PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 4, December 1991

# PRODUCT SHIFTS ON $B(H)$

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. A shift on $B(H)$ is a *-endomorphism $\alpha$ for which $\bigcap_r \alpha^r(B(H)) = \mathbb{C}P$ for some projection $P$. The paper discusses some aspects of the classification of shifts on $B(H)$ up to conjugacy by *-automorphisms, with a focus on the shifts arising from an infinite tensor product decomposition of $H$.

## 1. INTRODUCTION

Let $H$ be a separable Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on $H$. Slightly extending the definition given in [7] to include the nonunital case, a *-endomorphism $\alpha$ of $B(H)$ will be called a shift if $\bigcap_r \alpha^r(B(H)) = \mathbb{C}P$ for some (possibly zero) projection $P$ on $B(H)$. The present paper discusses some aspects of the classification of shifts on $B(H)$ up to conjugacy by *-automorphisms, with a focus on the shifts arising from an infinite tensor product decomposition of $H$.

## 2. SPATIAL DESCRIPTIONS

It was observed in [2, Proposition 2.1] that every nonzero normal *-endomorphism $\alpha$ of $B(H)$ is implemented by a sequence of isometries. An alternative proof of this result can be obtained by imitating the usual proof that *-automorphisms of $B(H)$ are inner, as described, for example, in [5, Lemma 7.5.3], and this alternative proof shows that the normality assumption can be omitted (and hence every *-endomorphism of $B(H)$ is normal). For later reference we now state this minor extension of [2, Proposition 2.1].

**Proposition 2.1.** *Let $\alpha$ be a nonzero *-endomorphism of $B(H)$. Then there is a (finite or infinite) sequence of isometries $V_1, V_2, \ldots$ in $B(H)$ having mutually orthogonal ranges such that*

$$\alpha(A) = \sum_n V_n A V_n^*$$

Received by the editors May 7, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L40; Secondary 46L55, 47A65.
*Key words and phrases.* Endomorphism, isometry, Hilbert space, shift, infinite tensor product, unitary equivalence.

*for each* $A \in B(H)$. *The linear space of operators*

$$E_\alpha = \{T \in B(H) \colon \alpha(A)T = TA \text{ for all } A \in B(H)\}$$

*is a Hilbert space relative to the inner product defined by*

$$T^*S = \langle S, T\rangle 1$$

*and* $\{V_1, V_2, \dots\}$ *is an orthonormal basis for* $E_\alpha$.

The following simple result relates the conjugacy of $\alpha$ and $\beta$ to the Hilbert spaces $E_\alpha$ and $E_\beta$.

**Proposition 2.2.** *Let* $\alpha, \beta$ *be *-endomorphisms of* $B(H)$ *and let* $U \in B(H)$ *be unitary. Then* $\alpha = (\operatorname{Ad} U)\beta(\operatorname{Ad} U^*)$ *if and only if* $E_\alpha = UE_\beta U^*$.

*Proof.* If $\alpha = (\operatorname{Ad} U)\beta(\operatorname{Ad} U^*)$ then an easy calculation yields $E_\alpha = UE_\beta U^*$. This same result shows that if $E_\alpha = UE_\beta U^*$ then $E_\alpha = E_\gamma$ where $\gamma = (\operatorname{Ad} U)\beta(\operatorname{Ad} U^*)$: hence there exist orthonormal bases $\{V_1, V_2, \dots\}$ and $\{W_1, W_2, \dots\}$ of $E_\alpha$ with $\alpha(A) = \sum V_n A V_n^*$ and $\gamma(A) = \sum W_n A W_n^*$ for each $A \in B(H)$. A simple calculation using the unitary transition matrix between the two orthonormal bases then yields the conclusion $\alpha = \gamma$. □

A natural question arising from Propositions 2.1 and 2.2 is the extent to which the specification of the isometries $V_i$ up to unitary equivalence determines the corresponding *-endomorphism up to conjugacy. The classification of an isometry $V$ up to unitary equivalence is given by the *Wold decomposition*, described, for example, in [4, Problem 118], which decomposes $V$ as a direct sum of copies of the unilateral shift and a unitary mapping on a subspace; when the unitary summand is absent the isometry is said to be *pure*. It will now be shown that if $\alpha$ is a shift, then at most one of any corresponding family of isometries is not pure; this result extends [1, Theorem 3] which is the case of two unitarily equivalent isometries. The proof is based on the following lemma in which Cuntz's notation $V_\mu V_\nu^*$ is used to denote $V_{\mu(1)} \cdots V_{\mu(r)} V_{\nu(s)}^* \cdots V_{\nu(1)}^*$, where $\mu = (\mu(1), \dots, \mu(r))$ has $r = |\mu|$ components and $\nu = (\nu(1), \dots, \nu(s))$ has $s = |\nu|$ components.

**Lemma 2.3.** *Let* $V_1, V_2, \dots$ *be isometries from a Hilbert space* $H$ *onto a family of mutually orthogonal subspaces;* $\alpha$ *be the *-endomorphism defined by* $\alpha(A) = \sum_n V_n A V_n^*$; *and* $h \in \bigcap_r V_i^r H$.

(i) *If* $P_h$ *is the orthogonal projection from* $H$ *onto the closed linear span* $S$ *of the vectors* $V_\mu V_\nu^* h$ *with* $|\mu| \geq 0$ *and* $|\nu| \geq 0$, *then* $\alpha(P_h) = P_h$ *and hence* $P_h \in \bigcap_r \alpha^r(B(H))$.

(ii) *If* $Q_h$ *is the orthogonal projection from* $H$ *onto the closed linear span* $T$ *of the vectors* $V_\mu V_\nu^* h$ *with* $|\mu| = |\nu| \geq 0$, *then* $Q_h \in \bigcap_r \alpha^r(B(H))$.

*Proof.* (i) Let $|\mu| = r$, $|\nu| = s$, and $h = V_i^{s+1} k$, where $k \in H$. Then $\alpha(1) V_\mu V_\nu^* h = \sum V_n V_n^* V_\mu V_\nu^* V_i^{s+1} k = V_\mu V_\nu^* h$ (even if $r = 0$) and hence $\alpha(1) \geq P_h$. Since $S$ is invariant under both $V_n$ and $V_n^*$, it follows that $P_h V_n = V_n P_h$

for each $n$ and hence $\alpha(P_h) = \sum V_n P_h V_n^* = P_h \sum V_n V_n^* = P_h \alpha(1) = P_h$.

(ii) Since $T$ is invariant under each $V_\gamma V_\delta^*$ with $|\gamma| = |\delta|$, it follows that $Q_h V_\gamma V_\delta^* = V_\gamma V_\delta^* Q_h$. Thus

$$\begin{aligned}\alpha^r(V_1^{*r} Q_h V_1^r) &= \sum_{|\gamma|=r} V_\gamma V_1^{*r} Q_h V_1^r V_\gamma^* \\ &= \sum_{|\gamma|=r} V_\gamma V_1^{*r} V_1^r V_\gamma^* Q_h = \sum_{|\gamma|=r} V_\gamma V_\gamma^* Q_h \\ &= \alpha^r(1) Q_h .\end{aligned}$$

However, using part (i), $Q_h \le P_h = \alpha^r(P_h) \le \alpha^r(1)$ so $Q_h = \alpha^r(V_1^{*r} Q_h V_1^r)$ and hence $Q_h \in \bigcap_r \alpha^r(B(H))$. □

**Proposition 2.4.** *Let $V_1, V_2, \ldots$ be a sequence of isometries from a Hilbert space $H$ onto a family of orthogonal subspaces and let $\alpha$ be the *-endomorphism of $B(H)$ defined by $\alpha(A) = \sum_n V_n A V_n^*$. If $\alpha$ is a shift, then at most one of the isometries $V_n$ is not pure and if $V_n$ is not pure then the subspace $\bigcap_r V_n^r H$ associated with the unitary summand of $V_n$ is one-dimensional.*

*Proof.* Suppose, to obtain a contradiction, that there exist nonzero vectors $h \in \bigcap_r V_i^r H$ and $k \in \bigcap_r V_j^r H$ with $\langle h, k\rangle = 0$. Then, by part (ii) of the lemma, there exist associated nonzero projections $Q_h, Q_k$ in $\bigcap_r \alpha^r(B(H))$. Let $|\mu| = |\nu|$ and $|\delta| = |\gamma|$. The proof will be completed if it is shown that $\langle V_\mu V_\nu^* h, V_\gamma V_\delta^* k\rangle = 0$ for then $Q_h Q_k = 0$ and, therefore, $\alpha$ is not a shift. To this end let $V_\delta V_\gamma^* V_\mu V_\nu^* = V_\rho V_\sigma^*$; $|\rho| = |\sigma| = r$; $h = V_i^{r+1} h'$; and $k = V_j^{r+1} k'$. Then $\langle V_\mu V_\nu^* h, V_\gamma V_\delta^* k\rangle = \langle V_\rho V_\sigma^* h, k\rangle = \langle V_\sigma^* h, V_\rho^* k\rangle = \delta_{\sigma(i,\ldots,i)}\delta_{\rho(j,\ldots,j)}\langle V_i h', V_j k'\rangle$. But, when $i \ne j$, $\langle V_i h', V_j k'\rangle = \langle h', V_i^* V_j k'\rangle = 0$ and, when $i = j$, $\langle V_i h', V_j k'\rangle = \langle V_i^{*r} h, V_i k'\rangle = \langle h, V_i^{r+1} k'\rangle = \langle h, k\rangle = 0$, as required. □

Using Proposition 2.4, it is possible to give a complete classification up to conjugacy of the simplest shifts $\alpha$, for which $E_\alpha$ is one-dimensional.

**Proposition 2.5.** (i) *Let $V$ be an isometry on a separable infinite-dimensional Hilbert space $H$ and let $\alpha$ be the *-endomorphism of $B(H)$ defined by $\alpha(A) = VAV^*$. Then $\alpha$ is a shift if and only if $V$ is either a pure isometry or $\bigcap_r V^r H$ is one-dimensional.*

(ii) *Let $\alpha, \beta$ be shifts on $B(H)$ defined by $\alpha(A) = VAV^*$ and $\beta(A) = WAW^*$, where $V$ and $W$ are isometries on $H$. Then $\alpha$ is conjugate to $\beta$ if and only if $V$ is unitarily equivalent to $\lambda W$ for some $\lambda \in S^1$ i.e. if and only if $V$ and $W$ have the same Wold decomposition.*

*Proof.* (i) By Proposition 2.4, the conditions on $V$ are necessary for $\alpha$ to be a shift. Conversely, if $V$ satisfies the given conditions, then $V$ can be assumed to be a direct sum of copies of the unilateral shift and (possibly) a unitary map on a

one-dimensional subspace. Let $\{e_i : i \in \mathbb{N}\}$ be the orthonormal basis associated with one of the shift summands. If $T \in \alpha^r(B(H))$ then $T = V^r S V^{*r}$ for some $S \in B(H)$ and hence $Te_i = 0$ for $1 \le i \le r$. It follows that, if $T \in \bigcap_r \alpha^r(B(H))$ then the restriction of $T$ to $(\bigcap_r V^r H)^\perp$ is 0. The same argument applies to $T^*$ and hence $T$ leaves $\bigcap_r V^r H$ globally invariant. Therefore $\bigcap_r \alpha^r(B(H)) = \mathbb{C}P$ where $P$ is the projection from $H$ onto the zero or one-dimensional space $\bigcap_r V^r H$.

(ii) This follows from Proposition 2.2. □

The case in which $E_\alpha$ has dimension one is very special. For example, in contrast to the result of Proposition 2.5(i), if $E_\alpha$ has dimension greater than one, then each isometry $V$ in $E_\alpha$ is onto a space of infinite codimension and hence must be equivalent to an infinite direct sum of copies of the unilateral shift, possibly with an additional one-dimensional unitary summand.

Later in the paper an example will be produced to show that it may occur that $\alpha(A) = \sum V_n A V_n^* = \sum W_n A W_n^*$ where all the $V_n$ are pure and one of the $W_n$ is not. It will also be shown that the condition that all the $V_n$ are pure is not sufficient to ensure that $\alpha$ is a shift. Both these examples rely on the notion of a product shift, that will now be introduced.

## 3. Product shifts

Let $K$ be a Hilbert space and, for each $i \in \mathbb{N}$, let $k_i \in K$ be a unit vector such that $\sum_i |1 - \langle k_i, k_{i+1}\rangle| < \infty$. Recall that, as described in [3] or [6], the incomplete infinite tensor product $\bigotimes^{k_i} K$ is the Hilbert space inductive limit of the sequence $K \to K \otimes K \to K \otimes K \otimes K \cdots$, where the $i$th map takes $h$ to $h \otimes k_{i+1}$. Let $\phi_i$ be the natural map from the $i$-fold tensor product $K \otimes K \otimes \cdots \otimes K$ into $\bigotimes^{k_i} K$ and let $\psi_i$ be the corresponding map from $K \otimes K \otimes \cdots \otimes K$ into $\bigotimes^{k_{i+1}} K$. Then, by [3, Proposition 1.1], for each $h$ in the $r$-fold tensor product $K \otimes K \otimes \cdots \otimes K$, the sequence $\phi_r(h), \phi_{r+1}(h \otimes k_{r+2}), \phi_{r+2}(h \otimes k_{r+2} \otimes k_{r+3}), \ldots$ has a limit in $\bigotimes^{k_i} K$ and, by [3, Proposition 1.3], the map taking $\psi_r(h)$ to this limit extends to an isomorphism $\theta$ from $\bigotimes^{k_{i+1}} K$ onto $\bigotimes^{k_i} K$. Informally, we can describe $\theta$ as taking $\bigotimes x_i$ (where $x_i = k_{i+1}$ for almost all $i$) to $\bigotimes x_i$ (interpreting the last expression as the limit of $\phi_1(x_1), \phi_2(x_1 \otimes x_2), \ldots$).

**Proposition 3.1.** *Let $P$ be a projection on $K$; $k_i$ be a sequence of unit vectors in $K$ with $\sum |1 - \langle k_1, k_{i+1}\rangle| < \infty$; and $\theta$ be the isomorphism from $\bigotimes^{k_{i+1}} K$ onto $\bigotimes^{k_i} K$ taking $\bigotimes x_i$ (where $x_i = k_{i+1}$ for almost all $i$) to $\bigotimes x_i$. Then the map $\alpha_{k,P}$ defined on $B(H) = B(\bigotimes^{k_i} K)$ by $\alpha_{k,P}(T) = \theta(P \otimes T)\theta^{-1}$ is a shift.*

*Proof.* The map $\alpha_{k,P}$ is clearly a *-endomorphism. In the unital case, $\alpha_{k,1}^r(B(H))' = B(K \otimes \cdots \otimes K) \otimes \mathbb{C}1$ (where the tensor product has $r$ factors) and hence $\bigcup_r \alpha_{k,1}^r(B(H))'$ is weakly dense in $B(H)$, from which it follows that $\alpha_{k,1}$ is a shift. In the nonunital case let $q$ be the greatest lower bound

of the projections $\alpha^r_{k,P}(1)$. If $q = 0$ then, since $\alpha^r_{k,P}(1)T = T$ for each $T \in \bigcap_r \alpha^r_{k,P}(B(H))$, $T = qT = 0$ for all such $T$ and hence $\alpha_{k,P}$ is a shift. If $q \neq 0$ then $q\alpha^r_{k,1}(T)q = q\alpha^r_{k,P}(T)q$ for all $T$ because, as can be checked by induction, $\alpha^r_{k,P}(1)\alpha^r_{k,1}(T)\alpha^r_{k,P}(1) = \alpha^r_{k,P}(T)$. Thus, since $\alpha_{k,P}(q) = q$, $\alpha^r_{k,P}(qB(H)q) = q\alpha^r_{k,P}(B(H))q = q\alpha^r_{k,1}(B(H))q$ and so, when both algebras are restricted to $qH$, $[\alpha^r_{k,P}(q(B(H))q)]' = q\alpha^r_{k,1}(B(H))'q$. Hence, as in the unital case, $\bigcap_r \alpha^r_{k,P}(qB(H)q) = \mathbb{C}q$; however, $\bigcap_r \alpha^r_{k,P}(B(H)) \subseteq qB(H)q$ and, therefore, $\bigcap_r \alpha^r_{k,P}(B(H)) = \mathbb{C}q$, as required. □

If there exists a unitary $U$ from a Hilbert space $H$ onto an infinite tensor product $\bigotimes^{k_i} K$ with the properties above, then any *-endomorphism $\alpha$ of $B(H)$ of the form $(\operatorname{Ad} U^*)\alpha_{k,P}(\operatorname{Ad} U)$ will be called a *product shift* on $B(H)$.

Not every shift on $B(H)$ is a product as can be seen from Proposition 2.4. Any product shift $\alpha_{k,P}$ has $\alpha_{k,P}(1)$ of infinite codimension whereas if $\alpha$ is the shift corresponding to a single unilateral shift $V$ then $\alpha(1)$ has codimension 1. Nevertheless the author is not aware of examples of nonproduct unital shifts on $B(H)$: the following proposition shows that any sequence of isometries defining such a shift must be all pure.

**Proposition 3.2.** *Let $\alpha$ be a unital shift on $B(H)$ defined by $\alpha(A) = \sum_n V_n A V_n^*$ where $V = V_1$ is not pure. Then $\alpha$ is a product shift.*

*Proof.* Let $h$ be a unit vector in $\bigcap_r V^r H$ and note that, by Proposition 2.4, $\bigcap_r V^r H$ is one-dimensional. It follows that $Vh = \lambda h$ for some $\lambda \in S^1$ and, replacing $V$ by $\bar{\lambda} V$, we can assume that $Vh = h$.

Let $c_{00}$ denote the space of functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(r) = 1$ for all sufficiently large $r$ and, for each $f \in c_{00}$, let $b_f = \phi_s(V_{f(1)} \otimes \cdots \otimes V_{f(s)}) \in \bigotimes^V E_\alpha$, where $\phi_s$ is the natural map from the $s$-fold tensor product $E_\alpha \otimes \cdots \otimes E_\alpha$ into $\bigotimes^V E_\alpha$ and where $s$ is chosen sufficiently large so that $f(r) = 1$ for all $r \geq s$. Then, by [6, Lemma 4.1.4], $\{b_f\colon f \in c_{00}\}$ is an orthonormal basis for $\bigotimes^V E_\alpha$. Define $\psi\colon \bigotimes^V E_\alpha \to H$ extending $\psi(b_f) = V_{f(1)} \cdots V_{f(s)} h$ (which is well defined since $Vh = h$). Then, for $t \geq s$, $\langle \psi(b_f), \psi(b_g)\rangle = \langle V_{f(1)} \cdots V_{f(s)} h, V_{g(1)} \cdots V_{g(t)} h\rangle = \delta_{f(1)g(1)} \cdots \delta_{f(s)g(s)} \langle V^*_{g(t)} \cdots V^*_{g(s+1)} h, h\rangle = \delta_{fg}\langle h, h\rangle = \delta_{fg}$ so that $\psi$ defines a unitary mapping from $\bigotimes^V E_\alpha$ onto a subspace $K$ of $H$.

For each $\mu, \nu$ with $|\mu| \geq 0$ and $|\nu| = r \geq 0$, $V_\mu V_\nu^* h = V_\mu V_\nu^* V^r h = \delta_{\nu(1,\ldots,1)} V_\mu h$, so that $K$ is equal to the closed linear span of the vectors $V_\mu V_\nu^* h$ with $|\mu| \geq 0$ and $|\nu| \geq 0$. By Lemma 2.3 (i) it follows that the projection onto $K$ belongs to $\bigcap_r \alpha^r(B(H))$ and hence, since $\alpha$ is a unital shift, $K = H$.

Note that

$$\begin{aligned}(\psi^* V_n \psi)(\phi_r(V_{f(1)} \otimes \cdots \otimes V_{f(r)})) &= \psi^* V_n V_{f(1)} \cdots V_{f(r)} h \\ &= \phi_{r+1}(V_n \otimes V_{f(1)} \otimes \cdots \otimes V_{f(r)}),\end{aligned}$$

from which it follows that

$$(\psi^* V_n^* \psi)(\phi_r(V_{f(1)} \otimes \cdots \otimes V_{f(s)})) = \langle V_{f(1)}, V_n \rangle \phi_{r-1}(V_{f(2)} \otimes \cdots \otimes V_{f(r)});$$

hence

$$\begin{aligned}(\operatorname{Ad}\psi^*)\alpha(\operatorname{Ad}\psi)(T_1 \otimes T_2 \otimes \cdots \otimes T_s \otimes 1) \\ = \sum_n (\psi^* V_n \psi)(T_1 \otimes \cdots \otimes T_s \otimes 1)(\psi^* V_n \psi)^* = {}^1 \otimes T_1 \otimes \cdots \otimes T_s \otimes 1\end{aligned}$$

and thus that $\alpha$ is a product shift. □

It will now be shown how product shifts can be defined to illustrate the comments made after Proposition 2.5.

**Example 3.3.** Let $K = \mathbb{C}^2$; $k_i = (1, 0)$ for each $i$; and the isometries $V_1$, $V_2$, $W_1$, $W_2$ be defined on $H = \bigotimes^{k_i} K$ by

$$\begin{aligned} V_1(h) &= (1/\sqrt{2}, 1/\sqrt{2}) \otimes h, & V_2(h) &= (1/\sqrt{2}, -1/\sqrt{2}) \otimes h, \\ W_1(h) &= (1, 0) \otimes h, & W_2(h) &= (0, 1) \otimes h. \end{aligned}$$

Then $\alpha_{k,1}(A) = V_1 A V_1^* + V_2 A V_2^* = W_1 A W_1^* + W_2 A W_2^*$ for each $A \in B(H)$. However, both $V_1$ and $V_2$ are pure isometries (as is $W_2$) but $\bigcap_n W_1^n H = \mathbb{C}k$, where $k = \bigotimes(1, 0)$. □

**Example 3.4.** Let $H$, $V_1$, $V_2$, and $\alpha_{k,1}$ be as defined in Example 3.3 and let $\mathscr{H}$ be any Hilbert space. Then $\alpha = \alpha_{k,1} \otimes \mathrm{id}$ on $B(H) \otimes B(\mathscr{H})$ is not a shift; however,

$$\alpha(A) = (V_1 \otimes 1)A(V_1^* \otimes 1) + (V_2 \otimes 1)A(V_2^* \otimes 1)$$

for each $A$, and both $V_1 \otimes 1$ and $V_2 \otimes 1$ are pure isometries. Hence the converse of Proposition 2.4 is false. □

The next main result gives a criterion for two unital product shifts to be conjugate. The proof is based on the following lemma:

**Lemma 3.5.** *Let* $W: \bigotimes^{k_i} K \to \bigotimes^{h_i} K$ *be a unitary for which* $(\operatorname{Ad} W)\alpha_{k,1}(\operatorname{Ad} W^*) = \alpha_{h,1}$. *Then there exists a unitary* $V$ *on* $K$ *such that, for each* $r \geq 1$ *and each* $T_1, \ldots, T_r \in B(K)$, $(\operatorname{Ad} W)(T_1 \otimes \cdots \otimes T_r \otimes 1) = (VT_1V^*) \otimes (VT_2V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1$.

*Proof*. For notational convenience, let $H_h = \bigotimes^{h_i} K$ and $H_k = \bigotimes^{k_i} K$. Then $\alpha_{h,1}(B(H_h))' = \operatorname{Ad}(W)[\alpha_{k,1}(B(H_k))']$ and hence, for each $T \in B(K)$,

$$(\operatorname{Ad} W)(T \otimes 1 \otimes 1 \otimes \cdots) = VTV^* \otimes 1 \otimes 1 \otimes \cdots$$

for some unitary $V$ on $K$. Then, for each $r \geq 0$,

$$\begin{aligned}(\operatorname{Ad} W)\alpha_{k,1}^r(T \otimes 1 \otimes 1 \otimes \cdots) &= \alpha_{h,1}^r(\operatorname{Ad} W)(T \otimes 1 \otimes 1 \otimes \cdots) \\ &= \alpha_{h,1}^r(VTV^* \otimes 1 \otimes 1 \otimes \cdots).\end{aligned}$$

Hence, for each $r \geq 0$, $(\operatorname{Ad} W)(T_1 \otimes \cdots \otimes T_r \otimes 1) = (VT_1V^*) \otimes (VT_2V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1$. □

**Theorem 3.6.** *The product shifts $\alpha_{k,1}$ and $\alpha_{h,1}$ are conjugate if and only if there exists a unitary map $V$ on $K$ with $\sum |1 - |\langle Vk_i, h_i\rangle|| < \infty$.*

*Proof.* If $\sum |1 - |\langle Vk_i, h_i\rangle|| < \infty$ then, by [3, Proposition 1.3], the map taking $\bigotimes x_i$ (where $x_i = k_i$ for almost all $i$) to $\bigotimes x_i$ extends to an isomorphism from $\bigotimes^{k_i} K$ onto $\bigotimes^{\alpha_i V^* h_i} K$, where $\alpha_i$ is the complex number of unit modulus such that $|\langle k_i, V^*h_i\rangle| = \langle k_i, \alpha_i V^* h_i\rangle$. However there is an isomorphism from $\bigotimes^{\alpha_i V^* h_i} K$ onto $\bigotimes^{h_i} K$ taking $\bigotimes x_i$ to $\bigotimes \overline{\alpha}_i V x_i$ and hence there is an isomorphism $W$ from $\bigotimes^{k_i} K$ onto $\bigotimes^{h_i} K$ taking $\bigotimes x_i$ (where $x_i = k_i$ for almost all $i$) to $\bigotimes \overline{\alpha}_i V x_i$. Then, for $T_1, \ldots, T_r \in B(K)$,

$$(\operatorname{Ad} W)\alpha_{k,1}(\operatorname{Ad} W^*)(T_1 \otimes \cdots \otimes T_r \otimes 1) = \alpha_{h,1}(T_1 \otimes \cdots \otimes T_r \otimes 1),$$

from which it follows, by linearity and continuity that $\alpha_{k,1}$ is conjugate to $\alpha_{h,1}$.

Conversely, if $\alpha_{k,1}$ is conjugate to $\alpha_{h,1}$ then, by Lemma 3.5, there exists an isomorphism $W$ from $\bigotimes^{k_i} K$ onto $\bigotimes^{h_i} K$ and a unitary $V$ on $K$ such that $(\operatorname{Ad} W)(T_1 \otimes \cdots \otimes T_r \otimes 1) = (VT_1V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1$ for all $T_1, \ldots, T_r \in B(K)$. In particular, this equation holds when $T_i$ is the one-dimensional projection onto the subspace spanned by $k_i$. In this case, the decreasing sequence of projections $T_1 \otimes \cdots \otimes T_r \otimes 1$ is weakly convergent as $r \to \infty$; by considering the effect of $T_1 \otimes \cdots \otimes T_r \otimes 1$ on an orthonormal basis $\{e_{1i_1} \otimes e_{2i_2} \otimes \cdots : i_1, i_2, \ldots \in \mathbb{N}\}$ (where $e_{r1} = k_r$ for each $r$ and $i_k = 1$ for all sufficiently large $k$) it can be seen that the weak limit is the one-dimensional projection onto the subspace spanned by $k = \bigotimes k_i$. Hence the projections $(VT_1V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1$ are weakly convergent to a one-dimensional projection in $\bigotimes^{h_i} K$; let $x$ be a unit vector in the corresponding one-dimensional subspace. If $\phi_r$ denotes the natural map from the $r$-fold tensor product $K \otimes \cdots \otimes K$ into $\bigotimes^{h_i} K$ then there exists $s \in \mathbb{N}$ and a unit vector $x_s$ in $\phi_s(K \otimes \cdots \otimes K)$ such that $\|x - x_s\| < \frac{1}{2}$. Then, for each $r$,

$$\begin{aligned} &\|[(VT_1V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1]x_s\| \\ &\qquad \geq \|x\| - \|x - [(VT_1V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1]x_s\| > \tfrac{1}{2}. \end{aligned}$$

It follows that $\prod_{i=s+1}^{r} |\langle k_i, V^*h_i\rangle| \nrightarrow 0$ as $r \to \infty$ and hence, by Lemma 2.4.1 (II) of [6], $\sum |1 - |\langle k_i, V^*h_i\rangle|| < \infty$, as required. □

Theorem 3.6 can be used to produce examples of nonconjugate but outer-conjugate unital shifts. Let $\log^k$ denote the $k$-fold composite $\log \circ \cdots \circ \log$ and, for each $r \in \mathbb{N}$, define a sequence $f_r$ by

$$f_r(n) = \begin{cases} 1 & \text{if } \log^r(n) \leq 0, \\ 1/(n \log(n) \log^2(n) \cdots \log^r(n)) & \text{if } \log^r(n) > 0. \end{cases}$$

Let $K$ be a separable Hilbert space (of dimension at least 2) with an orthonormal basis $\{e_i\}$ and, for each $n \in \mathbb{N}$, define $k_{r,n} \in K$ by $k_{r,n} = (1 - f_r(n))e_1 + [f_r(n)(2 - f_r(n))]^{1/2} e_2$.

**Lemma 3.7.** (i) $\|k_{r,n}\| = 1$ *for each* $r$ *and* $n$.

(ii) $\sum_n |1 - \langle k_{r,n}, k_{r,n+1}\rangle| < \infty$ *for each* $r$.

*Proof*. Claim (i) is easily verified. To check claim (ii), note that

$$0 \le \langle k_{r,n}, k_{r,n+1}\rangle \le \|k_{r,n}\| \, \|k_{r,n+1}\| = 1$$

and hence that

$$\begin{aligned} 0 &\le |1 - \langle k_{r,n}, k_{r,n+1}\rangle| \\ &= f_r(n) + f_r(n+1) - f_r(n) f_r(n+1) \\ &\quad - [f_r(n) f_r(n+1)(2 - f_r(n))(2 - f_r(n+1))]^{1/2} \\ &\le 2 f_r(n) - f_r(n) f_r(n+1) - f_r(n+1)(2 - f_r(n)) \\ &= 2(f_r(n) - f_r(n+1)). \end{aligned}$$

However $\sum_n 2(f_r(n) - f_r(n+1))$ is convergent, from which the result follows. □

**Proposition 3.8.** *For each* $r \in \mathbb{N}$, *let* $\alpha_r$ *be the product shift* $\alpha_{k_r,1}$ *on* $\bigotimes_{n=1}^{\infty} {}^{k_{r,n}} K$, *where* $k_{r,n}$ *is defined above. Then, if* $r \ne s$, $\alpha_r$ *is outer conjugate but not conjugate to* $\alpha_s$.

*Proof*. For definiteness suppose that $r > s$. By construction $\alpha_r$ and $\alpha_s$ have the same multiplicity $\dim(K)$ and hence are outer conjugate by [7, Theorem 2.4]. If they are conjugate then, by Theorem 3.6, there exists a unitary $V$ on $K$ such that $\sum_n |1 - |\langle V k_{r,n}, k_{s,n}\rangle|| < \infty$. It follows that $|\langle V k_{r,n}, k_{s,n}\rangle| \to 1$ as $n \to \infty$ and hence that $|\langle V e_1, e_1\rangle| = 1$, so that $V e_1 = \alpha e_1$ for some $\alpha \in S^1$. Hence,

$$\begin{aligned} |\langle V k_{r,n}, k_{s,n}\rangle| &= |(1 - f_r(n))(1 - f_s(n))\langle \alpha e_1, e_1\rangle \\ &\qquad + [f_r(n) f_s(n)(2 - f_r(n))(2 - f_s(n))]^{1/2} \langle V e_2, e_2\rangle| \\ &\le (1 - f_r(n))(1 - f_s(n)) + 2 f_s(n)^{1/2} f_r(n)^{1/2} \\ &= 1 - f_s(n)[1 - f_r(n) + f_r(n) f_s(n)^{-1} - 2 f_r(n)^{1/2} f_s(n)^{-1/2}]. \end{aligned}$$

However, for all sufficiently large $n$, $f_r(n) f_s(n)^{-1} = 1/(\log^{s+1}(n) \cdots \log^r(n)) \to 0$ as $n \to \infty$ and therefore, for all sufficiently large $n$, $|\langle V k_{r,n}, k_{s,n}\rangle| \le 1 - f_s(n)/2$ and therefore, $\sum |1 - |\langle V k_{r,n}, k_{s,n}\rangle||$ is not convergent, giving a contradiction. □

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