PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 4, December 1990

COVERS OF DEHN FILLINGS ON ONCE-PUNCTURED TORUS BUNDLES. II

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(Communicated by Frederick R. Cohen)

ABSTRACT. Let M be a compact, orientable 3-manifold that fibers over S^1 with fiber a once-punctured torus. We prove that infinitely many Dehn fillings on M yield manifolds with virtually $\mathbb{Z}$-representable fundamental groups.

1. INTRODUCTION

Let M be a compact, orientable 3-manifold that fibers over the circle with fiber a once-punctured torus. In this paper we use cut and paste techniques to prove that infinitely many Dehn fillings on M yield manifolds with virtually $\mathbb{Z}$-representable fundamental groups. This extends the results in [B].

This work was motivated by the question of whether every compact, orientable, irreducible 3-manifold with infinite fundamental group is finitely covered by a Haken manifold (is virtually Haken). It follows from work of Thurston, Waldhausen, and others that an affirmative answer would have important consequences for the classification of 3-manifolds with infinite fundamental group.

A group is virtually $\mathbb{Z}$-representable if it has a finite index subgroup with a nontrivial representation to the integers. If N is a compact 3-manifold, the virtual $\mathbb{Z}$-representability of $\pi_1(N)$ is equivalent to the existence of a finite cover $\widetilde{N} \to N$ with $\operatorname{rank} H_1(\widetilde{N}) \geq 1$. If N is also orientable and irreducible, it follows that $\widetilde{N}$ is Haken.

The manifolds obtained by Dehn filling on once-punctured torus bundles have been much studied (see [FH], [CJR], [T]). In particular they provided the first examples of hyperbolic (hence irreducible) non-Haken 3-manifolds.

2. STATEMENT OF RESULTS

Let M be a compact, orientable 3-manifold that fibers over S^1 with fiber a once-punctured torus, T_0, and characteristic homeomorphism $h\colon T_0 \to T_0$ which is the identity on ∂T_0, the boundary of T_0. Then

$$M \cong T_0 \times [0, 1]/(h(x), 0) \sim (x, 1),$$

Received by the editors June 16, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57M10, 57N10.

where we identify $T_0 \times \{1\}$ to $T_0 \times \{0\}$ by h.

Choose a point b in ∂T_0 and consider the loops $\alpha = b \times [0, 1]/\sim$ and $\beta = \partial T_0$ (see Figure 1).

Denote by $M(\mu, \lambda)$ the manifold obtained from Dehn filling on M with respect to the loop $\alpha^\mu \beta^\lambda$ in ∂M. By Dehn filling on a 3-manifold N with respect to a simple loop in a boundary torus we mean gluing a solid torus to ∂N so that this loop bounds a meridional disk in the solid torus.

Let D_x denote the right-handed Dehn twist about the loop x and D_y denote the left-handed Dehn twist about the loop y in T_0. Then any orientation preserving homeomorphism $h: T_0 \to T_0$ can be represented up to isotopy by a composition of powers of D_x and D_y. We prove:

Theorem 1. *Let M be a once-punctured torus bundle over S^1 with characteristic homeomorphism of the form $D_x^{r_1} \circ D_y^{s_1} \circ \cdots \circ D_x^{r_k} \circ D_y^{s_k}$ such that $4|s_i$ for each $i = 1, \ldots, k$. Let (μ, λ) be relatively prime integers with $|\lambda| \geq 2$, and let $R = r_1 + \cdots + r_k$ and $m = |R\mu + 2\lambda|$. Then $M(\mu, \lambda)$ has virtually $\mathbb{Z}$-representable fundamental group provided there is a prime $p|\lambda$ such that one of the following conditions on m and p holds:*

(a) $p = 2$ *and* $4|m$,
(b) $p \geq 3$ *and* $p|m$,
(c) $p \nmid m$, $p < m$, *and* $m \geq 4$.

Remarks. 1. Since $M(\mu, \lambda) = M(-\mu, -\lambda)$, we will assume that $\mu \geq 1$. For a given $\mu \geq 1$, $M(\mu, \lambda)$ has virtually $\mathbb{Z}$-representable fundamental group for all but finitely many λ.

2. We assume that g.c.d. $(r_1, \ldots, r_k) = 1$ since the complementary case was treated in [B]. Theorem 1 holds true if $4|r_i$ for each $i = 1, \ldots, k$ and g.c.d. $(s_1, \ldots, s_k) = 1$, provided we let $S = s_1 + \cdots + s_k$ and $m = |S\mu + 2\lambda|$.

We have the following consequence of Theorem 1:

Theorem 2. *Let M be a once-punctured torus bundle over S^1. Then infinitely many Dehn fillings on M yield manifolds with virtually $\mathbb{Z}$-representable fundamental groups. Specifically, there exists an integer $c \in \{1, 2, 3, 4, 6\}$, depending on M, such that for each μ a nonzero multiple of c, $M(\mu, \lambda)$ has virtually $\mathbb{Z}$-representable fundamental group for all but finitely many of the λ coprime to μ.*

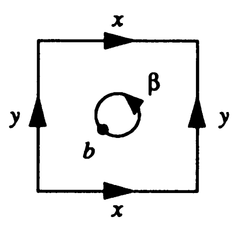

FIGURE 1

In §3 we prove Theorem 1. Section 4 is devoted to showing how Theorem 2 follows from Theorem 1.

3. Proof of Theorem 1

For a torus bundle M, with characteristic homeomorphism h, and Dehn filling parameters (μ, λ) as in Theorem 1, we prove our result by constructing a finite cover $\widetilde{M} \to M$ such that:

(i) The cover $\widetilde{M} \to M$ extends to a cover $N \to M(\mu, \lambda)$ by Dehn filling on $\widetilde{M}$ and on M.

(ii) $\operatorname{rank} H_1(\widetilde{M}) >$ number of components of $\partial \widetilde{M}$.

Property (ii) guarantees that any manifold obtained by Dehn filling on $\widetilde{M}$ (hence N) has positive first Betti number; thus $M(\mu, \lambda)$ has virtually $\mathbb{Z}$-representable fundamental group.

The cover $\widetilde{M}$ is obtained by constructing a cover $F \xrightarrow{\pi} T_0$ which h lifts to a homeomorphism $\tilde{h}: F \to F$. The mapping torus, $\widetilde{M}$, of the pair $(F, \tilde{h})$ is the desired cover of M.

3.1. Recall that the characteristic homeomorphism, h, of M is of the form $D_x^{r_1} \circ D_y^{s_1} \circ \cdots \circ D_x^{r_k} \circ D_y^{s_k}$ where $4 | s_i$ for $i = 1, \ldots, k$.

For each integer $d \geq 3$, we construct a $4d$-fold cover $S \xrightarrow{\pi} T_0$ corresponding to the kernel of the map

$$\theta: \pi_1(T_0) \to \mathbb{Z}/d \oplus \mathbb{Z}/4$$

defined by $\theta([x]) = (1, 0)$ and $\theta([y]) = (0, 1)$. The loop x (resp. y) in T_0 is covered by four loops $\tilde{x}_1, \ldots, \tilde{x}_4$ (resp. d loops $\tilde{y}_1, \ldots, \tilde{y}_d$) in S that project d to 1 onto x (resp. 4 to 1 onto y). The surface S for $d = 5$ is pictured in Figure 2.

3.2. We now alter S by cutting and pasting to obtain a cover $F \xrightarrow{\pi} T_0$ to which both Dehn twists D_x and D_y^4 lift.

It will be helpful to think of S as being divided into a grid by the curves $\{\tilde{x}_i\}$, $\{\tilde{y}_i\}$. Thus S consists of 4 rows numbered from bottom to top. Each row contains d squares, each with a boundary circle (see Figure 2).

Make $2d$ vertical cuts in S, one between each circle of row 1 (resp. row 3) and the circle directly above in row 2 (resp. row 4). Now identify the left edge of each cut to the right edge of the cut $d-2$ to the right $\pmod{d}$. Call the resulting surface F. The surface F for $d = 5$, with identifications numbered, is shown in Figure 3.

Lemma 3.1. *The surface F is a $4d$-fold cover of T_0. If d is odd, F has genus $2d - 1$ and four boundary circles, each of which projects d to 1 onto $\beta = \partial T_0$. If d is even, F has genus $2d - 3$ and eight boundary circles, each projecting $d/2$ to 1 onto β.*

The loop x (resp. y) in T_0 is covered by four loops $\tilde{x}_1, \ldots, \tilde{x}_4$ (resp. d loops $\tilde{y}_1, \ldots, \tilde{y}_d$) in F that project d to 1 onto x (resp. 4 to 1 onto y). F

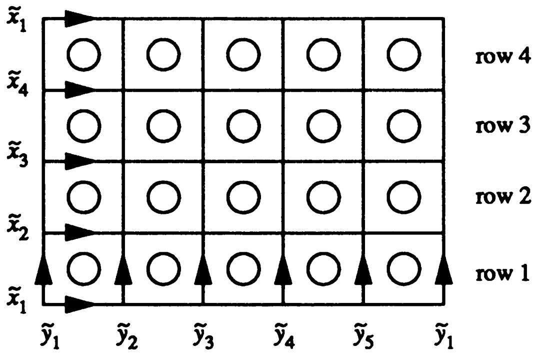

FIGURE 2

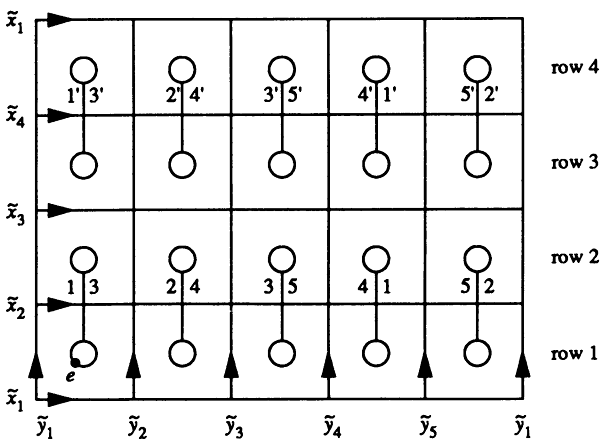

FIGURE 3

is divided into a grid by the curves $\{\tilde{x}_i\}$, $\{\tilde{y}_i\}$. We number the four rows of F from bottom to top (see Figure 3).

Since the loops $\{\tilde{y}_i\}$ in F project 4 to 1 onto y in T_0, the Dehn twist D_y^4 lifts to simultaneous Dehn twists about the $\{\tilde{y}_i\}$.

Moreover, D_x lifts to a homeomorphism, $\widetilde{D}_x$, of F which can be viewed as a $1/d$ "fractional" Dehn twist about each of the $\{\tilde{x}_i\}$. Indeed, if we require our lifts to fix the basepoint, e of F in the first square of row 1 (see Figure 3), then $\widetilde{D}_x$ fixes pointwise rows 1 and 3 while shifting rows 2 and 4 each $d-1$ squares to the right $(\bmod d)$.

Since D_x and D_y^4 both lift to F, it follows that any homeomorphism $h: T_0 \to T_0$ of the form $D_x^{r_1} \circ D_y^{s_1} \circ \cdots \circ D_x^{r_k} \circ D_y^{s_k}$ where $4|s_i$ also lifts to a homeomorphism $\tilde{h}: F \to F$. Denote by $\widetilde{M}$ the mapping torus of $(F, \tilde{h})$.

3.3. We prove that $H_1(\widetilde{M})$ satisfies property (ii) that was given at the beginning of §3:

Lemma 3.2. $\operatorname{rank} H_1(\widetilde{M}) >$ *number of components of* $\partial\widetilde{M}$.

Proof. This is equivalent to the existence of non-boundary classes in $H_1(F)$ that are fixed by $\tilde{h}_*$ (see [B] or [H]).

A portion of F is shown in Figure 4 (see p. 1104). The class $[\gamma]+[\delta]$ in $H_1(F)$ corresponding to the loops γ, δ is fixed by $\tilde{h}_*$ since $\widetilde{D}_x$ fixes γ and δ and they each intersect with opposite orientations the same two Dehn twist curves in the set $\{\tilde{y}_i\}$.

Finally $[\gamma]+[\delta]$ is not homologous in $H_1(F)$ to a sum of boundary circles in F, since γ and δ have nonzero intersection with the curves σ and ρ while all the boundary circles have null intersection with σ and ρ. □

Remark. This argument generalizes to show that $\operatorname{rank} H_1(\widetilde{M}) \geq$ (number of components of $\partial\widetilde{M}) + [d/3]$.

3.4. Now consider M with characteristic homeomorphism h as in Theorem 1. We use the covers $\widetilde{M} \to M$ constructed in 3.2 to prove our result in the case where the integers m, p in the theorem satisfy condition (a) or (b).

Let (μ, λ) be Dehn filling parameters for M, $\mu \geq 1$, $|\lambda| \geq 2$. Recall that $R = r_1 + \cdots + r_k$, $m = |R\mu + 2\lambda|$, and suppose $p \geq 3$ is a prime factor of λ dividing m (condition (b)).

Let $\widetilde{M}$ be the mapping torus of $(F, \tilde{h})$ for F corresponding to $d = p$. Given Lemma 3.2, all that remains to show is that the loop $\alpha^\mu\beta^\lambda$ in ∂M lifts to loops in $\partial\widetilde{M}$. Since p divides m and λ, p also divides R and it follows from the construction of F and the action of $\widetilde{D}_x$ on F that $\tilde{h}$ fixes pointwise the four boundary components of F, each of which projects p to 1 to $\beta = \partial T_0$. Thus α, β^p and hence β^λ in ∂M lift to loops in $\partial\widetilde{M}$ so that $\alpha^\mu\beta^\lambda$ lifts to loops in $\partial\widetilde{M}$. Choose one lift in each component of $\partial\widetilde{M}$ and label them $\{c_i\}$.

Lemma 3.3. *The cover* $\widetilde{M} \to M$ *extends to a cover* $N \to M(\mu, \lambda)$ *by Dehn filling on* $\widetilde{M}$ *and* M *with respect to* $\{c_i\}$ *in* $\partial\widetilde{M}$ *and* $\alpha^\mu\beta^\lambda$ *in* ∂M.

Lemma 3.4. $\operatorname{rank} H_1(N) \geq 1$.

Proof. Since $\operatorname{rank} H_1(\widetilde{M}) >$ number of components of $\partial\widetilde{M}$ (Lemma 3.2), it follows that all manifolds obtained by Dehn filling on $\widetilde{M}$ (hence N) have positive first Betti number. □

If $p = 2$ and $4|m$ (condition (a)), then $4|R$ and we let $\widetilde{M}$ be the mapping torus of $(F, \tilde{h})$ for F corresponding to $d = 4$. Now the above argument applies almost verbatim.

3.5. In this section we conclude the proof of Theorem 1 with the case where M, h, and (μ, λ) are such that the integers m and p satisfy condition (c): $p \nmid m$, $p < m$, and $m \geq 4$.

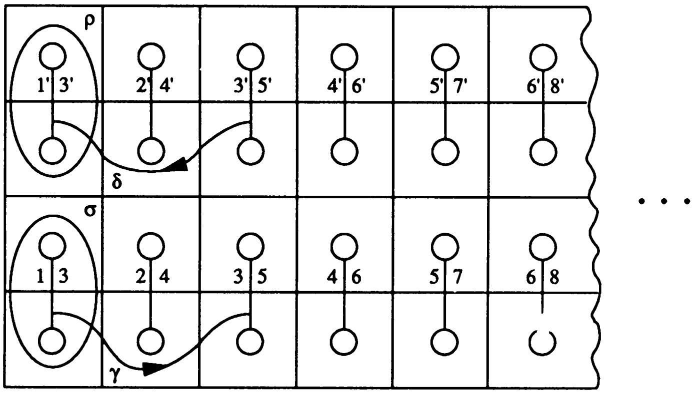

FIGURE 4

For clarity of exposition we will assume that m is odd. The proof for m even is analogous.

Consider the cover $\widetilde{M} \to M$ where $\widetilde{M}$ is the mapping torus of $(F, \tilde{h})$ for F corresponding to $d = m$ (see §3.2). By Lemma 3.2, all Dehn fillings on $\widetilde{M}$ yield manifolds with positive first Betti number.

We now show that the loop $\alpha^{\mu}\beta^{\lambda}$ in ∂M lifts to loops in two of the four boundary tori of $\widetilde{M}$, which we denote by T_i and index so that $\tilde{\beta}_i = T_i \cap F$ is the circle in the ith row of F for $i = 1, \dots, 4$. The circles $\tilde{\beta}_i$ of F project m to 1 onto $\beta = \partial T_0$ (see §3.2).

Here it is important to specify orientations. All bundles are constructed by identifying the back face, $F \times \{1\}$, to the front face, $F \times \{0\}$, via h $((h(x), 0) \sim (x, 1))$. All curves transverse to F are oriented from front to back and all boundary circles of F are given a counterclockwise orientation.

Lemma 3.5. *The loop $\alpha^{\mu}\beta^{\lambda}$ in ∂M lifts to loops in T_2 and T_4 of $\partial\widetilde{M}$.*

Proof. First note that $\tilde{h}$ shifts rows 2 and 4 of F, rotating $\tilde{\beta}_2$ and $\tilde{\beta}_4$ (recall action of $\widetilde{D}_x$, §3.2). One checks that the loops $\alpha\beta^{R(m-1)/2}$ and β^m lift to loops in T_2 and T_4. But then $\alpha^{\mu}\beta^{\lambda}$ also lifts to a loop, being a combination of $\alpha\beta^{R(m-1)/2}$ and β^m. Indeed, $m = |R\mu + 2\lambda| \Rightarrow R\mu + 2\lambda \equiv 0 \pmod{m} \Rightarrow$ $\mu R(m-1) - 2\lambda \equiv 0 \pmod{m} \Rightarrow \mu R(m-1)/2 \equiv \lambda \pmod{m}$. □

We would be done except for the fact that $\alpha^{\mu}\beta^{\lambda}$ does not lift to loops in T_1, and T_3 of $\widetilde{M}$. We resolve this problem by altering $\widetilde{M}$ to obtain a new cover $M' \to M$ that satisfies Lemma 3.2, and has the property that $\alpha^{\mu}\beta^{\lambda}$ lifts to loops in each component of $\partial M'$. M' is the mapping torus of (F', h') where F' is obtained from F by cut and paste alteration of the boundary circles $\tilde{\beta}_1$, $\tilde{\beta}_3$ in rows 1 and 3.

The pattern of cut and paste depends on the parity of λ. In each case we first illustrate the general procedure with an example. Consider M with homeomorphism $h = D_x \circ D_y^4$ and $(\mu, \lambda) = (1, -4)$. Then $R = 1$, $m = 7$, and $p = 2$. Starting with $F \xrightarrow{\pi} T_0$ for $d = m = 7$, we obtain $F' \xrightarrow{\pi} T_0$ by making eight horizontal cuts and identifications in F as pictured in Figure 5. Rows 1 and 3 of F' each contain two boundary circles that project 2 to 1 onto β and three circles that project 1 to 1 onto β. It is clear from the construction of F' that D_x and D_y^4 lift to F' and hence that h lifts to a homeomorphism $h' \colon F' \to F'$. The loop $\alpha\beta^{-4}$ in ∂M lifts to loops in the boundary tori of M' so that $M' \to M$ extends to a cover $N' \to M(1, -4)$. The class $[\gamma] + [\delta]$ in $H_1(F')$ corresponding to the curves γ, δ in F' that are in Figure 5 is nonboundary and fixed by h'_*. Hence M' satisfies Lemma 3.2 and rank $H_1(N') \geq 1$.

In general if λ is even, let $p = 2$ and construct (F', h') from (F, h) by making $2m - 6$ horizontal cuts in F: $m - 3$ cuts each in rows 1 and 3 between the adjacent boundary circles in the last $m - 2$ squares. In each row number the bottom edges of the $m - 3$ cuts from left to right and the top edges from right to left and then identify edges accordingly (see Figure 5). Rows 1 and 3 of F' each contain two boundary circles that project 2 to 1 onto β and $m - 4$ boundary circles that project 1 to 1 onto β.

By construction, h lifts to a homeomorphism $h' \colon F' \to F'$, the curves γ, δ in F carry over to F' giving a nonboundary class $[\gamma] + [\delta]$ in $H_1(F')$ that is fixed by h'_*, and the loop $\alpha^\mu \beta^\lambda$ in ∂M lifts to loops in $\partial M'$. Thus $M' \to M$ extends to a cover $N' \to M(\mu, \lambda)$ with rank $H_1(N') \geq 1$.

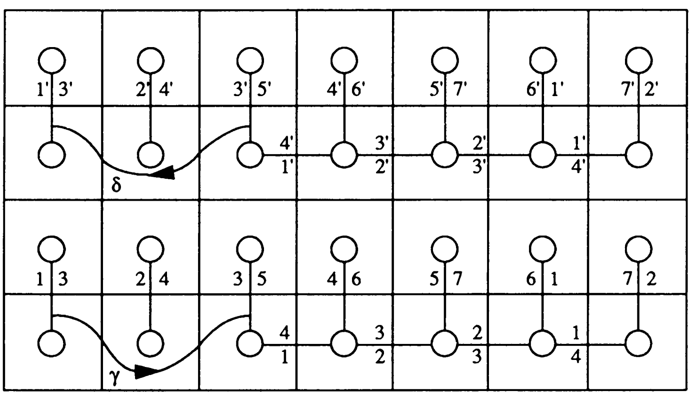

FIGURE 5

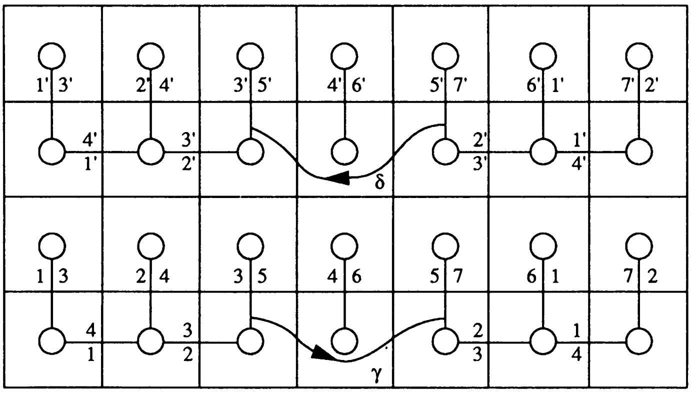

FIGURE 6

We illustrate the case λ odd by M with $h = D_x \circ D_y^4$ and $(\mu, \lambda) = (1, 3)$. Hence $R = 1$, $m = 7$, and $p = 3$. The desired F' is pictured in Figure 6. Note the position of the eight cuts as well as the curves γ, δ.

In general if λ is odd, then p is odd and $m - p$ is even. We construct F' by making $2(m-p)$ horizontal cuts: $m-p$ cuts each in rows 1 and 3 between adjacent circles in all but the $p-2$ middle squares. Now number the bottom edges of the $m-p$ cuts from left to right and the top edges from right to left and identify. Rows 1 and 3 of F' each contain $m-p$ boundary circles that project 1 to 1 onto β and one circle that projects p to 1 onto β. The curves γ, δ are in the center of rows 1 and 3 as in Figure 6.

Remark. If m is even, the arguments are similar. The surface F' for $m = 8$, $p = 2$ is illustrated in Figure 7 and for $m = 8$, $p = 3$ in Figure 8. Generalizations are left to the reader.

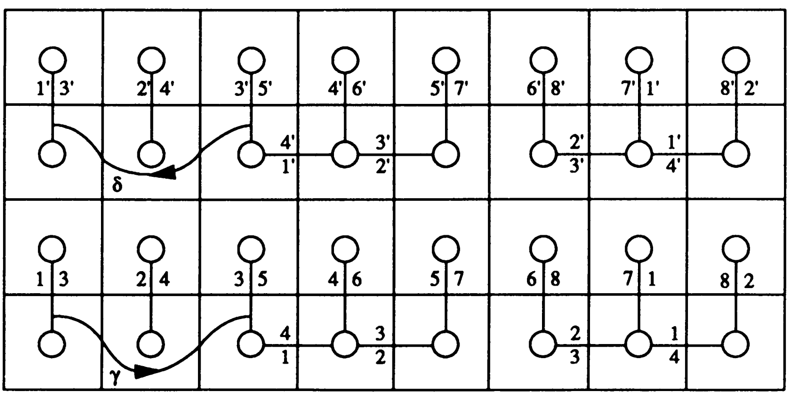

FIGURE 7

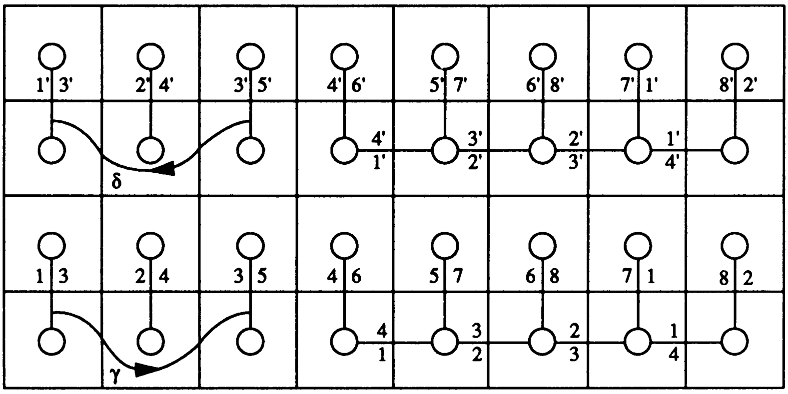

FIGURE 8

4. PROOF OF THEOREM 2

Let M be an oriented once-punctured torus bundle over S^1 with characteristic homeomorphism h, and let $[h] \in \mathrm{SL}_2(\mathbb{Z})$ be the monodromy matrix representing the action of h_* on $H_1(T_0)$.

Fact 4.1. M is determined up to oriented S^1 bundle equivalence by the conjugacy class of $[h]$ in $\mathrm{SL}_2(\mathbb{Z})$ (see [CJR]).

In particular $[D_x] = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $[D_y] = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ generate $\mathrm{SL}_2(\mathbb{Z})$.

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ denote the subgroup generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$. Then Theorem 1 applies to bundles with monodromy $[h] \in \Gamma$.

Fact 4.2. Γ contains $\Gamma(4)$ the 4-congruence subgroup, and $\mathrm{SL}_2(\mathbb{Z})/\Gamma(4) \cong \mathrm{SL}_2(\mathbb{Z}/4)$ consists of elements of order $c \in \{1, 2, 3, 4, 6\}$.

Now given M with monodromy $[h]$, by Fact 4.2 there exists a smallest positive integer $c \in \{1, 2, 3, 4, 6\}$ for which the c-fold cyclic cover, M', of M has monodromy $[h^c]$ conjugate to an element of Γ, and by Fact 4.1, M' is bundle equivalent to an M'' to which Theorem 1 applies.

Note that $M'' \to M$ is a c-fold cover and let α'', β'' (resp. α, β) be the basis for $H_1(\partial M'')$ (resp. $H_1(\partial M)$) described in §2. It is clear that α'' projects to a loop of the form $\alpha^c\beta^t$ and β'' projects to β. Thus Theorem 1 applied to M'' guarantees that for each μ a nonzero multiple of c, $M(\mu, \lambda)$ has virtually $\mathbb{Z}$-representable fundamental group for all but finitely many of the λ coprime to μ.

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