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THE JONES POLYNOMIAL OF PERIODIC KNOTS

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ABSTRACT. We give the conditions for the Jones polynomial of periodic knots which are the improvement of Traczyk's and Murasugi's results.

1. INTRODUCTION

A knot K in S^3 is said to have period $r > 1$, if there exists an orientation preserving homeomorphism f on S^3 of period r which preserves K with $\text{Fix}(f) \cong S^1$ and $\text{Fix}(f) \cap K = \varnothing$. By the positive solution of Smith Conjecture, $\text{Fix}(f)$ is unknotted. Let Σ^3 be the quotient space under f and $\varphi : S^3 \to \Sigma^3$ the quotient map. Then Σ^3 is a 3-sphere. We call $\varphi(K)$, denoted by k, the factor knot of K.

Recently, some results concerning the Jones polynomial have been applied to the study of periodic knots [6, 7]. In this paper, we give more precise conditions of the Jones polynomial of periodic knots. In fact, we will prove the following theorems. Here we denote the Jones polynomial of a knot K by $V_K(t)$.

Theorem 1. *For an odd prime r, let K be an r periodic knot and f the periodic map on S^3 realizing the period.*

(i) *If* $\text{lk}(K, \text{Fix}(f)) \equiv 1 \mod 2$, *then*

$$V_K(t) - V_K(t^{-1}) \equiv 0 \text{ mod } (r, t^{2r} - 1).$$

(ii) *If* $\text{lk}(K, \text{Fix}(f)) \equiv 0 \mod 2$, *then*

$$V_K(t) - V_K(t^{-1}) \equiv 0 \text{ mod } (r, t^r - 1)$$

and

$$V_K(t) + V_K(t^{-1}) \equiv 0 \text{ mod } (r, (t^r + 1)/(t + 1)).$$

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Theorem 2. *Under the same assumption as in Theorem* 1, *let* k *be the factor knot of* K.

(i) *If* $\operatorname{lk}(K, \operatorname{Fix}(f)) \equiv 1 \bmod 2$, *then*

$$V_K(t) \equiv [V_k(t)]^r \bmod (r, t^{2r} - t^{r+1} - t^{r-1} + 1).$$

(ii) *If* $\operatorname{lk}(K, \operatorname{Fix}(f)) \equiv 0 \bmod 2$, *then*

$$V_K(t) \equiv (t^{1/2} + t^{-1/2})^{r-1}[V_k(t)]^r \bmod (r, (t^{2r} - t^{r+1} - t^{r-1} + 1)/(t+1)).$$

Section 2 gives the proof of Theorem 1 and Theorem 2. In §3, we will present some examples and remarks.

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2. Proof of Theorems 1 and 2

Definition 1 [1, 4]. Fix a nonzero complex number v and a positive integer n. Then the Jones algebra J_n is defined as a C-algebra with generators $1, e_1, e_2, \dots, e_{n-1}$ and relations

$$e_i^2 = -(v^2 + v^{-2})e_i,$$
$$e_i e_{i\pm1} e_i = e_i,$$
$$e_i e_j = e_j e_i \quad \text{if } |i-j| > 1.$$

It is well known that J_n is semisimple when v is not a root of unity. Let $\rho_{n,i}$ $(0 \le i \le [n/2])$ be the irreducible representations of J_n, and $\chi_{n,i}$ $(0 \le i \le [n/2])$ their characters.

Definition 2 [4]. Let G_n be the free semigroup generated by $1, \epsilon_i, \sigma_i, \sigma_i^{-1}$ $(1 \le i \le n-1)$. For $\xi \in G_n$, we define an n-string tangle as in Figure 1. We identify ξ and this n-string tangle if there is no fear of confusion. By the analogy with braids, we define the closure of ξ, denoted by $\hat{\xi}$, naturally.

Let $\pi_n : G_n \to J_n$ be the semigroup homomorphism defined by $\pi_n(\epsilon_i) = e_i$, $\pi_n(\sigma_i) = v^{-1} + ve_i$ and $\pi_n(\sigma_i^{-1}) = v + v^{-1}e_i$. In [4], J. Murakami has shown the following. For $\xi \in G_n$,

$$\langle \hat{\xi} \rangle = \sum_{i=0}^{[n/2]} a_{n,i}(v)\chi_{n,i}(\pi_n(\xi)),$$

where $\langle \hat{\xi} \rangle$ is the bracket polynomial [2] of $\hat{\xi}$ and

$$a_{n,i}(v) = (-1)^{n+1}(v^{2n+2-4i} - v^{-2n-2+4i})/(v^4 - v^{-4}).$$

Now, we begin the proof of the theorems. From a simple observation, we can assume that there exists $\xi \in G_n$ such that $k = \hat{\xi}$ and $K = \hat{\xi^r}$. Then, n is

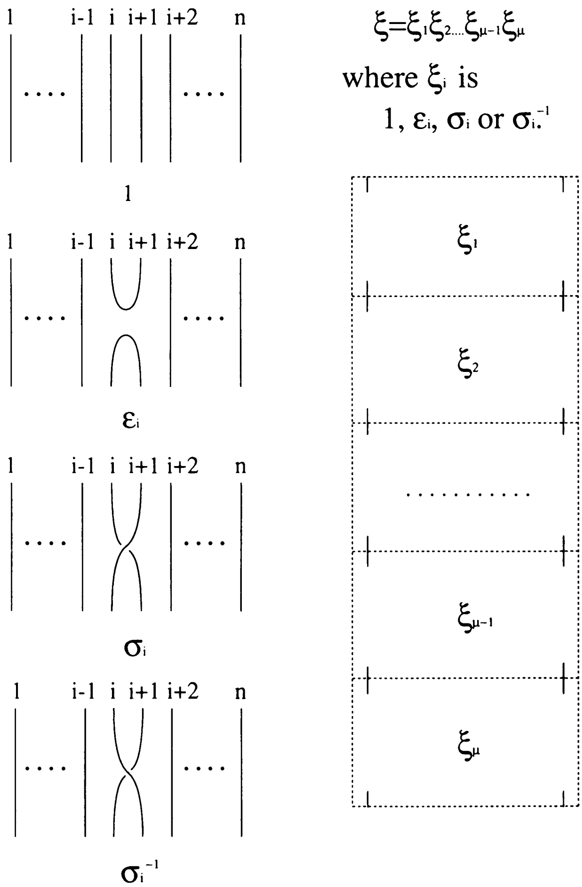

FIGURE 1

odd or even according as $\operatorname{lk}(K, \operatorname{Fix}(f))$ is odd or even. Let $w(K)$ and $w(k)$ denote the writhes of $\hat{\xi^r}$ and $\hat{\xi}$ respectively. Then we have $w(K) = rw(k)$ and

$$V_K(v^4) = (-v^3)^{w(K)}\langle\hat{\xi^r}\rangle = \sum_{i=0}^{[n/2]} a_{n,i}(v)(-v^3)^{rw(k)}\chi_{n,i}(\pi_n(\xi^r)),$$

$$V_k(v^4) = (-v^3)^{w(k)}\langle\hat{\xi}\rangle = \sum_{i=0}^{[n/2]} a_{n,i}(v)(-v^3)^{w(k)}\chi_{n,i}(\pi_n(\xi)).$$

Let $f_i(v) = (-v^3)^{w(k)}\chi_{n,i}(\pi_n(\xi)) \in Z[v^{\pm 1}]$.

Claim. $(-v^3)^{rw(k)}\chi_{n,i}(\pi_n(\xi^r)) \equiv f_i(v^r) \mod r$.

Proof. As $\rho_{n,i}(\pi_n(\xi))$ is a matrix over $Z[v^{\pm 1}]$, we have

$$\begin{aligned}\chi_{n,i}(\pi_n(\xi^r)) &= \operatorname{tr}(\rho_{n,i}(\pi_n(\xi^r))) \\ &= \operatorname{tr}((\rho_{n,i}(\pi_n(\xi)))^r) \\ &\equiv (\operatorname{tr}(\rho_{n,i}(\pi_n(\xi))))^r \mod r.\end{aligned}$$

That is, $\chi_{n,i}(\pi_n(\xi^r)) \equiv (\chi_{n,i}(\pi_n(\xi)))^r \mod r$. Therefore,

$$\begin{aligned}(-v^3)^{rw(k)}\chi_{n,i}(\pi_n(\xi^r)) &\equiv ((-v^3)^{w(k)}\chi_{n,i}(\pi_n(\xi)))^r \\ &\equiv (f_i(v))^r \\ &\equiv f_i(v^r) \mod r.\end{aligned}$$

We now consider the two cases of theorems.

Case 1. $\operatorname{lk}(K, \operatorname{Fix}(f)) \equiv 1 \mod 2$.

Note that n is odd and $a_{n,i}(v) \in Z[v^{\pm 4}]$ for every i. Write $f_i(v) = g_i(v) + h_i(v)$ where $g_i(v)$ and $h_i(v)$ are elements of $Z[v^{\pm 4}]$ and $Z[v^{\pm 1}] - Z[v^{\pm 4}]$ respectively. By the claim, we have

$$V_K(v^4) \equiv \sum_{i=0}^{[n/2]} a_{n,i}(v)g_i(v^r) + \sum_{i=0}^{[n/2]} a_{n,i}(v)h_i(v^r) \bmod r$$

and

$$V_k(v^4) = \sum_{i=0}^{[n/2]} a_{n,i}(v)g_i(v) + \sum_{i=0}^{[n/2]} a_{n,i}(v)h_i(v).$$

As K and k are knots, $V_K(v^4)$ and $V_k(v^4)$ are in $Z[v^{\pm 4}]$. That is,

$$\sum_{i=0}^{[n/2]} a_{n,i}(v)h_i(v^r) \equiv \sum_{i=0}^{[n/2]} a_{n,i}(v)h_i(v) \equiv 0 \mod r.$$

From $a_{n,i}(v) = a_{n,i}(v^{-1})$, we have

$$V_K(v^4) - V_K(v^{-4}) \equiv \sum_{i=0}^{[n/2]} a_{n,i}(v)(g_i(v^r) - g_i(v^{-r})) \mod r.$$

Since each $g_i(v^r)$ is an element of $Z[v^{\pm 4r}]$, $v^{4r} - v^{-4r}$ divides $g_i(v^r) - g_i(v^{-r})$. By replacing t for v^4, we obtain Theorem 1(i). Furthermore

$$V_K(v^4) - [V_k(v^4)]^r \equiv \sum_{i=0}^{[n/2]} (a_{n,i}(v) - (a_{n,i}(v))^r)g_i(v^r) \mod r.$$

It is easy to show $a_{n,i}(v) - (a_{n,i}(v))^r \equiv 0 \mod (r, v^{8r} - v^{4r+4} - v^{4r-4} + 1)$. A substitution t for v^4 gives Theorem 2(i).

Case 2. $\operatorname{lk}(K, \operatorname{Fix}(f)) \equiv 0 \mod 2$.

Note that n is even and that $a_{n,i}(v) \notin Z[v^{\pm 4}]$ for every i. However $(v^2+v^{-2})a_{n,i}(v)$ is in $Z[v^{\pm 4}]$ for every i. By the argument similar to that of Case 1, we obtain

$$(v^2+v^{-2})V_K(v^4) \equiv \sum_{i=0}^{[n/2]}(v^2+v^{-2})a_{n,i}(v)y_i(v^r) \mod r,$$

$$(v^2+v^{-2})V_k(v^4) = \sum_{i=0}^{[n/2]}(v^2+v^{-2})a_{n,i}(v)y_i(v)$$

where each $y_i(v)$ is in $Z[v^{\pm 2}] - Z[v^{\pm 4}]$. Therefore we have

$$\begin{aligned}(v^2+v^{-2})&(V_K(v^4)-V_K(v^{-4}))\\ &\equiv \sum_{i=0}^{[n/2]}(v^2+v^{-2})a_{n,i}(v)(y_i(v^r)-y_i(v^{-r})) \mod r,\\ (v^2+v^{-2})&(V_K(v^4)+V_K(v^{-4}))\\ &\equiv \sum_{i=0}^{[n/2]}(v^2+v^{-2})a_{n,i}(v)(y_i(v^r)+y_i(v^{-r})) \mod r.\end{aligned}$$

Since each $y_i(v^r)$ is an element of $Z[v^{\pm 2r}]-Z[v^{\pm 4r}]$, $v^{2r}-v^{-2r}$ and $v^{2r}+v^{-2r}$ divide $y_i(v^r)-y_i(v^{-r})$ and $y_i(v^r)+y_i(v^{-r})$ respectively. By replacing t for v^4, we obtain Theorem 1(ii). Furthermore,

$$\begin{aligned}(v^2+v^{-2})&V_K(v^4)-[(v^2+v^{-2})V_k(v^4)]^r\\ &\equiv \sum_{i=0}^{[n/2]}((v^2+v^{-2})a_{n,i}(v)-((v^2+v^{-2})a_{n,i}(v))^r)y_i(v^r) \mod r.\end{aligned}$$

It is also easy to show that

$$(v^2+v^{-2})a_{n,i}(v)-((v^2+v^{-2})a_{n,i}(v))^r \equiv 0 \operatorname{mod}(r, v^{8r}-v^{4r+4}-v^{4r-4}+1).$$

A substitution t for v^4 gives Theorem 2(ii).

3. Examples

We begin with the following proposition.

Proposition. *For any knot K, either*

(i) $V_K(t)-V_K(t^{-1}) \equiv 0 \mod (t^6-1)$ *or*

(ii) $V_K(t)+V_K(t^{-1}) \equiv 0 \mod (t^2-t+1)$.

Proof. Let $V_K(t) = (t^2-t+1)Q(t)+at+b$, $a, b \in Z$. As $V_K(e^{\pi i/3})$ is a power of $i\sqrt{3}$ [3], we have $a=0$ or $a+2b=0$. If $a+2b=0$, a simple calculation shows (ii). If $a=0$, it is clear that t^2-t+1 divides $V_K(t)-V_K(t^{-1})$. Furthermore t^3-1 and $t+1$ also divide $V_K(t)-V_K(t^{-1})$ [**1**]. This shows (i).

By the proposition, it seems that Theorem 1 does not work in the case $r = 3$. But we know Murasugi's result [5]:

$$\Delta_K(t) \doteq \Delta_k(t)^r (1 + t + \cdots + t^{\lambda-1})^{r-1} \mod r,$$

where $\Delta_K(t)$ and $\Delta_k(t)$ are the Alexander polynomials of K and k, respectively, and $\lambda = \mathrm{lk}(K, \mathrm{Fix}(f))$. So we can evaluate $\mathrm{lk}(K, \mathrm{Fix}(f))$ from Δ_K and there is a possibility that Theorem 1 works in the case $r = 3$. But the author does not have such an example. In the cases $r \geq 5$, Theorem 1 works well. Here we give two examples.

Example 1. Consider $K = 10_{24}$. Traczyk's criterion does not work for $r = 5$ because $V_K(t) - V_K(t^{-1}) \equiv 0 \mod (5, t^5 - 1)$. From $\Delta_K(t) \doteq (1+t)^4 \mod 5$, if K has period 5, $\mathrm{lk}(K, \mathrm{Fix}(f))$ must be 2. But we have $V_K(t) + V_K(t^{-1}) \not\equiv 0 \mod (5, (t^5+1)/(t+1))$. By Theorem 1, K cannot have period 5.

Example 2. Consider $K = 10_{55}$. Traczyk's criterion also does not work for $r = 5$. From $\Delta_K(t) \doteq 1 \mod 5$, if K has period 5, $\mathrm{lk}(K, \mathrm{Fix}(f))$ must be 1. But we have $V_K(t) - V_K(t^{-1}) \not\equiv 0 \mod (5, t^{10} - 1)$. By Theorem 1, K can not have period 5.

Finally we remark that Theorem 2 holds for the Jones polynomial of periodic links.

References

1. V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126** (1987), 335–388.
2. L. H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), 395–407.
3. W. B. R. Lickorish and K. C. Millett, *Some evaluations of link polynomials*, Comment. Math. Helv. **61** (1986), 349–359.
4. J. Murakami, *The parallel version of polynomial invariants of links*, Osaka J. Math. **26** (1989), 1–55.
5. K. Murasugi, *On periodic knots*, Comment. Math. Helv. **46** (1971), 162–174.
6. ———, *Jones polynomials of periodic links*, Pacific J. Math. **131** (1988), 319–329.
7. P. Traczyk, 10_{101} *has no period* 7: *a criterion for periodicity of links*, Proc. Amer. Math. Soc. **108** (1990), 845–846.

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