CONVEX BODIES WITH FEW FACES

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ABSTRACT. It is proved that if $u_1, \ldots, u_n$ are vectors in $\mathbf{R}^k$, $k \leq n$, $1 \leq p < \infty$ and

$$r = \left(\frac{1}{k} \sum_1^n |u_i|^p \right)^{1/p}$$

then the volume of the symmetric convex body whose boundary functionals are $\pm u_1, \ldots, \pm u_n$, is bounded from below as

$$|\{x \in \mathbf{R}^k : |\langle x, u_i \rangle| \leq 1 \text{ for every } i\}|^{1/k} \geq 1/\sqrt{p}\, r.$$

An application to number theory is stated.

0. INTRODUCTION

In [V], Vaaler proved that if $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ is the central unit cube in $\mathbf{R}^n$ and U is a subspace of $\mathbf{R}^n$ then the volume $|U \cap Q_n|$, of the section of Q_n by U is at least 1. This result may be reformulated as follows: if $u_1, \ldots, u_n$ are vectors in $\mathbf{R}^k$, $1 \leq k \leq n$ whose Euclidean lengths satisfy $\sum_1^n |u_i|^2 \leq k$ then

$$|\{x \in \mathbf{R}^k : |\langle x, u_i \rangle| \leq 1 \text{ for every } i\}|^{1/k} \geq 2.$$

A related theorem, (Theorem 1, below) in which the condition $\sum |u_i|^2 \leq k$ is replaced by $\max_i |u_i| \leq 1$ was proved by Carl and Pajor [CP] and Gluskin [G]. Gluskin's methods enable him to obtain sharp results in limiting cases which in turn have applications in harmonic analysis. Results closely related to Theorem 1 were also obtained by Bárány and Füredi [BF] and Bourgain, Lindenstrauss and Milman [BLM].

Theorem 1. *There is a constant $\delta > 0$ so that if $u_1, \ldots, u_n \in \mathbf{R}^k$, $1 \leq k \leq n$ are vectors of length at most 1 then*

$$|\{x \in \mathbf{R}^k : |\langle x, u_i \rangle| \leq 1 \text{ for every } i\}|^{1/k} \geq \delta / \sqrt{1 + \log(n/k)}.$$

The estimate is best possible if n is at most exponential in k, apart from the value of the constant δ. This is demonstrated by an example which had

appeared some time earlier in a paper of Figiel and Johnson [FJ]. Theorem 1 gives a lower bound on the volume ratios of the unit balls of k-dimensional subspaces of l_∞^n and hence on the distance of these subspaces from Euclidean space.

Regarding Theorem 1 as a $p = \infty$ version of Vaaler's $p = 2$ result, Kashin asked whether a similar result holds for $2 < p < \infty$. This question is answered in the affirmative by the following theorem.

Theorem 2. *Suppose* $u_1, \ldots, u_n \in \mathbf{R}^k$ *with* $k \le n$, $1 \le p < \infty$ *and let*

$$r = \left(\frac{1}{k} \sum_1^n |u_i|^p \right)^{1/p}.$$

Then

$$\left| \{ x \in \mathbf{R}^k : |\langle x, u_i \rangle| \le 1 \text{ for every } i \} \right|^{1/k} \ge \begin{cases} 2\sqrt{2}/\sqrt{p}\,r & \text{if } p \ge 2 \\ 1/r & \text{if } 1 \le p \le 2. \end{cases}$$

The lower bound is best possible (up to a constant) provided $e^p k \le n \le e^k$.

Remark. The slightly stronger result for $p \ge 2$ is isolated since for $p = 2$ it gives back exactly Vaaler's result.

Theorem 1 follows immediately from Theorem 2 by a standard optimization argument. If $(u_i)_1^n$ in $\mathbf{R}^k$ all have norm at most 1 then for any $p \in [1, \infty)$,

$$\left(\frac{1}{k} \sum_k^n |u_i|^p \right)^{1/p} \le \left(\frac{n}{k} \right)^{1/p}$$

so that

$$|\{ x : |\langle x, u_i \rangle| \le 1 \text{ for every } i \}|^{1/k} \ge 2\sqrt{2}/\sqrt{p} \left(\frac{n}{k} \right)^{1/p}$$

(for $p \ge 2$) and the latter is at least $2/\sqrt{e}\sqrt{1 + \log(n/k)}$ when $p = 2(1 + \log(n/k))$.

With the careful use of well-known methods for estimating the entropy of convex bodies it is possible to obtain more general (but less precise) estimates than that provided by Theorem 2; (see [BP]). The purpose of this paper is to provide a very short proof of Theorem 2, and a fortiori, Theorem 1.

Vaaler originally proved his theorem because of its applications to the geometry of numbers. The last section of this paper includes a statement of the generalization of Siegel's lemma which follows from Theorem 2.

1. THE LOWER BOUND

The proof of Theorem 2 makes use of the following result from [MeP] which was designed to extend Vaaler's theorem in a different direction: it estimates

the volumes of sections of the unit balls of the spaces l_p^n, $1 \le p \le \infty$. For $1 \le p \le \infty$, $n \in \mathbf{N}$ let

$$B_p^n = \left\{ x \in \mathbf{R}^n : \sum_1^n |x_i|^p \le 1 \right\}$$

be the unit ball of l_p^n.

Theorem 3. *Let U be a k-dimensional subspace of $\mathbf{R}^n$; if $1 \le p \le q \le \infty$ then*

$$\frac{|B_p^n \cap U|}{|B_p^k|} \le \frac{|B_q^n \cap U|}{|B_q^k|}. \qquad \square$$

Remark. The case $p = 2$, $q = \infty$ is Vaaler's theorem since then, the left side is 1 and the inequality states that

$$|B_\infty^n \cap U| \ge |B_\infty^k| \ge 2^k.$$

For notational convenience, the proof of Theorem 2 is divided into several short lemmas. The first is no more than a convenient form of Hölder's inequality. For $k \in \mathbf{N}$, S^{k-1} will denote the Euclidean sphere in $\mathbf{R}^k$ and $\sigma = \sigma_{k-1}$, the rotationally invariant probability measure on S^{k-1}. Also let v_k be the volume of the Euclidean unit ball in $\mathbf{R}^k$.

Lemma 4. *Let C and B be symmetric convex bodies in $\mathbf{R}^k$ with Minkowski gauges $\|\cdot\|_C$ and $\|\cdot\|_B$, respectively. Then for $p > 0$*

$$\left(\frac{|C|}{|B|} \right)^{1/k} \ge \left(\frac{k+p}{k|B|} \int_B \|x\|_C^p \, dx \right)^{-1/p}.$$

Proof.

$$\left(\frac{|C|}{|B|} \right)^{1/k} = \left(\frac{v_k}{|B|} \int_{S^{k-1}} \|\theta\|_C^{-k} \, d\sigma(\theta) \right)^{1/k}$$

$$= \left(\frac{k v_k}{|B|} \int_{S^{k-1}} \left(\frac{\|\theta\|_B}{\|\theta\|_C} \right)^k \int_0^{\|\theta\|_B^{-1}} r^{k-1} \, dr \, d\sigma(\theta) \right)^{1/k}$$

$$= \left(\frac{1}{|B|} \int_B \left(\frac{\|x\|_B}{\|x\|_C} \right)^k \, dx \right)^{1/k}$$

$$\ge \left(\frac{1}{|B|} \int_B \left(\frac{\|x\|_B}{\|x\|_C} \right)^{-p} \, dx \right)^{-1/p}$$

$$= \left(\frac{k+p}{k|B|} \int_B \|x\|_C^p \, dx \right)^{-1/p}. \qquad \square$$

(Lemma 4 appears in [MiP] as Corollary 2.2.)

Lemma 5. *Suppose $u_1, \ldots, u_n \in \mathbf{R}^k$ with $k \leq n$ and $1 \leq p < \infty$. Then*

$$|\{x \in \mathbf{R}^k : |\langle x_i, u_i \rangle| \leq 1 \text{ for every } i\}|^{1/k}$$

$$\geq 2 \left(\frac{k+p}{k} \sum_{i=1}^{n} \frac{1}{|B_p^k|} \int_{B_p^k} |\langle x, u_i \rangle|^p \, dx \right)^{-1/p}.$$

Proof. Define $T : \mathbf{R}^k \to \mathbf{R}^n$ by $(Tx)_i = \langle x, u_i \rangle$, $1 \leq i \leq n$ and let $U = T(\mathbf{R}^k)$. The problem is to estimate from below

$$|T^{-1}(B_\infty^n)|^{1/k} = |T^{-1}(U \cap B_\infty^n)|^{1/k}.$$

By Theorem 3,

$$|U \cap B_\infty^n|^{1/k} \geq 2 \left(\frac{|U \cap B_p^n|}{|B_p^k|} \right)^{1/k}$$

and so

$$|T^{-1}(B_\infty^n)|^{1/k} \geq 2 \left(\frac{|T^{-1}(B_p^n)|}{|B_p^k|} \right)^{1/k}.$$

Regard T as an operator: $l_p^k \to l_p^n$. Then by Lemma 4,

$$2 \left(\frac{|T^{-1}(B_p^n)|}{|B_p^k|} \right)^{1/k} \geq 2 \left(\frac{k+p}{k|B_p^k|} \int_{B_p^k} \|Tx\|^p \, dx \right)^{-1/p}$$

$$= 2 \left(\frac{k+p}{k|B_p^k|} \int_{B_p^k} \sum_{1}^{n} |\langle x, u_i \rangle|^p \, dx \right)^{-1/p}. \qquad \square$$

Proof of Theorem 2. Let $(u_i)_1^n$ and p be as above. For each i let v_i be the unit vector in the direction of u_i. By Lemma 5,

$$|\{x \in \mathbf{R}^k : |\langle x, u_i \rangle| \leq 1 \text{ for every } i\}|^{1/k}$$

$$\geq 2 \left(\frac{k+p}{k} \sum_{i=1}^{n} \frac{1}{|B_p^k|} \int_{B_p^k} |\langle x, u_i \rangle|^p \, dx \right)^{-1/p}$$

$$= 2 \left(\frac{k+p}{k} \sum_{1}^{n} |u_i|^p \cdot \frac{1}{|B_p^k|} \int_{B_p^k} |\langle x, v_i \rangle|^p \, dx \right)^{-1/p}$$

$$\geq 2 \left(\frac{1}{k} \sum_{i=1}^{n} |u_i|^p \right)^{-1/p} \min \left(\frac{k+p}{|B_p^k|} \int_{B_p^k} |\langle x, v \rangle|^p \, dx \right)^{-1/p}$$

where the minimum is taken over all vectors v of Euclidean length 1. So to complete the proof it suffices to show that for such a vector v,

$$\left(\frac{k+p}{|B_p^k|} \int_{B_p^k} |\langle x, v \rangle|^p \, dx \right)^{1/p} \leq \begin{cases} \sqrt{p/2} & \text{if } p \geq 2 \\ 2 & \text{if } 1 \leq p < 2. \end{cases}$$

Let $(x^{(j)})_1^k$ and $(v^{(j)})_1^k$ be the coordinates of the vectors x and v in $\mathbf{R}^k$. For $p \geq 2$, observe that the functions $(x^{(j)}v^{(j)})$ on B_p^k form a conditionally symmetric sequence, so by Khintchine's inequality and Hölder's inequality (for $\sum_1^n v^{(j)2} = 1$),

$$\left(\frac{k+p}{|B_p^k|} \int_{B_p^k} \left| \sum_1^k x^{(j)} v^{(j)} \right|^p dx \right)^{1/p}$$

$$\leq \sqrt{\frac{p}{2}} \left(\frac{k+p}{|B_p^k|} \int_{B_p^k} \left(\sum_1^k x^{(j)2} v^{(j)2} \right)^{p/2} dx \right)^{1/p}$$

$$\leq \sqrt{\frac{p}{2}} \left(\frac{k+p}{|B_p^k|} \int_{B_p^k} \sum_1^n |x^{(j)}|^p \cdot v^{(j)2} \, dx \right)^{1/p}$$

$$= \sqrt{\frac{p}{2}} \left(\frac{k+p}{|B_p^k|} \int_{B_p^k} |x^{(1)}|^p \, dx \right)^{1/p}$$

$$= \sqrt{\frac{p}{2}}.$$

For $1 \leq p < 2$ it is easily checked that

$$\left(\frac{k+p}{|B_p^k|} \int_{B_p^k} |\langle x, v \rangle|^p \, dx \right)^{1/p}$$

$$\leq (k+p)^{1/p} \left(\frac{1}{|B_p^k|} \int_{B_p^k} \langle x, v \rangle^2 \, dx \right)^{1/2}$$

$$= (k+p)^{1/p} \left(\frac{1}{|B_p^k|} \int_{B_p^k} (x^{(1)})^2 \, dx \right)^{1/2}$$

and the last expression can be (rather roughly) estimated by 2 using standard inequalities involving logarithmically concave functions. □

Remark. The proof of Theorem 2 can be simplified even further if the integration over B_p^k is replaced by integration over S^{k-1} (and Hölder's inequality applied here). The proof was presented as above because Lemma 5 has some intrinsic interest: for example it may be used to recover Gluskin's precise estimate as follows. Suppose $m \in \mathbf{N}$ and the vectors $(z_i)_1^m \in \mathbf{R}^k$ satisfy

$$|z_i| \leq (\log(1 + m/k))^{-1/2}, \qquad 1 \leq i \leq m.$$

For $\varepsilon > 0$, let $W(\varepsilon)$ be the set

$$\left\{ x \in \mathbf{R}^k : \max_j |x^{(j)}| \leq 1, \, \max_i |\langle x, z_i \rangle| \leq \frac{1}{\varepsilon} \right\};$$

that is, $W(\varepsilon)$ is the intersection of the cube B^k_∞ with m "bands" of width at most $(2/\varepsilon)\sqrt{\log(1 + m/k)}$. Then $|W(\varepsilon)|^{1/k} \to 2$ as $\varepsilon \to 0$, uniformly in k and m. To see this, apply Lemma 5 with $n = k + m$, the first k, u_i's being the standard basis vectors $\mathbf{R}^k$ and the remaining m being the vectors $(\varepsilon z_i)^m_1$. If e_j is a standard basis vector,

$$\frac{k+p}{|B^k_p|} \int_{B^k_p} |\langle x, e_j \rangle|^p \, dx = 1$$

and so Lemma 5 (and the proof of Theorem 2) show that for each $p \geq 2$, $|W(\varepsilon)|^{1/k} \geq 2(1 + m/k(p/2)^{p/2}\varepsilon^p(\log(1 + m/k))^{-p/2})^{-1/p}$ and the latter is at least $2/(1 + \sqrt{e} \cdot \varepsilon)$ if $p = \max(2, 2\log(1 + m/k))$. □

As was briefly mentioned earlier, more general estimates than that of Theorem 2 are obtained in [BP] (for entropy numbers instead of volumes). It is worth noting however that even the argument of Theorem 2 can be used to give the following: there is a constant c so that if $u_1, \ldots, u_n \in \mathbf{R}^k$, $k \leq n$ and $T: l^k_2 \to l^n_\infty$ is given by $(Tx)_i = \langle u_i, x \rangle$, $1 \leq i \leq n$, then the kth entropy number of T satisfies

$$e_k(T) \leq \frac{c\sqrt{p}}{\sqrt{k}} \left(\frac{1}{k} \sum_1^n |u_i|^p \right)^{1/p} \left(1 + \log \frac{n}{k} \right)^{1/p}$$

and hence

$$e_k(T) \leq \frac{ec}{\sqrt{k}} \sqrt{1 + \log \frac{n}{k}} \cdot \|T\|$$

(taking $p = 2(1 + \log(n/k))$). To obtain this, one uses Schütt's estimates [S] for the entropy numbers of the formal identity from l^n_p to l^n_∞ in place of the result of Meyer and Pajor, an the dual Sudakov inequality of Pajor and Tomczak, [PT] in place of the application of Hölder's inequality.

2. An application to linear forms

As stated in the Introduction, Vaaler's original result has applications to the geometry of numbers. One such, a sharpened form of Siegel's lemma, is given in [BV]. Using the arguments of Bombieri and Vaaler and Theorem 2, one can obtain the generalization of their result, contained in Theorem 6, below. Some notation is needed. If A is a $k \times n$ matrix of reals with independent rows ($1 \leq k \leq n$), denote by $v_j = v_j(A)$, $1 \leq j \leq k$, the rows of A. Let $(e_i)^n_1$ be the standard basis of $\mathbf{R}^n$ and denote by c_i, the distance (in the Euclidean norm) of e_i from the span of the v_j's in $\mathbf{R}^n$. (So if A_i is the matrix with $k + 1$ rows, $v_1, \ldots, v_n, e_i$ then

$$c_i^2 = \frac{\det(A_i A_i^*)}{\det(A A^*)} \quad \text{for } 1 \leq i \leq n.)$$

Theorem 6. *Let A be a $k \times n$ matrix with rank k and integral entries. With the notation above, the system $Ax = 0$ admits $n - k$ linearly independent solutions*

$$z^{(r)} = (z_1^{(r)}, \ldots, z_n^{(r)}) \in \mathbf{Z}^n, \qquad 1 \le r \le n - k$$

so that for every $p \ge 2$,

$$\prod_{1 \le r \le n-k} \max_i |z_i^{(r)}| \le D^{-1} \sqrt{\frac{p}{2}} \left(\frac{1}{n-k} \sum_1^n c_i^p \right)^{1/p} \sqrt{\det AA^*}$$

where D denotes the $G - C - D$ of all $k \times k$ determinants extracted from A. $\square$

Remark. The principal importance of such a generalization of Bombieri and Vaaler's result is that it takes into account, more strongly, the form of the matrix A. If the c_i's are all about the same size, then for $p > 2$, the expression

$$\left(\frac{1}{n-k} \sum c_i^p \right)^{1/p}$$

is small compared with the corresponding expression in which p is replaced by 2.

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