ in $\widehat{N}_{k-1}$ meeting $f_{k-1}(T)$ transversely in one point. In particular, $f_{k-1}$ is not an embedding. As $f_k$ is an embedding and $M_k$ is a double cover of $\widehat{N}_{k-1}$, it follows that $f_{k-1}$ has double curves but has no triple points. Thus we can choose any double curve $C$ of $f_{k-1}(T)$ and alter $f_{k-1}$ by performing a cut and paste along $C$ and smoothing the angle along $C$ to obtain an immersion $g$ of the torus $T$ into $\widehat{N}_{k-1}$ that has area strictly less than $A(f_0)$ and meets the loop $\lambda$ in $\widehat{N}_{k-1}$ transversely in a single point. It follows that $g$ is an incompressible map of the torus $T$ into $\widehat{N}_{k-1}$ and that this map is not homotopic into $\partial\widehat{N}_{k-1}$. The composite $p_{k-1}\circ g\colon T \to M_0$ is an incompressible map of the torus into $M$ with area strictly less than $A(f_0)$. This implies that $p_{k-1}\circ g$ is homotopic into $\Sigma$, by the least area property of $f_0$. But this implies that $p_{k-1}\circ g$ is homotopic into $\partial\widehat{N}$ and hence that $g$ is homotopic into $\partial\widehat{N}_{k-1}$, which is the required contradiction.

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Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109
*E-mail address*: peter.scott@um.cc.umich.edu

Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia