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THE VIRTUAL Z-REPRESENTABILITY OF 3-MANIFOLDS WHICH ADMIT ORIENTATION REVERSING INVOLUTIONS

SHICHENG WANG

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ABSTRACT. We prove a result which supports the Waldhausen Conjecture, i.e., suppose M is an irreducible orientable 3-manifold with $|\pi_1(M)| = \infty$; if M admits an orientation reversing involution τ, and M has a nontrivial finite cover, then some finite cover $\widetilde{M}$ of M has positive first Betti number.

An important conjecture in 3-manifolds theory is the so-called

Waldhausen Conjecture. *Suppose M is an irreducible orientable 3-manifold with $|\pi_1(M)| = \infty$, then M is finitely covered by a Haken manifold.*

Waldhausen Conjecture is a corollary of the so-called

Strong Waldhausen Conjecture. *Suppose M is an orientable 3-manifold with $|\pi_1(M)| = \infty$, then M is finitely covered by some 3-manifold with positive first Betti number.*

Remark. Any closed nonorientable 3-manifold has positive first Betti number (see [2]).

In this note we prove the following theorem which supports the Strong Waudhausen Conjecture.

Theorem. *Suppose M is an irreducible orientable 3-manifold with $|\pi_1(M)| = \infty$. If M admits an orientation reversing involution τ, and M has a nontrivial finite cover, them some finite cover $\widetilde{M}$ of M has positive first Betti number.*

A rational homology 3-sphere is a 3-manifold M such that $H_*(M; Q) = H_*(S^3; Q)$; here Q is the field of rational numbers and S^3 is the 3-sphere. Since we only consider compact 3-manifolds in this note, the fact that the first Betti number of a closed 3-manifold M is zero is equivalent to the fact that M is a rational homology 3-sphere. We say a finite group G acts freely on M, if for any $g \in G$, $\mathrm{fix}(g) \neq \emptyset$ implies that $g = \mathrm{id}$; here $\mathrm{fix}(g)$ is the fixed point set of g.

Proof of theorem. Suppose a closed irreducible orientable 3-manifold M is not S^3 or RP^3, and M admits an orientation reversing involution τ such that

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$\operatorname{fix}(\tau) = \varnothing$ or $\operatorname{fix}(\tau)$ contains a 2-dimensional component, then M or a double cover of M has positive first Betti number. (See [3]). Hence to prove theorem, we need only to prove the following Theorem 1.

Theorem 1. *If a rational homology 3-sphere M admits an orientation reversing involution τ with only isolated fixed points, and M has a nontrivial finite cover, then some finite cover $\widetilde{M}$ of M has positive first Betti number.*

Remark 1. Theorem 1 was first proved in [3] under the additional condition that $\pi_1(M)$ has a subgroup of even index. In [4], it was proved that if Theorem 1 is not true, then the commutator subgroup of $\pi_1(M)$ is an infinite group with no proper subgroup of finite index and which has odd index in $\pi_1(M)$. We realize that the work in [4] which contains the argument used at the end of our proof is a significant step toward Theorem 1.

Remark 2. The conditions that M admits an orientation reversing involution τ with only isolated fixed points, and M has a nontrivial finite cover force $\pi_1(M)$ to be an infinite group.

Proof of Theorem 1. In [1], Epstein proved that the fundamental group of a nonorientable 3-manifold, if finite, must be Z_2. The proof of Theorem 1 is based on the following generalization of Epstein's result observed by G. Mess. The elementary proof given here was discussed with Q. Zhou.

Proposition (Epstein-Mess). *If a finite group G acts freely on a compact orientable 3-manifold $\widetilde{M}$, with $H_1(\widetilde{M}, Z) = 0$, such that some element of G is orientation reversing on $\widetilde{M}$, then $G = Z_2$.*

Proof of the proposition. We list the facts which were proved in [1] or Chapter 9 of [2]. The (1), (2), (3), (4) below can be found in the (i), (ii), (iv), (v) of the proof of Theorem 9.5 in [2], (5) is Lemma 9.1 in [2].

(1) Let $M = \widetilde{M}/G$, then M has two RP^2 boundary components.
(2) Let $M_1 = RP^\infty \cup M \cup RP^\infty$ and $\widetilde{M}_1$ is the covering of M_1 corresponding to $\pi_1(\widetilde{M}) \subset \pi_1(M) = \pi_1(M_1)$. Then $H_1(\widetilde{M}_1, Z) = 0$, $H_2(\widetilde{M}_1, Z) = 0$, and $H_3(M_1, Z) = Z_2 \oplus Z_2$.
(3) If G is not Z_2, then G contains a dihedral group $D_{2q} = \langle s, t\colon s^2 = t^q = 1, sts = t^{-1}\rangle$; here s is orientation reversing and $q > 1$ is an odd prime number. So we may assume that $G = D_{2q}$, $q > 1$.
(4) $H_3(D_{2q}, Z) = Z_{2q}$.
(5) If $\pi_1(X) = G$ and $\pi_2(X) = 0$, then $H_3(G, Z)$ is a quotient group of $H_3(X, Z)$. If G is not Z_2, then we may assume that $G = D_{2q}$, $q > 1$, by (3). We want to build a complex M^* from M_1 by attaching cells such that

$$\pi_1(M^*) = D_{2q},$$
$$\pi_2(M^*) = 0, \quad \text{and}$$
$$H_3(M^*, Z) = H_3(M_1, Z) = Z_2 \oplus Z_2.$$

Then $H_3(D_{2q}, Z) = Z_{2q}$ would be a quotient group of $H_3(M^*, Z) = Z_2 \oplus Z_2$ by (4) and (5). It is impossible since $q > 1$. So we must have $G = Z_2$.

Construction of M^*:

Step 1. Attaching 2-cells to $\widetilde{M}_1$ to get a $\widetilde{M}_2 = \widetilde{M}_1 \cup \{g(D_i), g \in G, i = 1, 2, \dots, k$ for *some* $k\}$ such that

(I) $g(D_i) \cap \widetilde{M}_1 = \partial g(D_i)$ and $g(D_i) \cap g'(D_j) \subset \partial g(D_i)$, for any $g, g' \in G$.

(II) $\pi_1(\widetilde{M}_2) = 0$.

(III) The action G can be extended to $\widetilde{M}_2$ freely.

Let D be any $g(D_i)$. From the exact sequence of pair $(\widetilde{M}_1 \cup D, \widetilde{M}_1)$, Excision Theorem, and the fact ∂D is homological zero in $\widetilde{M}_1$, we get a short exact sequence

$$0 \to H_2(\widetilde{M}_1, Z) \to H_2(\widetilde{M}_1 \cup D, Z) \to Z \to 0$$

which is split; therefore every disk we attached yields a free generator of second homology group. Then from $H_2(\widetilde{M}_1, Z) = 0$, we get

(a) $\pi_2(\widetilde{M}_2) = H_2(\widetilde{M}_2, Z)$ is a free Z-module of rank $k|G|$.

Again, since $H_1(\widetilde{M}_1, Z) = 0$, for each D_i, ∂D_i bounds a surface F_i in $\widetilde{M}_1$. Since $\pi_1(\widetilde{M}_2) = 0$, we can surgery F_i along some immersed disks in $\widetilde{M}_2$ to get an immersed disk $\overline{D}_i$. Then $D_i \cup \overline{D}_i$ is an image of a sphere, $h_i(S^2)$, which hits $g(D_j)$ algebraical zero times for each $g(D_j) \neq D_i$, here $g \in G$, $j = 1, \dots, k$, and h_i is a map from S^2 to $\widetilde{M}_2$. Now it is easy to see that $g(h_i(S^2))$ is exactly a generator corresponding to $g(D_i)$, so $\{g(h_i(S^2)), g \in G, i = 1, 2, \dots, k\}$ is a basis of $\pi_2(\widetilde{M}_2) = H_2(\widetilde{M}_2, Z)$ as a free Z-module. Hence we get

(b) There is a *ZG-basis* $\{h_i : S^2 \to \widetilde{M}_2, i = 1, 2, \dots, \}$ of $\pi_2(\widetilde{M}_2) = H_2(\widetilde{M}_2, Z)$.

Remark. If we use a weak condition $H_1(\widetilde{M}_1, Q) = 0$ instead of the condition $H_1(M_1, Z) = 0$, then (a) is still true, however (b) is no longer true.

Now G acts freely on $\widetilde{M}_2$. Let $M_2 = \widetilde{M}_2/G$, M_2 is obtained from M_1 by attaching 2-cells which does not change H_3. So we have $\pi_1(M_2) = G$ and $H_3(M_2, Z) = H_3(M_1, Z)$.

By (b), for each i, the orbit $G(h_i(S^2))$ in $\widetilde{M}_2$ yields an immersed sphere S_i in M_2. If some nontrivial linear combination of S_i's is ∂C for some 3-chain C in M_2, then some nontrivial linear combination of $g(h_i(S^2))$'s would be $\partial(p^{-1}(C))$, here $p: \widetilde{M}_2 \to \widetilde{M}_1$ is the covering map. It is impossible. This shows that $\{S_i, i = 1, \dots, k\}$ are linearly independent in $H_2(M_2, Z)$.

Step 2. For each i, attaching a 3-cell D_i^3 to S_i via

$$D_i^3 \to S^2 \to \widetilde{M}_2 \to M_2.$$

From the exact sequence of the pair $(M_2 \cup D_i^3, M_2)$, the Excision Theorem and the fact that S_i is an element of infinite order in $H_2(M_2, Z)$, one gets $H_3(M_2 \cup D_i^3, Z) = H_3(M_2, Z)$.

Let $M_3 = M_2 \cup \bigcup_{i=1}^{k} D_i^3$, then $\pi_1(M_3) = G$ (since the way 3-cells attached does not change π_1) and $H_3(M_3, Z) = H_3(M_2, Z) = H_3(M_1, Z) = Z_2 \oplus Z_2$.

Let $\widetilde{M}_3$ be the universal cover of M_3, then $\widetilde{M}_3$ is obtained from $\widetilde{M}_2$ by attaching $k|G|$ 3-cells to those $\{g(h_i(S^2)), g \in G, i = 1, \dots, k\}$ and hence $\pi_2(\widetilde{M}_3) = 0$, finally we get $\pi_2(M_3) = 0$.

Set $M^* = M_3$, we have finished the proof of the proposition.

Let us come back to the proof of the Theorem 1.

Suppose a rational homology sphere M admits an orientation reversing involution τ with only isolated fixed points. Remove a suitable open ball neighborhood of each fixed point, denote the resulting manifold by N, then τ acts freely on N. To prove the Theorem 1, we need only to show that if N has a nontrivial finite cover, then some finite cover $\widetilde{N}$ of N has positive first Betti number.

The proof of next two lemmas can be found in [1] or in the proof of Theorem 9.5 in [2].

Lemma 1. *If an orientation reversing involution τ acts freely on an orientable compact 3-manifold N, where $H_i(N, Q) = 0$, then two and only two sphere boundary components S_1, S_2 of N are invariant under τ.*

Lemma 2. *Suppose τ and N are as in Lemma 1. Let $\widetilde{N} \to N$ be a covering of odd degree and $\tilde{\tau}$ be a lift of τ. If $H_1(\widetilde{N}, Q) = 0$, then one invariant sphere boundary component $\widetilde{S}_1$ of $\tilde{\tau}$ lies in the preimage of S_1, another invariant sphere boundary component $\widetilde{S}_2$ of $\tilde{\tau}$ lies in the preimage of S_2.*

Let $G = \pi_1(N)$. Suppose $\infty > |H_1(N, Z)| > 1$. Then the commutator subgroup G_1 of G is a proper characteristic subgroup of G. Let $\widetilde{N}_1$ be the covering of N corresponding to G_1. The nontrivial deck transformation group of the covering $\widetilde{N}_1 \to N$ and $\tilde{\tau}$, a lift of τ on $\widetilde{N}_1$, forms a group which is not Z_2 and acts freely on $\widetilde{N}_1$. (Since each element in this group is either in the deck transformation group or a lift of the free involution.) By the proposition above, $|H_1(\widetilde{N}_1, Z)| > 1$. If $|H_1(\widetilde{N}_1, Z)|$ is infinite, then the theorem is proved. Otherwise $\infty > |H_1(N, Z)| > 1$ implies that G_1 has a proper characteristic subgroup of finite index. (For example, the second commutator subgroup.)

Suppose $|H_1(N, Z)| = 1$, then $G = G_1$. By the condition of the Theorem, G_1 has a proper characteristic subgroup of finite index.

So we may assume that the commutator subgroup G_1 has a proper characteristic subgroup G_2 of finite index.

Let $\widetilde{N}$ be the nontrivial covering of N corresponding to G_2. If $|H_1(\widetilde{N}, Z)|$ is infinite, then the theorem is proved. So we may assume that $|H_1(\widetilde{N}, Z)|$ is finite. The nontrivial deck transformation group of the covering $\widetilde{N} \to N$ is G/G_2 which is not abelian by the choice of G_2.

Let $\tilde{\tau}$ be a lift of τ on $\widetilde{N}$, $\widetilde{S}_1$, and $\widetilde{S}_2$ be the two (and only two) invariant sphere boundary components of $\tilde{\tau}$. Then $\tilde{\tau}^2$ is a lift of $\tau^2 = \mathrm{id}$; therefore $\tilde{\tau}^2$ lies in the deck transformation group G/G_2. On the other hand, $\tilde{\tau}^2$ is an orientation preserving homeomorphism on $\widetilde{S}_1$ which has fixed points. So we have $\tilde{\tau}^2 = \mathrm{id} \in G/G_2$.

Now we define a self-homomorphism H on the deck transformation group G/G_2 as follows:

$$H(x) = \tilde{\tau}x\tilde{\tau}^{-1} = \tilde{\tau}x\tilde{\tau} \text{ for } x \in G/G_2.$$

For any $x \in G/G_2$, $x\tilde{\tau}x^{-1}(x(\widetilde{S}_1)) = x(\widetilde{S}_1)$ and $x(\widetilde{S}_1) \neq \widetilde{S}_2$ by Lemma 2. Further for any $x \in G/G_2$, $x \neq \mathrm{id}$, since x acts freely and preserves orientation of $\widetilde{S}_1$, we have $x(\widetilde{S}_1) \neq \widetilde{S}_1$. This implies that $x\tilde{\tau}x^{-1} \neq \tilde{\tau}^{-1}$ for every $x \in G/G_2$, $x \neq \mathrm{id}$. (Otherwise $\tilde{\tau}$ has three different invariant sphere boundary components $\widetilde{S}_1$, $\widetilde{S}_2$, $x(\widetilde{S}_1)$). In other words,

$$x\tilde{\tau}x^{-1}\tilde{\tau}^{-1} \neq \mathrm{id} \text{ for every } x \in G/G_2,\ x \neq \mathrm{id}.$$

Now it is easy to verify that $x \to x\tilde{\tau}x^{-1}\tilde{\tau}^{-1}$ is a 1-1 and onto map on the finite set G/G_2. Hence for every $y \in G/G_2$, $y = x\tilde{\tau}x^{-1}\tilde{\tau}^{-1}$ for *some* $x \in G/G_2$. Now

$$H(y) = \tilde{\tau}y\tilde{\tau} = \tilde{\tau}x\tilde{\tau}x^{-1}\tilde{\tau}^{-1}\tilde{\tau} = \tilde{\tau}x\tilde{\tau}x^{-1} = y^{-1}.$$

It is possible only if G/G_2 is abelian group. This is a contradiction. We have proved Theorem 1.

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REFERENCES

1. D. B. A. Epstein, *Projective planes in* 3-*manifolds*, Proc. London Math. Soc. **11** (1961), 469–484.
2. J. Hempel, 3-*manifolds*, Annals of Math Study, no. 81, Princeton Univ. Press, 1976.
3. ——, *Orientation reversing involutions and the first Betti number for finite coverings of* 3-*manifolds*, Invent. Math. **67** (1982), 133–142.
4. ——, *Virtually Haken* 3-*manifolds. Combinatorial methods in topology and algebraic geometry*, Contemp. Math. **44** (1985), 149–157.

PEKING UNIVERSITY, BEIJING, 100871, PEOPLE'S REPUBLIC OF CHINA