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GENUS TWO HEEGAARD SPLITTINGS

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ABSTRACT. It is shown that a 3-manifold has a finite number of genus two Heegaard splittings. A corollary is that the mapping class group is finite if the manifold is non-Haken.

This paper explores a relation between a rigidity property in the theory of minimal surfaces and the problem of classifying Heegaard surfaces and diffeomorphisms of 3-dimensional manifolds.

Theorem 4 states that a closed 3-manifold has a finite number of genus two Heegaard splittings up to homeomorphism. This result contrasts with a recent result of Sakuma, who found a closed 3-manifold that possesses an infinite number of genus two Heegaard splittings, distinct up to isotopy [S]. In fact, Sakuma found infinitely many such manifolds. Since genus two manifolds have geometric decompositions [Th2] and the genus two Seifert fiber spaces and connect sums of lens spaces are known to have only finitely many distinct Heegaard splittings of genus two, it is a consequence of Theorem 4 that only manifolds with nontrivial torus decompositions can have an infinite number of genus two Heegaard splittings that are distinct up to isotopy. By combining Theorem 4 with some recent work of Hass-Scott [HS], we obtain Corollary 8, which states that the mapping class group of a non-Haken irreducible 3-manifold of genus two is finite.

A *minimal embedding* of a 2-dimensional surface in a Riemannian 3-manifold is an embedding with mean curvature zero. There is no known one-parameter family of minimal embeddings in a negative curvature 3-manifold, and it is reasonable to guess that the space of minimal embeddings of a given genus surface is finite. This is true for a generic metric of negative curvature by results of White [Wh], since if false, there would exist in M a minimal embedding that has a nontrivial Jacobi field, and White's result rules this out for generic metrics. Closely related is the following conjecture of Waldhausen [W], which we will solve for genus two.

Conjecture 1. *For each $g \in Z_+$, a closed 3-manifold has finitely many Heegaard splittings of genus g up to homeomorphism.*

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A *Heegaard splitting* of a closed orientable 3-manifold M is a decomposition of M into two handlebodies. More precisely, it is a triple (F, H_1, H_2) where F is an embedded connected orientable surface in M, H_1, and H_2 are handlebodies embedded in M, $M = H_1 \cup H_2$, and $H_1 \cap H_2 = F = \partial H_1 = \partial H_2$. F is called a *Heegaard surface.* We say that two Heegaard splittings of M, (F, H_1, H_2), and (F', H_1', H_2'), are *equivalent* if there is a homeomorphism $f: M \to M$ such that $f(F) = F'$, $f(H_1) = H_1'$, and $f(H_2) = H_2'$, and similarly that (F, H_1, H_2) and (F', H_1', H_2') are *equivalent up to isotopy* if in addition f is isotopic to the identity. Any closed orientable 3-manifold admits a Heegaard splitting. Of particular interest are the splittings with Heegaard surface of smallest possible genus. The genus of such a splitting is called the *genus of* M.

Let M be a Riemannian 3-manifold and let k be a positive integer. Let $E_k(M)$ denote the space of minimal embeddings of a surface of genus k together with the space of minimal immersions of a surface of genus k, which double cover an embedded minimal nonorientable surface. Let $I_k(M)$ denote the space of minimal immersions of a surface of genus k. We work with the C^∞-topology on the space of immersed minimal surfaces, so that a sequence of surfaces $\{S_i\}$ converges to S if for each $p \in S$ there is a coordinate neighborhood U_p of p in M such that each of the S_i is represented by a smooth graph in U_p, converging smoothly to S. $E_k(M)$ is a closed subset of $I_k(M)$, since limits of smooth minimal embeddings are either embeddings or immersions that double cover an embedded surface.

The following lemma gives a compactness statement for the space of genus two and three minimal surfaces in certain manifolds. The proof of Lemma 2 is essentially contained in work of Uhlenbeck [U] and Choi-Schoen [CS].

Lemma 2. *Let M be a closed 3-manifold with negative sectional curvature.*

(a) *$E_2(M)$ and $I_2(M)$ are compact.*
(b) *If there does not exist an embedded nonorientable surface of genus 3 in M then $E_3(M)$ is compact.*

This should be compared to the results of Choi-Schoen, who showed that $E_k(M)$ is compact for all k if M is a closed simply connected 3-manifold with positive Ricci curvature. A similar argument also appears in [A, Corollary 4.3].

Before proving Lemma 2 we state a well-known compactness result.

Lemma 3. *Let $\{S_i\}$ be a sequence of smooth immersed minimal surfaces of genus g in a compact Riemannian manifold. Suppose that the area and second fundamental forms of $\{S_i\}$ are uniformly bounded. Then a subsequence converges smoothly to a minimal immersion.*

Proof of Lemma 3. This standard result follows from Ascoli's theorem when the surfaces are expressed as graphs in local charts.

Proof of Lemma 2. Since M is a closed 3-manifold with negative sectional curvature, there is a negative constant K_0 such that the sectional curvatures K_M of M satisfy $K_M \leq K_0$. Let $\{F_i\}$ be a sequence of minimal embeddings of genus g. Then the Gauss-Bonnet theorem states that $\int_{F_i} K = 2\pi(2-2g)$ where K is the curvature of the induced metric on F_i. Let λ_1 and λ_2 be the principle normal curvatures of F_i. The mean curvature H is given by $H = \lambda_1 + \lambda_2 = 0$, and K is given by $K = K_M + \lambda_1\lambda_2$ by Gauss's theorem. We then have that $K = K_M + \lambda_1\lambda_2 \leq K_0 - (\lambda_1)^2 \leq K_0 < 0$, so that

$$2\pi(2-2g) = \int_{F_i} K \leq \int_{F_i} K_0 = \text{Area}(F_i)K_0 \quad \text{and} \quad \text{Area}(F_i) \leq 2\pi(2-2g)/K_0.$$

It is proved in [CS] that an L^2-bound on the second fundamental form of a sequence of minimal surfaces $\{F_i\}$ implies that the pointwise norm of the second fundamental form is uniformly bounded away from a neighborhood of a finite number of isolated points, where handles are pinching off in the limit. This implies the existence of minimal immersions of smaller genus, which give rise to the original surfaces F_i by a process of taking a number of copies of the boundary of a regular neighborhood and tubing them together with at least one tube. In negative curvature manifolds, no minimal immersions of nonnegative Euler characteristic exist, implying that the second fundamental form of a sequence of genus two surfaces $\{F_i\}$ is uniformly bounded. For genus three embeddings, we have to assume there is no minimal embedding homeomorphic to $P^2 \sharp P^2 \sharp P^2$ in M to get the same conclusion, as the boundary of a regular neighborhood of such a surface is a genus two surface, which becomes genus three with the addition of a tube. So in each case we can conclude that the second fundamental form and the area of the sequence of surfaces $\{F_i\}$ are uniformly bounded, and so a subsequence converges to a minimal immersion by Lemma 3. A minimal surface that is the limit of a sequence of minimal embeddings is either embedded or a double covering of an embedding. This proves Lemma 2.

We now apply Lemma 2 to obtain a finiteness result for the number of Heegaard splittings of certain 3-manifolds.

Theorem 4. *A closed* 3*-manifold* M *has a finite number of genus* 2*-Heegaard splittings.*

Proof. If the Heegaard genus of M is zero or one, then the conclusion follows from the classification theorems of Waldhausen [W] and Bonahon-Otal [BO]. If the manifold is Haken, then Johannsen has recently shown that M admits only finitely many Heegaard splittings [J], so we suppose that M is non-Haken. Thurston's orbifold theorem [Th2] implies that any genus two 3-manifold has a decomposition along spheres and incompressible tori into pieces with geometric structure. Since M is non-Haken it follows that there are no tori in the decomposition, and that M is either hyperbolic, a Seifert fiber space with three exceptional fibers and orbit space the 2-sphere, or a connect sum of two lens spaces.

If M is a connect sum of two lens spaces, there exists an essential 2-sphere intersecting the Heegaard surface in one curve and splitting M into two genus one manifolds, by a theorem of Haken [H]. This 2-sphere is unique up to homeomorphism by Milnor [M]. Bonahon showed that lens spaces have unique Heegaard splittings, and it follows that M also has a unique splitting [B].

If M is a Seifert fiber space with three exceptional fibers and orbit space the 2-sphere, results of Boileau-Otal [BoO] show that there are only finitely many genus two splittings of M.

The final case is that M is hyperbolic. Given a Heegaard splitting, results of Pitts-Rubinstein [PR1, PR2] imply that either the Heegaard surface is isotopic to a minimal embedding or there is a smaller genus minimal embedding. The Gauss-Bonnet theorem implies that there are no minimal embeddings of genus zero or one in a negative sectional curvature manifold. Lemma 2 implies that only a finite number of surfaces can result from the first possibility, up to isotopy.

Note. The same methods can be used to show that a 3-manifold with a positive Ricci curvature metric admits only finitely many Heegaard splittings of genus zero, one or two, and with some additional argument, genus three. It follows from a theorem of Hamilton [Ha] that these manifolds are all Seifert fiber spaces with orbit space the 2-sphere and 3 exceptional fibers and so most of these results are covered by known theorems on the classification of Heegaard splittings of genus two Seifert fiber spaces [B, BO, BoO].

We next discuss another conjecture of Waldhausen to which the method developed above has some application [W].

Conjecture 5. *Let M be irreducible and $f: M \to M$ a diffeomorphism. If f is homotopic to the identity then f is isotopic to the identity.*

This conjecture is known to hold for Seifert fiber spaces and for Haken manifolds, as well as in some other cases [W, Sc, BO]. It is false for reducible manifolds [FW]. We will show that it is close to true when the manifold is genus two.

Theorem 6. *An irreducible genus two manifold has a finite number of isotopy classes of diffeomorphisms homotopic to the identity.*

Proof. If M is Haken this is a result of Waldhausen, so we assume that M is non-Haken. Since M is irreducible and genus two, [Th2] implies that it is hyperbolic, as shown in Theorem 4. Let $\{h_i\}$ be an infinite sequence of diffeomorphisms of M, each homotopic to the identity. It suffices to show that two of the diffeomorphisms are isotopic. By passing to a subsequence we can assume that each h_i leaves invariant a Heegaard surface H, since there are only finitely many such surfaces up to isotopy. Moreover we can find h_i and h_j such that $h = h_j h_i^{-1}$ does not interchange the handlebodies on either side of H. We will show that h is isotopic to the identity.

A genus two Heegaard splitting defines an involution τ on M with a 1-dimensional fixed point set σ, and τ leaves H invariant. The action of τ on H lies in the center of the mapping class group of H, and thus commutes with $h|_H$. Thus h can be isotoped so that h and τ commute on H. We now show that h can be isotoped so that h and τ commute on all of M. Since h and τ commute on H, it suffices to show that a diffeomorphism h of a handlebody X to itself which commutes with τ on $H = \partial X$ can be isotoped rel boundary to commute with τ on X. To see this we pick a nonseparating τ-invariant disk E in X, so that $\tau E = E$. Then $h(E)$ is a disk in X whose boundary $h(\partial E)$ satisfies $\tau h(\partial E) = h\tau(\partial E) = h(\partial E)$, so $h(\partial E)$ is invariant under τ. $h(\partial E)$ bounds a τ-equivariant disk E' isotopic to $h(E)$ and we can isotop h, rel boundary, so that $h(E)$ coincides with E' and so that τ and h commute on $H \cup E$. Repeating on a second nonseparating τ-invariant disk in X, we cut H into a 3-ball so that h and τ commute on the boundary of this ball. A τ-equivariant map of a 2-sphere to a 2-sphere can be extended to a τ-equivariant map of the 3-ball by coning. This extension defines an isotopic map which commutes with τ on X. Repeating on each of the two handlebodies, we arrive at an isotopy of h on M that commutes with τ. It follows that, after an isotopy, h leaves σ pointwise fixed. By Thurston's orbifold theorem, a hyperbolic manifold with an involution having a 1-dimensional fixed point set admits a hyperbolic metric such that the diffeomorphism is a hyperbolic isometry, and the fixed point set is a hyperbolic geodesic. So the fixed point set of τ is a hyperbolic geodesic in a hyperbolic metric. Theorem 3.3 of [HS] states that if M is a hyperbolic 3-manifold, L an essential unlinked link in M and h a homeomorphism of M that is homotopic to the identity and fixes L, then h is isotopic to the identity. A hyperbolic geodesic is a special case of an essential unlinked link and it follows that h is isotopic to the identity, concluding the proof of Theorem 6.

Closely related to Conjecture 5 is:

Conjecture 7. *Let M be an irreducible non-Haken 3-manifold with infinite π_1. Then the mapping class group of M is finite.*

The mapping class group of M is the group of diffeomorphisms modulo the equivalence of isotopy. Even for a hyperbolic manifold, it is not known whether this group is finite. It is known that $\mathrm{Out}(\pi_1(M))$, the group of automorphisms of the fundamental group modulo inner automorphisms, is finite, but it is unknown whether these two groups coincide. Conjecture 7 holds for genus two manifolds.

Corollary 8. *The mapping class group of a non-Haken irreducible infinite π_1 3-manifold of genus two is finite.*

Proof. Since M is hyperbolic, $\mathrm{Out}(\pi_1(M))$ is finite by Mostow's Rigidity Theorem. The result then follows from Theorem 6.

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